RESEARCH STATEMENT

KENTARÔ YAMAMOTO

Overview

Most of my work involves various Stone-type dualities for algebras. Most often, we axiomatically define a class $\mathcal{V}$ of algebras, and paradigmatic examples X^+ in $\mathcal{V}$ are constructed in a specific manner from some mathematical object X in some other class $\mathcal{F}$. For instance, with the class of Boolean algebras, X is a pure set, and X^+ is the powerset of X; with the class of groups, X is again a pure set, and $X^+ = \mathrm{Sym}(X)$. One is then interested in obtaining an arbitrary member $A \in \mathcal{V}$ as a subalgebra of $(\mathrm{Cst}\, A)^+$, where $\mathrm{Cst}\, A$ is in $\mathcal{F}$, and encoding which subalgebra of $(\mathrm{Cst}\, A)^+$ it is by using an appropriate topology on $\mathrm{Cst}\, A$. Stone showed that this is possible with $\mathcal{F}$ being the class of Boolean algebras, where $\mathrm{Cst}\, A$ is the set of ultrafilters of A, and inspired important ideas in modern mathematics from Gelfand duality for C^*-algebras to Zariski topology. Duality allows one to concretely construct algebras on one hand and to complete them by supplementing them with missing points on the other.

In one article [15], I develop a novel Stone-type duality for spectral spaces and ortholattices, a class of algebras generalizing Boolean algebras and important in alternative foundations of quantum theory. In multiple other articles [23, 24], I apply Stone-type dualities in model theory of generic algebras, to prove results on the connection between logic and other areas of mathematics from combinatorics to permutation group theory to topological dynamics. In the most recent work [16], I obtain a syntax-free, algebraic characterization of logical equivalence in model theory of generic structures (this is an analogue of the Keisler-Shelah theorem, which gives a structural necessary and sufficient condition for two structures to satisfy the same sentences of first-order logic). In my ongoing project, I use Stone-type representation of partial orders by not just totally disconnected spaces or spectral spaces, but of *all* compacta and aim to develop model theory for structures with compact topology in general.

1. Research achievements

1.1. **Model theory.**

1.1.1. *Positive logic.* Positive logic [17] can be motivated as a common generalization of Tarski's model theory, which focuses on elementary embeddings, and Robinson's model theory, which focuses on general embeddings. (Other, better-known sources for positive logic include the *continuous logic* [2] in the sense of Chang and Keisler, in which sentences need not have corresponding negations.) In a recently finished work [16], I investigate the counterparts of ultraproducts in positive logic, which I call *prime products*, and in particular prove the analogue of the Keisler-Shelah theorem, showing that prime products characterize elementary equivalence in positive logic:

Theorem 1.1. (1) *Two structures have the same positive theory, i.e., they satisfy the same sentences formed from atomic formulas by applying $\exists$ and $\wedge$, if and only if they have isomorphic prime products of ultrapowers.*

(2) *(Generalized Continuum Hypothesis) Two structures have the same positive theory if and only if they have isomorphic prime* powers *of ultrapowers.*

This result also has an application in algebraic logic:

Corollary 1.2. *Every pair of members of K have isomorphic prime products of ultrapowers if and only if all the vacuously admissible rules of L are derivable in L, where K is a quasivariety algebrizing a logic L.*

1.1.2. *Automorphism groups of countable ultrahomogeneous structures.* A countable structure M is *ultrahomogeneous* if any isomorphism between finitely generated substructures thereof can be extended to an isomorphism of M. Paradigmatic examples include the total order of rationals and the (unique) countable atomless Boolean algebra. The automorphism group of such a countable structure can be regarded as a Polish (topological) group inheriting the topology of the Baire space $\mathbb{N}^{\mathbb{N}}$. Ultrahomogeneous structures have many automorphisms, and their properties manifest themselves in their automorphism groups.

I study the automorphism group of the countable ultrahomogeneous Heyting algebra L into which every finite Heyting algebra embeds, which exists and is unique up to isomorphism. Heyting algebras are mild generalizations of Boolean algebras motivated by intuitionistic logic, and L is a natural object of study in second-order intuitionistic propositional logic [7]. Motivations for studying the automorphism group of L include the fact that L is similar to better-known structures like the countable atomless Boolean algebra B while it is different from them in a number of different key points. For instance, L has infinitely many 1-generated substructures up to isomorphism and thus is not ω-categorical, while better-studied ultrahomogeneous structures in natural finite signatures historically tend to be ω-categorical.

