Analysis of a Newsvendor Model
Mapping Supply and Demand from *real-world* to $(0,1)^2$ space

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1 Introduction

The Newsvendor model is one of the extensively studied models in the *Operations Research* Literature. The following analysis tries to quantify and verify the model to emulate the real world supplies and demand, by mapping them to $(0,1)^2$ space, as discussed by Dashi, in the recitation section on Tuesday, March 16^{th}, 2010.

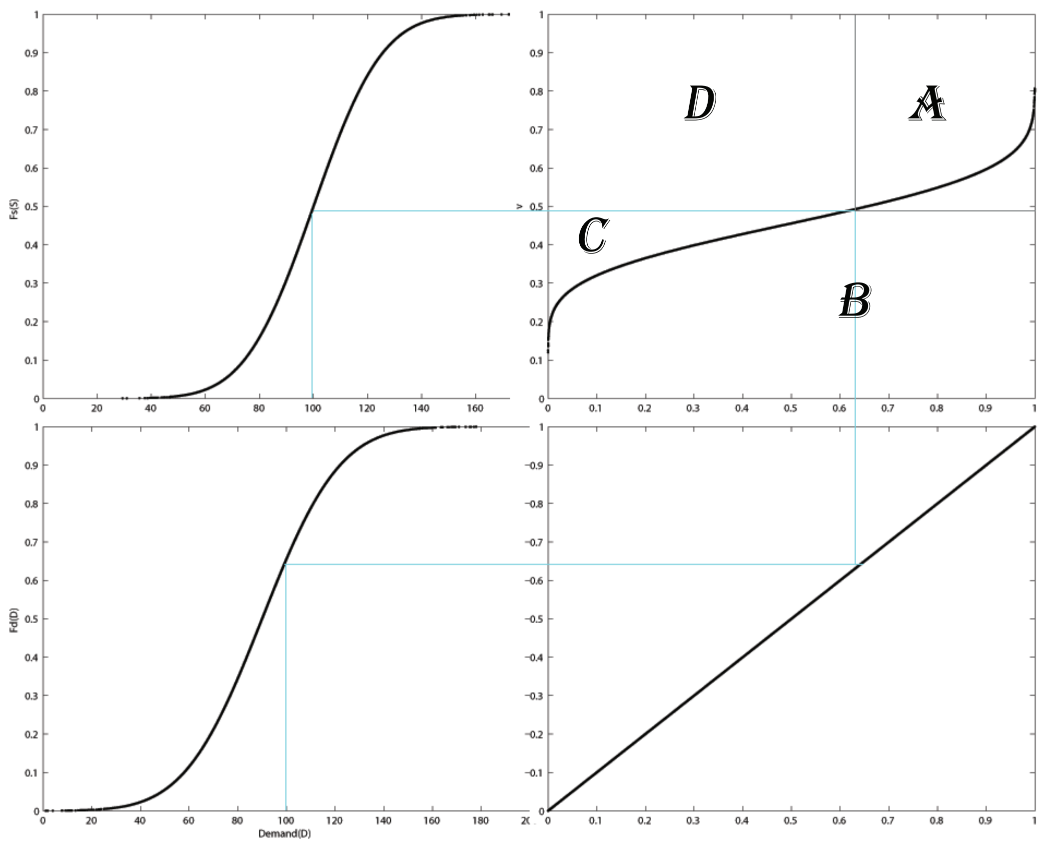

Figure 1: Mapping of Real World to $(0,1)^2$ space

Figure 1 depicts how the supply and demand from real world can be mapped in the $(0,1)^2$ space. The curve that represents the equivalued inverse $CDF's$ with respect to the Demand and Supply distributions is given by:

$$v = F_s({F_d}^{-1}(u))$$

where $F_s(\cdot)$ and $F_d(\cdot)$ represent the Cumulative Distribution functions of the supply and demand distribution respectively, whilst u and v are the parameters representing the values of $F_s(\cdot)$ and $F_d(\cdot)$ in the $(0,1)^2$ space, $u = F_s(S)$ and $v = F_d(D)$. With this understanding, it is easy to see that the expected return $E[R(y)]$, as a function of order size, y can be written as:

