

# DECIDING POLYHEDRALITY OF SPECTRAHEDRA

AVINASH BHARDWAJ, PHILIPP ROSTALSKI, AND RAMAN SANYAL

ABSTRACT. Spectrahedra are linear sections of the cone of positive semidefinite matrices which, as convex bodies, generalize the class of polyhedra. In this paper we investigate the problem of recognizing when a spectrahedron is polyhedral. We generalize and strengthen results of Ramana (1998) regarding the structure of spectrahedra and we devise a normal form of representations of spectrahedra. This normal form is effectively computable and leads to an algorithm for deciding polyhedrality.

## 1. INTRODUCTION

A **polyhedron**  $P^a$  is the intersection of the convex cone of non-negative vectors  $\mathbb{R}_{\geq 0}^n$  with an affine subspace. By choosing coordinates for the affine subspace, we can abuse notation and write

$$P^a = \{\mathbf{x} \in \mathbb{R}^{d-1} : b_i - \mathbf{a}_i^T \mathbf{x} \geq 0 \text{ for } i = 1, 2, \dots, n\}$$

for some  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^{d-1}$  and  $b_1, b_2, \dots, b_n \in \mathbb{R}$ . Polyhedra represent the geometry underlying linear programming [23] and, as a class of convex bodies, enjoy a considerable interest throughout pure and applied mathematics. A proper superclass of convex bodies that inherits many of the favorable properties of polyhedra is the class of spectrahedra.

A **spectrahedron**  $S^a$  is the intersection of the convex cone of positive semidefinite matrices with an affine subspace. Identifying the affine subspace with  $\mathbb{R}^{d-1}$  we write

$$S^a = \{\mathbf{x} \in \mathbb{R}^{d-1} : A_0 + x_1 A_1 + \dots + x_{d-1} A_{d-1} \succeq 0\}$$

where  $A_0, \dots, A_d \in \mathbb{R}^{n \times n}$  are symmetric matrices. Thus, a spectrahedron is to a semidefinite program, what a polyhedron is to a linear program. The associated map  $A : \mathbb{R}^{d-1} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$  given by  $A(\mathbf{x}) = A_0 + x_1 A_1 + \dots + x_{d-1} A_{d-1}$  is called a (symmetric) **matrix map** and a (symmetric) matrix  $A \in \mathbb{R}^{n \times n}$  is **positive semidefinite**  $A \succeq 0$  if  $v^T A v \geq 0$  for all  $v \in \mathbb{R}^n$ . Hence, the set of points  $S^a \subseteq \mathbb{R}^{d-1}$  at which  $A(\mathbf{x})$  is positive semidefinite is determined by a quadratic family of halfspaces

$$v^T A(\mathbf{x}) v = v^T A_0 v + x_1 v^T A_1 v + x_2 v^T A_2 v + \dots + x_{d-1} v^T A_{d-1} v \geq 0.$$

Spectrahedra and their projections have received considerable attention in combinatorial optimization [15], polynomial optimization [7], and convex algebraic geometry [9]. To see that

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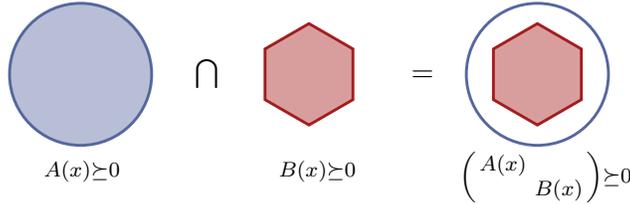
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polyhedra are spectrahedra, observe that a diagonal matrix is positive semidefinite if and only if the diagonal is non-negative. Thus, we have

$$P^a = \{\mathbf{x} \in \mathbb{R}^{d-1} : D(x) \succeq 0\}$$

where  $D(x) = \text{Diag}(b_1 - \mathbf{a}_1^T \mathbf{x}, \dots, b_n - \mathbf{a}_n^T \mathbf{x})$  is a diagonal matrix map.

It is a theoretically interesting and practically relevant question to recognize when a spectrahedron is a polyhedron. The diagonal embedding of  $\mathbb{R}_{\geq 0}^n$  into the cone of positive semidefinite matrices suggests that a spectrahedron is a polyhedron if  $A(\mathbf{x})$  can be diagonalized, i.e.,  $UA(x)U^{-1}$  is diagonal for some (orthogonal) matrix  $U$ . By basic linear algebra this is possible if and only if  $A(\mathbf{p})$  and  $A(\mathbf{q})$  commute for all  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{d-1}$ . While this is certainly a sufficient condition, observe that by Sylvester's law of inertia  $S^a = \{\mathbf{x} : LA(\mathbf{x})L^T \succeq 0\}$  for any non-singular matrix  $L$  and  $LA(\mathbf{x})L^T$  is in general not commuting. A more serious situation is when a polyhedron is *redundantly* presented as the intersection of a proper 'big' spectrahedron and a 'small' polyhedron contained in it.