I prove the following in my article [23]:

Theorem 1.3. (1) *$G := \mathrm{Aut}(L)$ is not* Roelcke precompact*.*
(2) *G is not amenable.*
(3) *G is simple.*

A topological group is Roelcke precompact if the completion of the group with respect to the meet of its left and right uniformities, respectively, is compact. Since the direct limit of the automorphism groups of any ω-categorical countable ultrahomogeneous structures is Roelcke precompact [19], this result amounts to G being strictly different from better-known topological groups arising as the automorphisms groups of countable ω-categorical structures. The non-amenability follows from combinatorics of finite Heyting algebras, and the simplicity from the Craig interpolation theorem for intuitionistic propositional logic [14]. Furthermore, in a different article [24], I show:

Theorem 1.4. *G has the* small index property*.*

This amounts to saying that the topology of G is uniquely determined by its abstract group structure and the property that G is a closed subgroup of $\mathrm{Sym}(\mathbb{N})$. The proof method was based on the Stone-type representation, or the Esakia dual [9], of L by a certain partial order $\leq$ on the Cantor set 2^{ω} such that the group of homeomorphisms of 2^{ω} preserving $\leq$ is exactly $\mathrm{Aut}(L)$.

1.2. **Stone-type duality.** In [15], I extend the choice-free analogue [4] of Stone duality of Boolean algebras, and I develop a choice-free duality theory for ortholattices. Ortholattices are of interest in alternative mathematical foundations of quantum physics [8] because the family of closed subspaces of any Hilbert space is an ortholattice. I prove without any choice principle:

Theorem 1.5 (Zermelo-Fraenkel set theory (ZF))**.** *The category of ortholattices* **OL** *is dually equivalent to a category* **UVO** *of certain spectral spaces and certain spectral maps.*

To describe **UVO**, we say that a subset S of a set with a irreflexive symmetric relation $\perp$ is *orthoregular* whenever $S^{\perp\perp} = S$, where $T^{\perp}$ is the image of a set T under $\perp$.

Definition 1.6. The objects of **UVO** are topological spaces X with a irreflexive symmetric relation with the following properties:

- X is T_0;
- The compact open orthoregular subsets of X consist a basis τ;
- τ is closed under $\cap$ and ${}^{\perp}$;
- Every proper filter of $\perp$ is of the form $\{U \in \tau \mid x \in U\}$ for some $x \in X$;
- If $x \perp y$, then for some $U \in \tau$, $x \in U$ and $y \in U^{\perp}$.

The morphisms of **UVO** are spectral *weak bounded morphisms.*

Here, weak bounded morphisms are generalizations of bounded morphisms with respect to the negation of $\perp$.

My result is a choice-free analogue of the work by Goldblatt [11], which I examine and compare with my approach:

Theorem 1.7.
(1) *(ZF and the Axiom of Choice) The category of ortholattice is dually equivalent to a category of certain Stone spaces and certain continuous maps.*
(2) *(ZF) That this functor is essentially surjective is equivalent to the Boolean Prime Ideal Theorem.*

Note that the Boolean Ideal Prime Ideal Theorem is not a theorem of ZF. In the same paper, I also develop a dictionary of how algebraic notions and constructions can be translated topologically in terms of their dual under the choice-free duality. For instance:

Theorem 1.8. *Let F : **OL** $\to$ **UVO** be the functor yielding the equivalence of categories in the previous theorem.*
(1) *$F(f)$ is surjective if and only if f is an embedding. $F(f)$ is an embedding if and only if f is surjective.*
(2) *L is complete if and only if for every open set U in $F(L)$, $U^{\perp\circ\perp}$ is compact, open, and orthoregular.*

I also have publications [22, 21] on other applications of Stone-type duality in logic.