$$E[R(y)] = E[R(y)]_A + E[R(y)]_B + E[R(y)]_C + E[R(y)]_D$$

where,

$$E[R(y)]_A = (p-c) * y * (1 - F_s(y)) * (1 - F_d(y)) \tag{1}$$

$$E[R(y)]_B = \int_0^{F_d(y)} \int_0^{F_s({F_d}^{-1}(u))} ((p-c) * {F_s}^{-1}(v))\, dv\, du + \int_0^{F_s(y)} ((p-c) * {F_s}^{-1}(v))\, dv \tag{2}$$

$$E[R(y)]_C = \int_0^{F_d(y)} \int_{F_s({F_d}^{-1}(u))}^{F_s(y)} (p * {F_d}^{-1}(u) - c * {F_s}^{-1}(v))\, dv\, du \tag{3}$$

$$E[R(y)]_D = (1 - F_s(y)) * \int_0^{F_d(y)} (p * {F_d}^{-1}(u) - c * y)\ du \tag{4}$$

Here, p and c are the associated price and the cost respectively.

2 Unimodality of Expected Return and finding the maximum

2.1 Independent Demand and Supply Distributions

Theorem 1. *If Demand is independent of Supply then expected return as a function of the order quantity attains a global maximum at*

$$y* = F_d^{-1}(\frac{p-c}{p})$$

Proof. We can analyze the behavior of the expected return, E[R(y)] by looking at the first and second order derivatives with respect to the order quantity, y.

$$\frac{d}{dy}E[R(y)] = \frac{d}{dy}E[R(y)]_A + \frac{d}{dy}E[R(y)]_B + \frac{d}{dy}E[R(y)]_C + \frac{d}{dy}E[R(y)]_D \tag{5}$$

$$\frac{d^2}{dy^2}E[R(y)] = \frac{d}{dy}\left(\frac{d}{dy}E[R(y)]\right) = \frac{d^2}{dy^2}E[R(y)]_A + \frac{d^2}{dy^2}E[R(y)]_B + \frac{d^2}{dy^2}E[R(y)]_C + \frac{d^2}{dy^2}E[R(y)]_D \tag{6}$$

The first order derivatives $\frac{d}{dy}E[R(y)]_A$, $\frac{d}{dy}E[R(y)]_B$, $\frac{d}{dy}E[R(y)]_C$, $\frac{d}{dy}E[R(y)]_D$ can be evaluated using *Leibniz's Rule*

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\, dx = \frac{db(\alpha)}{d\alpha}\, f(b(\alpha),\alpha) - \frac{da(\alpha)}{d\alpha}\, f(a(\alpha),\alpha) + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}\, f(x,\alpha)\, dx \tag{7}$$

Using (7), with (1), (2), (3) and (4) yields:

$$\frac{d}{dy}E[R(y)_A] = (p-c) * \{(1 - F_s(y)) * (1 - F_d(y)) - y * F_s'(y) * (1 - F_d(y)) - y * (1 - F_s(y)) * F_d'(y)\} \tag{8}$$

$$\begin{aligned}\frac{d}{dy}E[R(y)_B] &= (p-c) * \left\{ F_d'(y) * \left\{ \int_0^{F_s(y)} ({F_s}^{-1}(v))\, dv - \int_0^{F_s(y)} ({F_s}^{-1}(v))\, dv \right\} + y * F_s'(y) * (1 - F_d(y)) \right\} \\ &= (p-c) * y * F_s'(y) * (1 - F_d(y))\end{aligned} \tag{9}$$

$$\frac{d}{dy}E[R(y)_C] = \frac{d}{dy}\left\{\int_0^{F_d(y)} G(y,u)\,du\right\}$$

$$\left\{G(y,u) = \int_{F_s({F_d}^{-1}(u))}^{F_s(y)} (p * {F_d}^{-1}(u) - c * {F_s}^{-1}(v))\,dv\right\}$$

$$= \left\{\int_0^{F_d(y)} \frac{d}{dy}G(y,u)\,du\right\} + G(y, F_d(y)) * F'_d(y) + G(y,0) * 0$$