In this case, the diagonalizability criterion is genuinely lost.

In this paper we consider the question of algorithmically telling polyhedra from spectrahedra. This question was first addressed by Ramana [18] with a focus on the computational complexity. Our results regarding the structure of spectrahedra strengthen and generalize those of [18] and we present a simple algorithm to test if a spectrahedron  $S = \{x : A(x) \succeq 0\}$  is a polyhedron. The algorithm we propose consists of two main components:

- (Approximation) Calculate polyhedron  $\widehat{S}^a \supseteq S^a$  from  $A(\mathbf{x})$ , and
- (Containment) Determine whether  $\widehat{S}^a \subseteq S^a$ .

Finding a *fast* algorithm is not to be expected: Ramana [18] showed that deciding whether a spectrahedron is polyhedral is NP-hard. As detailed later, the 'Containment' step, which is coNP-hard by the results in [10], is done by enumerating all vertices/rays of  $\widehat{S}$ . This is clearly not feasible in practise and we make no claim that our algorithm is suitable for preprocessing semidefinite programs. However, as in the case of the 'vertex enumeration problem' for polyhedra, it is of considerable interest to have a *practical* algorithm for exploration, experimentation, hypothesis testing with spectrahedra. Our motivation arose in exactly this context. We nevertheless anticipate applications of our algorithm in the area of (combinatorial) optimization in particular in connection with *semidefinite extended formulations*; see, for example, [5, 8]. In Section 3 our algorithm is discussed in some detail and illustrated along an example. We close with some remarks regarding implementation and the complexity of the approximation step.

As for the approximation step, note that if there is a point  $\mathbf{p} \in S^a$  with  $A(\mathbf{p})$  positive definite, then the *algebraic boundary*, the closure of  $\partial S^a$  in the Zariski topology, is contained in the vanishing locus of  $f(\mathbf{x}) = \det A(\mathbf{x}) \neq 0$ . Thus, if  $F \subset S^a$  is a face of codimension one, the unique supporting hyperplane is a component of the algebraic boundary of  $S^a$  and hence yields a linear

factor of  $f$ . Therefore, isolating linear factors in  $f$  gives rise to a polyhedral approximation  $\widehat{S}^a$  of  $S^a$ . However, factoring a multivariate polynomial is computationally expensive and an alternative is the use of *numerical algebraic geometry* such as Bertini [1] to isolate the codimension one components of degree one (possibly with multiplicities). Our approach avoids calculating the determinant of the matrix map altogether by pursuing more algebro-geometric considerations. Ramana [18] showed that if  $S$  is a polyhedron, then in particular coordinates  $A(\mathbf{x})$  reveals the relevant linear factors in block diagonal form. In Section 2 we recall and strengthen Ramana's results with very short proofs that highlights the underlying geometry. In particular, our proof emphasizes the role played by eigenspaces of the matrix map. From this, we define a normal form with stronger properties and we prove that the polyhedral approximation can be obtained by essentially computing the *joint invariant subspace* of two generic points in the image of  $A(\mathbf{x})$ .

**Convention.** For reasons of clarity and elegance, we will work in a linear instead of an affine setting. That is, our main objects are exclusively *spectrahedral cones* and hence all matrix maps are *linear* maps  $\mathbb{R}^d \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$ . Clearly, all results can be translated between the linear and affine picture. The spectrahedral cone  $S$  associated to the prototypical spectrahedron  $S^a$  above is

$$S = \text{cone}\left\{(\mathbf{p}, 1) \in \mathbb{R}^d : \mathbf{p} \in S^a\right\} = \left\{(\mathbf{x}, x_d) \in \mathbb{R}^d : x_d A\left(\frac{1}{x_d}\mathbf{x}\right) \succeq 0, x_d \geq 0\right\}.$$

The affine picture is recovered by intersecting  $S$  with the hyperplane  $x_d = 1$ .

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## 2. NORMAL FORMS AND JOINT INVARIANT SUBSPACES

Let  $S = \{\mathbf{x} \in \mathbb{R}^d : A(\mathbf{x}) \succeq 0\}$  be a full-dimensional spectrahedral cone. Throughout this section, we will assume that  $A(\mathbf{x})$  is of full rank, i.e., that there is a point  $\mathbf{p} \in S$  with  $A(\mathbf{p}) \succ 0$ . As explained in the next section, this is not a serious restriction. We are interested in the codimension one faces of  $S$  and how they manifest in the presentation of  $S$  given by  $A(\mathbf{x})$ . Let us recall the characterization of faces of a spectrahedral cone.