2. Current & Future research plans

2.1. **Positive logic.** I plan to work on open problems left by the project on prime powers and the Keisler-Shelah theorem, regarding obtaining a ZFC theorem in the simpler form seen in Theorem 1.1.(2). In fact, Theorem 1.1.(2) is a corollary of the following even simpler theorem of ZFC:

Theorem 2.1. *Suppose that each of A and B is ω-saturated and has an endomorphism factoring through an embedding from a negatively sufficient [17] substructure. Then A and B have the same positive theory if and only if they have isomorphic prime powers.*

Here, negative sufficiency of a structure M means that if a finite tuple in a structure M does not satisfy a positive formula ϕ, then there is another positive formula ϕ' such that no tuple in M satisfy $\phi \wedge \phi'$. What is important to us is that saturated structures (or more precisely, any structure universal with respect to models of cardinality *at most* its cardinality) have such endomorphisms, so Theorem 1.1.(2) follows from the Generalized Continuum Hypothesis. This suggests the following problem that I plan to work on:

Problem 2.2. Is there a positive-logical analogue of the finite cover property [18] whose negation, in model theory of first-order logic, implies saturation of ultraproducts of countable models?

A positive solution to this problem would not only fit within the research program (e.g., [3]) of finding "dividing lines" in positive logic analogous to those in first-order logic but also lead to a ZFC version of Theorem 1.1.(2) for models of theories lacking that property.

I also plan to study the following problem:

Problem 2.3. Characterize structures that have endomorphisms that factor through embeddings from a negatively sufficient substructure.

Such structures turn out to be important in the theory of constraint satisfaction problems in theoretical computer science [5]. This suggests that this problem is of an independent interest in addition to its application to a simpler ZFC version of Theorem 1.1.

A related problem I would like to solve is the following.

Problem 2.4. Is there an analogue of ultraproducts that preserves negative sufficiency and has the important properties of ultraproducts such as Łoś's Theorem, saturation, the Keisler-Shelah theorem, interaction with the finite cover property, etc.?

Prime products in the sense of Theorem 1.1 do not preserve negative sufficiency. My definition of prime products are justifiable because it is naturally obtained by replacing the Stone duality latent in the definition of ultraproducts with the Esakia duality, and in the intended application in algebraic logic, the construction must work not just for negatively sufficient structures but for *all* structures. A solution to this problem may be more amenable to a positive solution to Problem 2.2 as well as types with parameters are better behaved over negatively sufficient structures.

In carrying out this project, I am planning to collaborate with algebraists, so I may find algebraic applications of positive logic such as Corollary 1.2.

2.2. **Logic for compact structures.** Countable ultrahomogeneous structures can be constructed as "limits" of some sort of categories of finite structures and embeddings satisfying certain properties [10]. One can replace those embeddings between finite objects with quotients and turn the theory into that on metrizable Stone spaces and relations on it (this construction was used in my proofs on the countable ultrahomogeneous Heyting algebra [24]). These relations can be made into equivalence relations, quotients with respect to which turn out to be, e.g., the pseudo-arc [12]. This new construction of the pseudo-arc is used in the new proof of the continuum's homogeneity reported by Solecki. Bartoš, Bice, and Vignati independently has yet another proof, which is based on their immediate profinite construction of compacta [1], which is reminiscent of Stone-type dualities of lattices but, unlike them, is able to produce Hausdorff yet non-zero-dimensional spaces. I now work on a systematic logical study of these profinite objects in general, focusing on their homogeneity and its logical characterization.

Problem 2.5. Develop a logic for compact structures in general that admits:

(1) the compactness theorem,
(2) the downward Löwenheim-Skolem theorem,

(3) Vaught's theorem on countable atomic models and homogeneous models,
(4) the Engeler–Svenonius–Ryll-Nardzewski theorem, and
(5) the Scott-Karp analysis.

I have a preliminary result on the first two tasks. The downward Löwenheim-Skolem theorem guarantees that models obtained by compactness can be assumed to be metrizable. Items (3) and (4) will be then applied systematically to prove homogeneity of compacta and compact structures. The Scott sentences will yield new invariants for classifying compacta.

My new logic can be understood in terms of the so-called cologic (e.g., [13]), a categorical logic jointly generalizing various frameworks from the ordinary first-order logic to coalgebraic logic (e.g., [20]). Historically, cologic has been proved successful in the study of profinite groups [6]. In Kruckman's parlance, ordinary first-order logic is the logic of finitely presentable structures and homomorphisms, and Solecki works in the logic of finite sets and partitions. My approach would then be similar to the logic of finite graphs and relational morphisms, but it addresses the problem that general compacta, unlike metrizable Stone spaces, do not have canonical countable bases and that categories of relations tend not to have many limits.

Problem 2.6. Investigate the categorical formulation of the logic in Problem 2.5 to see its relationship with Kruckman's cologic. If it is not a special case of cologic, devise a common generalization of the two logics.

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