$$= F'_s(y) * \left\{\int_0^{F_d(y)} (p * {F_d}^{-1}(u) - c * y)\,du\right\} \tag{10}$$

$$\frac{d}{dy}E[R(y)_D] = -F'_s(y) * \left\{\int_0^{F_d(y)} (p * {F_d}^{-1}(u) - c * y)\,du\right\} + (1 - F_s(y)) * \left\{\int_0^{F_d(y)} -c\,du + (p-c) * y * F'_d(y)\right\}$$

$$= -F'_s(y) * \left\{\int_0^{F_d(y)} (p * {F_d}^{-1}(u) - c * y)\,du\right\} + (1 - F_s(y)) * \{(p-c) * y * F'_d(y) - F_d(y) * c\} \tag{11}$$

From (5), (8), (9), (10), and (11), we get:

$$\frac{d}{dy}E[R(y)] = (1 - F_s(y)) * \{(p-c) * (1 - F_d(y)) - c * F_d(y)\} \tag{12}$$

furthermore, from 6

$$\frac{d^2}{dy^2}E[R(y)] = \frac{d}{dy}\left(\frac{d}{dy}E[R(y)]\right) = \quad c * F'_s(y) - p * \{F'_s(y) * (1 - F_d(y)) + F'_d(y) * (1 - F_s(y))\} \tag{13}$$

The Expected Return, $E[R(y)]$, as a function of order size, y attains maximum when,

$$\frac{d}{dy}E[R(y)] = (1 - F_s(y)) * \{(p-c) * (1 - F_d(y)) - c * F_d(y)\} = 0$$

$$F_s(y*) = 1; \qquad F_d(y*) = \frac{p-c}{p}$$

$$y* = F_s^{-1}(1),\ F_d^{-1}(\frac{p-c}{p}) \tag{14}$$

Thus, we have two stationary points for the function, namely $y*_1 = F_s^{-1}(1)$ and $y*_2 = F_d^{-1}(\frac{p-c}{p})$. It can be seen that

$$\left.\frac{d^2}{dy^2}E[R(y)]\right|_{y*_1} = 0 \qquad and \qquad \left.\frac{d^2}{dy^2}E[R(y)]\right|_{y*_2} < 0$$

since, the E[R(y)] has two two stationary points, with the second stationary point lying at the extremum of the supply distribution, we conclude that the Expected return as a function of y has a global maximum as

$$y* = F_d^{-1}(\frac{p-c}{p}),$$

Hence *unimodal*. □

This immediately leads to the following corollary.

Corollary 1. *Supply* $\perp\!\!\!\perp$ *Demand* $\Rightarrow$ *Optimal Order Quantity* $\perp\!\!\!\perp$ *Supply distribution*

2.2 The Correlated Case, Supply and Demand not independent

A simulation scheme to analyze this scenario can be proposed as below.

1. Generate a large sample (≥ 1000 values) of the pair (supply,demand) from the joint distribution, that is generate
$$(s,d) \sim F_{s,d}$$
.

2. Populate the $(0,1)^2$ space using $(u,v) = (F_s(s), F_d(d))$.

3. Calculate the expected return as a function of y, P(y), over the $(0,1)^2$ space using the analysis used in the previous section.