**Lemma 2.1** ([16, Thm. 1]). *Let  $S = \{\mathbf{x} : A(\mathbf{x}) \succeq 0\}$  be a full-dimensional spectrahedral cone. For every face  $F \subseteq S$  there is an inclusion-maximal linear subspace  $\mathcal{L}_F \subset \mathbb{R}^n$  such that*

$$F = \{\mathbf{p} \in S : \mathcal{L}_F \subseteq \ker A(\mathbf{p})\}.$$

For the case of faces of codimension one, this characterization in terms of kernels implies strong restrictions on the describing matrix map.

**Theorem 2.2.** *Let  $S = \{\mathbf{x} : A(\mathbf{x}) \succeq 0\}$  be a full-dimensional spectrahedral cone and let  $F \subset S$  be a face of codimension one. Then there is a non-singular matrix  $M \in \mathbb{R}^{n \times n}$  such that*

$$MA(\mathbf{x})M^T = \begin{bmatrix} A'(\mathbf{x}) \\ \ell(\mathbf{x})\text{Id}_k \end{bmatrix}$$

where  $k \geq 1$  and  $\ell(\mathbf{x})$  is a supporting linear form such that  $F = \{\mathbf{x} \in S : \ell(\mathbf{x}) = 0\}$ .

*Proof.* Let  $B = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_d)$  be a basis of  $\mathbb{R}^d$  such that  $\mathbf{b}_1 \in \text{int} S$  and  $\mathbf{b}_2, \dots, \mathbf{b}_d \in F$ . By applying a suitable congruence, we can assume that  $A(\mathbf{b}_1) = \text{Id}$ . In light of Lemma 2.1,

let  $U^T = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n) \in \mathbb{R}^{n \times n}$  be an orthonormal basis of  $\mathbb{R}^n$  such that  $\mathcal{L}_F$  is spanned by  $\mathbf{u}_{n-k+1}, \dots, \mathbf{u}_n$  with  $k = \dim \mathcal{L}_F$ . It is easily seen that  $UA(B\mathbf{x})U^T$  is of the form

$$\begin{bmatrix} A'(B\mathbf{x}) & \\ & x_1 \text{Id}_k \end{bmatrix}.$$

Reverting to the original coordinates ( $\mathbf{x} \mapsto B^{-1}\mathbf{x}$ ) replaces  $x_1$  by  $\ell(\mathbf{x})$ .  $\square$

The form of the matrix map as given in the previous Lemma expresses  $S$  as the intersection of a linear halfspace and a spectrahedral cone  $S' = \{\mathbf{x} : A'(\mathbf{x}) \succeq 0\}$ . Repeating the process for  $S'$  proves

**Corollary 2.3.** *Let  $S = \{\mathbf{x} : A(\mathbf{x}) \succeq 0\}$  be a full-dimensional spectrahedral cone. Then there is a non-singular matrix  $M \in \mathbb{R}^{n \times n}$  such that*

$$(\star) \quad MA(\mathbf{x})M^T = \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}$$

where  $D(\mathbf{x})$  is a diagonal matrix map of order  $m \geq 0$ . Moreover, if  $F \subset S$  is a face of codimension one, then  $F = \{\mathbf{x} \in S : D_{ii}(\mathbf{x}) = 0\}$  for some  $1 \leq i \leq m$ .  $\square$

If  $S$  is a polyhedral cone then all inclusion-maximal faces have codimension one and hence  $S$  is determined by  $D(\mathbf{x})$  alone. This recovers Ramana's result.

**Corollary 2.4** ([18, Thm. 1]). *Let  $S$  be a full-dimensional spectrahedral cone. Then  $S$  is polyhedral if and only if there is an  $M \in \text{GL}_n(\mathbb{R})$  such that*

$$MA(\mathbf{x})M^T = \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}$$

where  $D(\mathbf{x})$  is a diagonal matrix map and  $S = \{\mathbf{x} : D(\mathbf{x}) \succeq 0\}$ .  $\square$

The following example illustrates the results.

**Example 2.5.** The two dimensional spectrahedral cone given by

$$A(x, y) = x \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} + y \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \succeq 0$$

is the polyhedral cone generated by  $\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ . A congruence that takes  $A(x, y)$  to diagonal form is given by

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & -1 \end{bmatrix}.$$

which is unique up to left-multiplication with  $\text{Diag}(a, b)$  and  $a, b \in \mathbb{R} \setminus \{0\}$ .  $\diamond$

The example shows that a rationally presented spectrahedron cannot be transformed into  $(\star)$  of Corollary 2.3 in rational arithmetic, an issue that is commonly encountered with semidefinite programs. More importantly it shows that the congruence is not necessarily an orthogonal transformation. However, this can be remedied under an additional assumption. A matrix map  $A(\mathbf{x})$  is **unital** if there is a  $\mathbf{p}_0 \in \mathbb{R}^d$  such that  $A(\mathbf{p}_0) = \text{Id}$ .