4. Now
$$y^* = \operatorname*{argmax}_y P(y),$$
and maximum expected return $= P(y^*)$

This was implemented in MATLAB for *Normally distributed Demand and Supply* using the following code to analyze the behavior of optimal order quantity and corresponding expected return with the correlation between demand and supply. The unimodal nature of the expected return function was observed as expected.

```
function findOrderSizeNormal(mu_s,sigma_s,mu_d,sigma_d,p,c)
n = 1000;        %number of y's to evaluate = n%
m=1000;          %number of points in the (0,1)*(0,1) lattice = m*m%
rho_l=-1;
rho_u=1;
rho_Δ=0.01;
rho_n=(1+((rho_u-rho_l)/rho_Δ));
sigma_sd=sigma_s*sigma_d;
var_s=(sigma_s)^2;
var_d=(sigma_d)^2;
rho=zeros(11,1);
profit=zeros(m,1);
y=zeros(n+1,1);
avg_profit=zeros(n+1,1);
p_star=zeros(rho_n,1);
y_star=zeros(rho_n,1);
for k=1:rho_n
    rho(k) = rho_l+(k-1)*rho_Δ;
    disp(rho(k));
    rho_sd=rho(k)*sigma_sd;
    mu = [mu_s mu_d];
    SIGMA = [var_s,rho_sd;rho_sd,var_d];
    data = mvnrnd(mu,SIGMA,m);
    supply=data(:,1);
    demand=data(:,2);
    h1=figure('Visible','off');
    plot(supply,demand,'+');
    xlabel('Supply');
    ylabel('Demand');
    title(strcat('Correlation of Demand and Supply: rho = ',num2str(rho(k))));
    v = normcdf(supply,mu_s,sigma_s);
    u = normcdf(demand,mu_d,sigma_d);
    for j=1:(n+1)
        Fsy=((j-1)/n);
        y(j) = norminv(Fsy,mu_s,sigma_s);
        Fdy = normcdf(y(j),mu_d,sigma_d);
        for i=1:m
            if ((v(i) ≥ Fdy) && (u(i) ≥ Fsy))
                profit(i) = (p-c)*y(j);
            elseif (v(i) ≤ min(Fsy,normcdf(demand(i),mu_s,sigma_s)))
                profit(i) = (p-c)*supply(i);
```

```
            elseif ((v(i) > min(Fsy,normcdf(demand(i),mu_s,sigma_s))) && (v(i) ≤ Fsy))
                profit(i) = p*demand(i) - c*supply(i);
            else
                profit(i) = p*demand(i) - c*y(j);
            end
        end
        avg_profit(j) = mean(profit);
    end
    h2=figure('Visible','off');
    plot(y,avg_profit);
    xlabel('Order Quantity');
    ylabel('Expected Return');
    title(strcat('Expected Return vs Order Quantity: rho = ',num2str(rho(k))));
    p_star(k) = max(avg_profit);
    x = find(avg_profit == p_star(k));
    y_star(k) = y(x(1));
end
h3=figure('Visible','off');
plot(rho,y_star);
xlabel('Correlation Coefficient(rho)');
ylabel('Optimal Order Quantity');
title('Optimal Order Quantity vs. Correlation Coefficient(rho)');
exportfig(h3,'Y''_vs_rho.eps','bounds','tight');
h4=figure('Visible','off');
plot(rho,p_star);
xlabel('Correlation Coefficient(rho)');
ylabel('Optimal Expected Return');
title('Optimal Expected Return vs. Correlation Coefficient(rho)');
exportfig(h4,'E[Return'']_vs_rho.eps','bounds','tight');
y_star_non_stochastic = norminv((p-c)/p,mu_d,sigma_d);
save('normal_analysis_data');
end
```

Table 1: Optimal Order Quantity & Expected Return as a function of correlation between Demand & Supply