**Proposition 2.6.** *Let  $A(\mathbf{x})$  be a unital matrix map. Then there is an orthogonal  $n \times n$ -matrix  $U$  such that  $UA(\mathbf{x})U^T$  is in the form of Corollary 2.3.*

*Proof.* If  $M$  is a positive definite matrix, then, by the existence of a Cholesky decomposition, there is a matrix  $L \in \mathbb{R}^{n \times n}$  such that  $LML^T = \text{Id}_n$ . Moreover, such an  $L$  is unique up to left multiplication by an orthogonal matrix element, i.e., if  $L'$  also satisfies the condition, then  $(L')^{-1}L$  is orthogonal.

Let  $M$  be such that  $MA(\mathbf{x})M^T$  has the form of Corollary 2.4 and, since  $Q(\mathbf{p}_0)$  and  $D(\mathbf{p}_0)$  are both positive definite, let  $L_Q$  and  $L_D$  be respective Cholesky inverses such that  $L_D$  is diagonal. Now,

$$L = \begin{bmatrix} L_Q & \\ & L_D \end{bmatrix} M$$

also satisfies the condition of Corollary 2.4 and is a Cholesky inverse for  $M(\mathbf{p}_0)$ . However, a Cholesky inverse for  $M(\mathbf{p}_0) = \text{Id}_n$  is given by  $L' = \text{Id}_n$  and by the above remark, we have that  $L = (L')^{-1}L$  is orthogonal.  $\square$

The benefit of unital matrix maps is that they afford a normal form which can be effectively computed. Let us call a matrix map  $Q(\mathbf{x})$  **proper** if there is no non-zero  $v \in \mathbb{R}^n$  such that  $v$  is an eigenvector of  $Q(\mathbf{p})$  for every  $\mathbf{p} \in \mathbb{R}^d$ .

**Definition 2.7.** A unital matrix map  $A(\mathbf{x})$  is in **normal form** if

$$A(\mathbf{x}) = \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}$$

so that  $Q(\mathbf{x})$  is proper and  $D(\mathbf{x})$  is diagonal.

At this point it is tempting to think that the normal form of a unital map can be determined by analyzing the eigenstructure of a single (generic) point  $A(\mathbf{p})$ . The next example shows that this unfortunately is not the case.

**Example 2.8.** The spectrahedral cone  $S$  given by

$$A(x, y, t) = \begin{bmatrix} t & x & y \\ x & t & \\ y & & t \\ & & & t \end{bmatrix} \succeq 0$$

is the redundant intersection of the *second order cone*  $\{(x, y, t) : t \geq 0, t^2 \geq x^2 + y^2\}$  and the halfspace  $\{(x, y, t) : t \geq 0\}$ . Thus,  $S$  is determined by the principal  $3 \times 3$ -submatrix  $Q(x, y, t)$  and we claim that  $Q(x, y, t)$  is proper and that  $A(x, y, t)$  is already in normal form. To see this, note that both  $Q(1, 0, 0)$  and  $Q(0, 1, 0)$  have distinct eigenvalues but non-trivially intersecting eigenspaces. For any point  $(x_0, y_0, t_0)$ , the eigenspace of  $A_0 = A(x_0, y_0, t_0)$  associated to  $\lambda = t_0$  is 2-dimensional. Thus, there is no canonical choice of an eigenbasis for  $A_0$  which recovers the normal form.  $\diamond$

However, the normal form of a map  $A(\mathbf{x})$  is determined by the structure of the **joint invariant subspace**, the largest subspace  $\mathcal{N} \subseteq \mathbb{R}^n$  on which all elements in the image of  $A(\mathbf{x})$  commute. As the next result shows, the joint invariant subspace can be determined by considering two generic points in the image.