ρ	y*	p*	ρ	y*	p*	ρ	y*	p*	ρ	y*	p*
-1	74.81832	164.1622	-0.49	76.19764	161.4186	0.02	80.59813	149.3127	0.53	83.23891	145.2491
-0.99	72.81075	164.4309	-0.48	70.48418	158.4297	0.03	78.74961	147.4886	0.54	87.9848	149.6387
-0.98	74.13936	167.2829	-0.47	76.7976	155.5954	0.04	82.44207	147.2212	0.55	91.58671	147.0089
-0.97	76.69906	163.3136	-0.46	70.63232	151.466	0.05	76.29912	150.5431	0.56	89.91256	147.9181
-0.96	72.93652	166.2568	-0.45	72.93652	154.8545	0.06	80.19287	146.5711	0.57	80.67823	142.7827
-0.95	74.59525	171.747	-0.44	70.92387	155.3969	0.07	78.83757	147.5027	0.58	84.21617	142.2736
-0.94	75.88946	166.1103	-0.43	75.88946	160.6708	0.08	84.42069	150.7476	0.59	83.02427	148.1388
-0.93	71.34995	165.5375	-0.42	72.93652	156.7227	0.09	75.46944	151.7949	0.6	87.9848	145.9372
-0.92	75.46944	162.1896	-0.41	75.0383	157.0181	0.1	78.74961	147.5341	0.61	88.87383	143.723
-0.91	71.62693	166.7933	-0.4	72.16512	153.0543	0.11	85.22306	150.4411	0.62	89.62686	146.1592
-0.9	73.66963	163.1612	-0.39	76.39999	157.5691	0.12	76.99301	149.9518	0.63	95.13986	145.0035
-0.89	71.48912	166.5294	-0.38	68.90453	152.7435	0.13	84.14763	148.9286	0.64	82.735	144.3102
-0.88	71.20937	164.4184	-0.37	77.93875	149.7647	0.14	78.66125	147.2668	0.65	86.13013	144.3793
-0.87	74.36897	168.1245	-0.36	79.69556	156.4972	0.15	79.09901	148.2729	0.66	87.31752	144.7255
-0.86	73.06123	160.6478	-0.35	77.0899	160.6418	0.16	84.42069	147.0362	0.67	88.69783	146.7752
-0.85	71.7634	163.597	-0.34	76.99301	153.3618	0.17	75.36273	149.3097	0.68	88.75659	145.8398
-0.84	72.93652	160.5735	-0.33	75.68079	154.8412	0.18	80.91669	148.0447	0.69	92.67021	146.1345
-0.83	75.57546	165.7396	-0.32	73.66963	153.9035	0.19	84.3527	150.7405	0.7	84.28452	146.552
-0.82	74.70718	158.8706	-0.31	75.14717	155.0108	0.2	78.92511	145.2053	0.71	90.36546	145.4135
-0.81	73.5499	165.1162	-0.3	73.78842	151.5795	0.21	76.09554	147.4891	0.72	88.63897	147.432
-0.8	70.18293	161.6938	-0.29	74.70718	150.895	0.22	77.28208	148.9634	0.73	91.31206	143.6572
-0.79	76.59995	158.8049	-0.28	75.68079	154.0951	0.23	76.99301	149.4854	0.74	98.94673	145.3977
-0.78	73.78842	156.9062	-0.27	72.29657	155.9524	0.24	81.61635	142.8044	0.75	89.33903	145.3771
-0.77	70.92387	157.9013	-0.26	73.30755	152.0516	0.25	80.02847	146.9691	0.76	89.39677	144.4248
-0.76	69.71796	163.2052	-0.25	77.56647	152.9216	0.26	78.92511	143.6403	0.77	93.78525	150.7638
-0.75	75.46944	167.1078	-0.24	75.36273	155.7894	0.27	81.92017	152.6357	0.78	90.02626	146.3796
-0.74	70.48418	161.0832	-0.23	77.93875	155.8448	0.28	78.83757	148.9315	0.79	97.33511	146.9265
-0.73	74.25459	162.4248	-0.22	79.01226	153.096	0.29	82.36825	146.1915	0.8	92.67021	145.7263
-0.72	76.50026	157.0241	-0.21	71.62693	149.7098	0.3	82.36825	145.4356	0.81	98.79609	146.9054
-0.71	71.62693	156.5874	-0.2	75.36273	152.0203	0.31	86.57308	147.6347	0.82	96.98062	145.3318
-0.7	76.59995	154.4718	-0.19	73.90629	150.2333	0.32	79.94577	150.0524	0.83	98.4946	148.4211
-0.69	75.68079	156.7196	-0.18	77.37738	154.633	0.33	80.43699	148.9016	0.84	95.50053	145.6629
-0.68	75.0383	160.3604	-0.17	76.39999	152.2864	0.34	78.12205	145.124	0.85	95.91095	148.1925
-0.67	72.68389	158.0603	-0.16	76.7976	154.0966	0.35	87.31752	147.1279	0.86	97.73923	147.2521
-0.66	71.34995	162.0218	-0.15	81.30821	149.3452	0.36	80.43699	147.5015	0.87	95.96213	147.6979
-0.65	71.48912	157.2586	-0.14	77.37738	156.834	0.37	79.86271	145.8598	0.88	105.4302	147.0215
-0.64	71.34995	162.1416	-0.13	80.67823	151.4231	0.38	84.21617	151.3303	0.89	98.94673	148.8636
-0.63	76.29912	161.4095	-0.12	76.09554	152.1254	0.39	80.91669	146.7419	0.9	93.94289	141.4662
-0.62	73.5499	162.3971	-0.11	73.42921	150.4501	0.4	86.57308	146.3735	0.91	113.7426	143.4914
-0.61	73.78842	156.2379	-0.1	78.83757	149.0986	0.41	82.80765	143.7552	0.92	102.5132	146.6199
-0.6	74.25459	159.2195	-0.09	73.30755	150.8238	0.42	79.77931	145.7461	0.93	99.64905	148.1416
-0.59	75.46944	158.5814	-0.08	78.74961	152.162	0.43	82.36825	148.1303	0.94	104.0379	143.3478
-0.58	73.5499	157.854	-0.07	78.39361	152.3818	0.44	78.48325	144.7383	0.95	129.5158	148.5891
-0.57	72.03247	162.8022	-0.06	73.1849	147.9477	0.45	78.12205	142.6125	0.96	161.8046	145.1149
-0.56	73.5499	163.1042	-0.05	80.51772	148.0921	0.46	83.80208	147.0742	0.97	Infinity	147.3778
-0.55	74.25459	159.9453	-0.04	74.70718	150.1961	0.47	82.07053	149.187	0.98	Infinity	149.982
-0.54	74.70718	154.0325	-0.03	79.86271	149.5916	0.48	82.14533	145.7472	0.99	153.0414	151.576
-0.53	77.75357	159.2218	-0.02	80.11084	151.8921	0.49	87.31752	144.5938	1	161.8046	145.5395
-0.52	81.46283	158.1466	-0.01	76.59995	149.8323	0.5	85.87395	145.763			
-0.51	72.16512	158.4184	0	79.35692	150.4904	0.51	84.69088	145.2461			
-0.5	72.81075	154.4587	0.01	77.37738	150.6024	0.52	84.00998	145.9069			