**Proposition 2.9.** *Let  $A(\mathbf{x})$  be a unital matrix map and let  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$  be two distinct generic points. Let  $\mathcal{N} \subset \mathbb{R}^n$  the smallest subspace containing all eigenvectors common to  $A(\mathbf{p})$  and  $A(\mathbf{q})$ . Then  $\mathcal{N}$  is invariant under any matrix in the image of  $A(\mathbf{x})$  and  $\mathcal{N}^\perp$  is the largest invariant subspace on which  $A(\mathbf{x})$  restricts to a proper matrix map.*

*Proof.* Assuming that  $A(\mathbf{x})$  is in normal form, it is sufficient to prove that for generic  $\mathbf{p}$  and  $\mathbf{q}$  there is no eigenvector common to  $Q(\mathbf{p})$  and  $Q(\mathbf{q})$ . The collections  $V$  of ordered pairs of matrices  $(A, B) \in (\mathbb{C}^{n \times n})^2$  with a common eigenvector is an algebraic variety and hence nowhere dense. This can be seen either by elimination (cf. [3, Ch. 3]) on the set of tuples  $(A, \lambda_A, B, \lambda_B, v)$  where  $v$  is an eigenvector of  $A$  and  $B$  with eigenvalue  $\lambda_A$  and  $\lambda_B$ , respectively. Alternatively, it follows directly from Theorem 3.3 below. Since  $Q(\mathbf{x})$  is proper, it follows that  $V$  restricts to a proper subvariety in  $\{(Q(\mathbf{p}), Q(\mathbf{q})) : \mathbf{p}, \mathbf{q} \in \mathbb{R}^d\}$ .  $\square$

### 3. THE ALGORITHM

In this section we describe an algorithm for recognizing polyhedrality of a spectrahedral cone

$$S = \{\mathbf{x} \in \mathbb{R}^d : A(\mathbf{x}) \succeq 0\}$$

where  $A(\mathbf{x})$  is a linear, symmetric matrix map of order  $n$ . As already stated in the introduction, the algorithm consists of two steps: An ‘approximation’ step that constructs an outer polyhedral approximation  $\widehat{S}$  from the matrix map  $A(\mathbf{x})$  that coincides with  $S$  whenever  $S$  is polyhedral. This is then verified in the ‘containment’ step.

For the approximation step note that if  $A(\mathbf{x})$  is in normal form, then  $S$  is presented as the intersection of a proper spectrahedron and a polyhedron (both of which can be trivial).

**Proposition 3.1.** *Let  $S = \{\mathbf{x} \in \mathbb{R}^d : A(\mathbf{x}) \succeq 0\}$  be a full-dimensional spectrahedral cone with*

$$A(\mathbf{x}) = \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}$$

*in normal form. Then  $\widehat{S} = \{\mathbf{x} : D(\mathbf{x}) \geq 0\}$  is a polyhedral cone with  $S \subseteq \widehat{S}$ .*  $\square$

Towards a procedure to bring  $A(\mathbf{x})$  into normal form, we need to ensure that  $S$  is full-dimensional and  $A(\mathbf{x})$  of full rank. Lemma 2.1 implies that faces of the PSD cone are embeddings of lower-dimensional PSD cones into subspaces parametrized by kernels. Recall that the **linear hull**  $\text{lin}(C)$  of a convex cone  $C$  is the intersection of all linear spaces containing  $C$  and  $C$  is full-dimensional relative to  $\text{lin}(C)$ .

**Proposition 3.2** ([16, Cor. 5]). *Let  $S = \{\mathbf{x} : A(\mathbf{x}) \succeq 0\}$  be a spectrahedral cone and let  $\mathbf{p} \in \text{relint } S$  a point in the relative interior. Then the linear hull of  $S$  is given by*

$$\text{lin}(S) = \{\mathbf{x} \in \mathbb{R}^d : \ker A(\mathbf{p}) \subseteq \ker A(\mathbf{x})\}.$$

*If  $\bar{A}(\mathbf{x})$  is the restriction of  $A(\mathbf{x})$  to  $(\ker A(\mathbf{p}))^\perp$ , then*

$$S = \{\mathbf{x} \in \text{lin}(S) : \bar{A}(\mathbf{x}) \succeq 0\}$$

*and  $\bar{A}(\mathbf{p}) \succ 0$ .*

In concrete terms this means that if  $M$  is a basis for the kernel of  $A(\mathbf{p})$  at a relative interior point  $\mathbf{p} \in S$ , then  $\text{lin}(S)$  is the kernel for all points in the image of  $MA(\mathbf{x})M^T$ . The map  $\bar{A}(\mathbf{x})$  is given by  $M_0A(\mathbf{x})M_0^T$  up to a choice of basis  $M_0$  for the orthogonal complement of  $\ker A(\mathbf{p})$ . Since  $\bar{A}(\mathbf{p})$  is positive definite, we can choose  $M_0$  so that  $\bar{A}(\mathbf{p}) = \text{Id}$  and hence is unital. This, for example, can be achieved by taking advantage of the Cholesky decomposition. By choosing a basis  $B$  for  $\text{lin}(S)$ , we identify  $\text{lin}(S) \cong \mathbb{R}^k$  for  $k = \dim S$  which insures that  $S \subset \mathbb{R}^k$  is full-dimensional. The resulting spectrahedral cone

$$\bar{S} = \{\mathbf{z} \in \mathbb{R}^k : \bar{A}(B\mathbf{z}) \succeq 0\}$$

is linearly isomorphic to  $S$  (via  $B$ ).

In actual computations, a point in the relative interior of  $S$  may be found by interior point algorithms. In case the spectrahedral cone  $S$  is *strictly feasible*, i.e., a point  $\mathbf{p} \in \mathbb{R}^d$  with  $A(\mathbf{p}) \succ 0$  exists, an interior point algorithm finds a point arbitrarily close to the analytic center of a suitable dehomogenization of  $S$ . Viewed as a linear section of the cone of positive semidefinite matrices,  $S$  is not strictly feasible, if the linear subspace only meets the boundary of  $\{X \succeq 0\}$ . These are subtle but well-studied cases in which techniques from semidefinite and cone programming such as self-dual embeddings [22, Ch. 5], facial reduction [2], or an iterative procedure analogous to [11, Remark. 4.15] can be used to obtain a point  $\mathbf{p} \in \text{relint } S$ . For the purpose of this paper, we will simply follow the first approach as detailed in the implementation remarks below. After applying the above procedure and possibly after a change of basis and a transformation of the matrix map  $A(\mathbf{x})$  we may assume that the spectrahedral cone is indeed full-dimensional and described by a unital matrix map.

Utilizing Proposition 2.9, we compute the normal form of the unital matrix map  $A(\mathbf{x})$  by determining an orthonormal basis for the joint invariant subspace  $\mathcal{N}$ . The joint invariant subspace is given as the smallest subspace containing all eigenvectors common to matrices  $A(\mathbf{p})$  and  $A(\mathbf{q})$  for generically chosen  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$ . It can be computed either by pairwise intersecting eigenspaces of  $A(\mathbf{p})$  and  $A(\mathbf{q})$  or, somewhat more elegantly, by employing the following result followed by a diagonalization step.

**Theorem 3.3** ([20, Thm. 3.1]). *Let  $A$  and  $B$  be two symmetric matrices. Then the smallest subspace containing all common eigenvectors is given by*

$$\mathcal{N} = \bigcap_{i,j=1}^{n-1} \ker [A^i, B^j].$$

where  $[A, B] = AB - BA$  is the commutator.

These techniques originate from the theory of finite dimensional  $C^*$ -algebras and have been used in block-diagonalizations of semidefinite programs; see [4, 14]. After all  $(n-1)^2$  commutators have been computed, the intersection of their kernels can be computed effectively by means of simple linear algebra. By Proposition 2.9, the restriction of  $A(\mathbf{x})$  to  $\mathcal{N}$  is a map of pairwise commuting matrices, there is an orthogonal transformation  $M$  such that

$$MA(\mathbf{x})M^T = \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}$$

has the desired normal form with  $Q(\mathbf{x})$  proper and  $D(\mathbf{x})$  diagonal. The outer polyhedral approximation of  $S$  obtained from  $A(\mathbf{x})$  is given by

$$\widehat{S} = \{\mathbf{x} \in \mathbb{R}^d : D(\mathbf{x}) \geq 0\}.$$

It remains to check that  $\widehat{S} \subseteq S$ . While deciding containment of general (spectrahedral) cones is difficult, we exploit here the finite generation of polyhedral cones.

**Theorem 3.4** ([23, Thm. 1.3]). *For every polyhedral cone  $C$  there is a finite set  $R = R(C) \subseteq C$  such that*

$$C = \left\{ \sum_{\mathbf{r} \in R} \lambda_{\mathbf{r}} \mathbf{r} : \lambda_{\mathbf{r}} \geq 0 \text{ for all } \mathbf{r} \in R \right\}.$$

Thus, if  $R(\widehat{S}) \subseteq S$ , we infer that  $\widehat{S} \subseteq S \subseteq \widehat{S}$  and hence  $S$  is polyhedral. Let us remark that computationally expensive polyhedral computations may be avoided by inspecting the lineality spaces of  $S$  and  $\widehat{S}$  first. The lineality space of  $S$ , i.e. the largest linear subspaces contained in  $S$ , is given by the kernel of the linear map  $A(\mathbf{x})$ . The complete procedure is given in Algorithm 1. As a certificate the algorithm returns the collection of generators  $R(\widehat{S})$ . As we assume that  $A(\mathbf{x})$  is in normal form, it can be easily checked if  $S$  is polyhedral or not.

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**Algorithm 1** *Recognizing polyhedrality of a spectrahedral cone*

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**Input:** Spectrahedral cone  $S = \{\mathbf{x} \in \mathbb{R}^d : A(\mathbf{x}) \succeq 0\}$  given by symmetric matrix map  $A(\mathbf{x})$ .

- 1: Generate point  $\mathbf{a} \in \mathbb{R}^d$  in the relative interior of  $S$ .
- 2: Compute unital matrix map  $\bar{A}(\mathbf{z})$  of order  $m$  and linear isomorphism  $B$  such that

$$S = \{B\mathbf{z} : \bar{A}(\mathbf{z}) \succeq 0\}.$$

- 3: Determine the joint invariant subspace  $\mathcal{N} = \bigcap_{i,j=1}^{n-1} \ker [\bar{A}(\mathbf{p})^i, \bar{A}(\mathbf{q})^j]$  for two generic points  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^k$ .
- 4: Compute an orthonormal basis  $U$  corresponding to the decomposition  $\mathbb{R}^k = \mathcal{N}^\perp \oplus \mathcal{N}$  and compute

$$U\bar{A}(\mathbf{z})U^T = \begin{bmatrix} Q(\mathbf{z}) & \\ & D'(\mathbf{z}) \end{bmatrix}.$$

- 5: Obtain diagonal map  $D(\mathbf{z}) = VD'(\mathbf{z})V^T$  via an orthogonal eigenbasis of  $D'(\mathbf{z})$ .
- 6: Compute the extreme rays  $R = R(\widehat{S})$  of the polyhedral cone

$$\widehat{S} = \{\mathbf{z} \in \mathbb{R}^k : D(\mathbf{z})_{ii} \geq 0 \text{ for all } i = 1, \dots, \dim \mathcal{N}\}$$

- 7:  $S$  is polyhedral if and only if  $Q(\mathbf{r}) \succeq 0$  for all  $\mathbf{r} \in R$ .
- 

**Implementation details.** The algorithm is implemented in Matlab using the free optimization package Yalmip [13] and is available as part of the convex algebraic geometry toolbox *Bermeja* [19]. The SDP solver chosen for the computation of an interior point is SeDuMi [21], which implements a self dual embedding strategy and is thus guaranteed to find a point in the relative interior, even if the spectrahedral cone is not full-dimensional. Extreme rays of  $\widehat{S}$  are computed using the software *cdd/cddplus* [6].

In order to illustrate the algorithm, we consider the following example involving a variant of the ellipsope  $\mathcal{E}_3$  (also known as the ‘‘Samosa’’), cf. [12].

**Example 3.5.** The spectrahedral cone  $S = \{\mathbf{x} \in \mathbb{R}^4 : A(\mathbf{x}) \succeq 0\}$  with

$$A(\mathbf{x}) = \begin{bmatrix} 4x_4 & 2x_4 + 2x_1 & 2x_4 & 0 & 2x_3 \\ 2x_4 + 2x_1 & 2x_4 + 2x_1 & x_4 + x_1 & 0 & x_3 + x_2 \\ 2x_4 & x_4 + x_1 & 2x_4 + x_1 & x_3 - x_2 & x_3 \\ 0 & 0 & x_3 - x_2 & x_4 + x_1 & 0 \\ 2x_3 & x_3 + x_2 & x_3 & 0 & x_4 \end{bmatrix}.$$

is to be analyzed. Since the spectrahedral cone in context is full-dimensional and  $A(\mathbf{x})$  is of full rank, i.e.  $A(\mathbf{p}) \succ 0$  with  $\mathbf{p} = (0, 0, 0, 1)$ , the algorithm proceeds by first making the matrix map unital. This is facilitated by applying the Cholesky inverse, computed at the interior point  $\mathbf{p}$ . The congruence transformation  $U$ , thus obtained yields the unital matrix map  $\bar{A}(\mathbf{z})$ , allowing the use of orthogonal transformations thereafter.

The next step involves separating the invariant subspace from its orthogonal complement. This step is carried out using Theorem 3.3, by means of computing all commutator matrices and then intersecting their kernel. The following step involves (simultaneous) diagonalization of the commuting part of the matrix (here the lower right  $2 \times 2$  block) in order to arrive at the desired normal form. This transformation matrix  $V$  may be computed by diagonalizing any generic matrix in the image, restricted to the commuting part. The corresponding unital matrix map  $UA(\mathbf{x})U^T$  and its normal form  $MA(\mathbf{x})M^T$  with  $M = \begin{bmatrix} I & \\ & V \end{bmatrix} U$  are depicted below:

$$UA(\mathbf{x})U^T = \begin{bmatrix} x_4 & x_1 & x_3 & 0 & 0 \\ x_1 & x_4 & x_2 & 0 & 0 \\ x_3 & x_2 & x_4 & 0 & 0 \\ 0 & 0 & 0 & x_4 + x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_3 - x_2 & x_4 + x_1 \end{bmatrix},$$

$$MA(\mathbf{x})M^T = \begin{bmatrix} x_4 & x_1 & x_3 & 0 & 0 \\ x_1 & x_4 & x_2 & 0 & 0 \\ x_3 & x_2 & x_4 & 0 & 0 \\ 0 & 0 & 0 & x_4 + x_3 - x_2 + x_1 & 0 \\ 0 & 0 & 0 & 0 & x_4 - x_3 + x_2 + x_1 \end{bmatrix}$$

The normal form clearly shows that the spectrahedral cone has two polyhedral faces. The

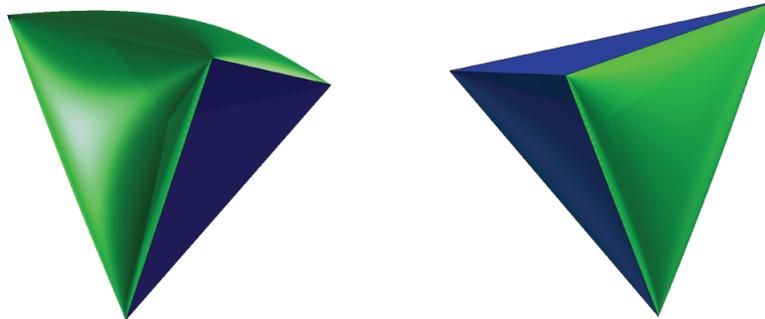


FIGURE 1. Dehomogenization  $S^a$  (at  $x_4 = 1$ ), of the spectrahedral cone  $S$

algorithm eventually terminates by confirming existence of a lineality space in the corresponding polyhedral cone, even though the initial spectrahedral cone was pointed. This ensures the non-polyhedrality of  $S$ . Figure 1 shows a dehomogenization ( $x_4 = 1$ ) of  $S$  with its two polyhedral facets.

**A word about complexity.** Although calculating the joint invariant subspace of a matrix map can be done in polynomial time, the transformation to an unital matrix map is more involved. The following example, adapted from [17, Example 23], shows that any such procedure may involve numbers with doubly-exponential bit complexity.

**Example 3.6.** Consider the family of spectrahedral cones

$$S_i = \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \begin{bmatrix} x_{i+1} & 2x_i \\ 2x_i & x_0 \end{bmatrix} \succeq 0 \right\} = \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \begin{array}{l} x_0 \geq 0 \\ x_0 x_{i+1} \geq 4x_i^2 \end{array} \right\}$$

for  $i = 0, \dots, d-1$ . The intersection  $S = S_0 \cap S_1 \cap \dots \cap S_{d-1}$  is strictly contained in the cone  $\{\mathbf{x} \in \mathbb{R}^{d+1} : x_i \geq 2^{2^{i-1}} x_0\}$ . Denote by  $A(\mathbf{x})$  the matrix map for  $S$ . Now assume that  $B(\mathbf{x})$  is a matrix map for  $S$  such that  $B(\mathbf{p}) = \text{Id}$  for some  $\mathbf{p} \in \text{int } S$ . Then  $B(\mathbf{x}) = UL A(\mathbf{x}) (UL)^T$  where  $L$  is the Cholesky inverse of  $A(\mathbf{p})$  and  $U$  is an orthogonal matrix. Denote by  $\mathbf{l} = (QL)_1$  the first column of  $QL$ . From the definition of the Cholesky decomposition we infer that  $0 < \|\mathbf{l}\|^2 = L_{11}^2 = \frac{1}{\mathbf{p}_n}$  has doubly-exponential bit complexity and hence for  $\mathbf{q} = (1, 0, \dots, 0)$ , we have that  $B(\mathbf{q}) = \mathbf{l}\mathbf{l}^T$  has doubly-exponential bit complexity.  $\diamond$

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AVINASH BHARDWAJ, DEPARTMENT OF INDUSTRIAL ENGINEERING AND OPERATIONS RESEARCH, UC BERKELEY, BERKELEY, USA

*E-mail address:* [avinash@ieor.berkeley.edu](mailto:avinash@ieor.berkeley.edu)

PHILIPP ROSTALSKI, INSTITUT FÜR MATHEMATIK, GOETHE-UNIVERSITÄT FRANKFURT, GERMANY

*E-mail address:* [rostalsk@math.uni-frankfurt.de](mailto:rostalsk@math.uni-frankfurt.de)

RAMAN SANYAL, FACHBEREICH MATHEMATIK UND INFORMATIK, FREIE UNIVERSITÄT BERLIN, GERMANY

*E-mail address:* [sanyal@math.fu-berlin.de](mailto:sanyal@math.fu-berlin.de)