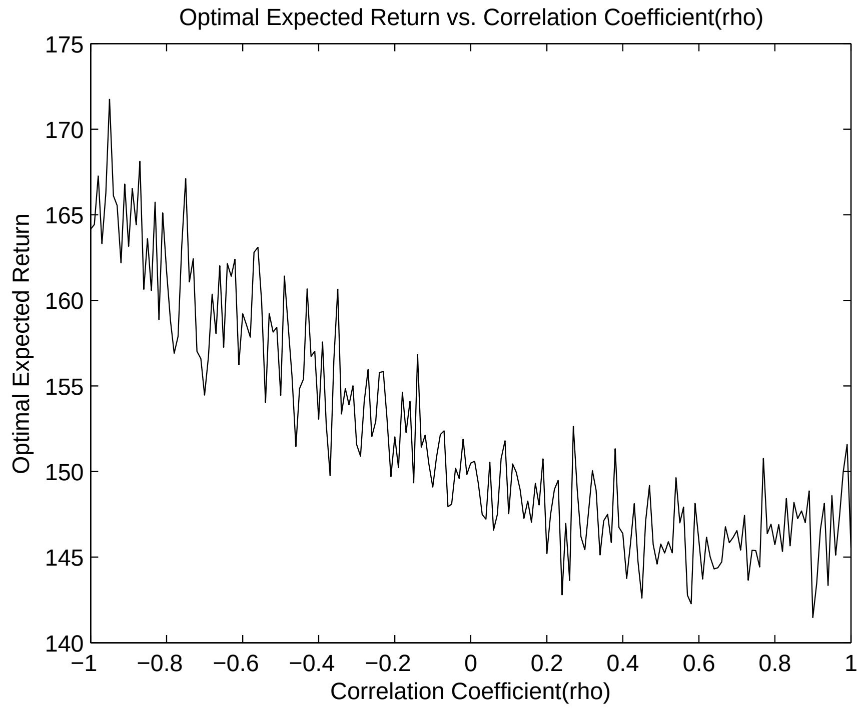

Figure 2: Optimal Expected Return as a function of correlation between Demand and Supply

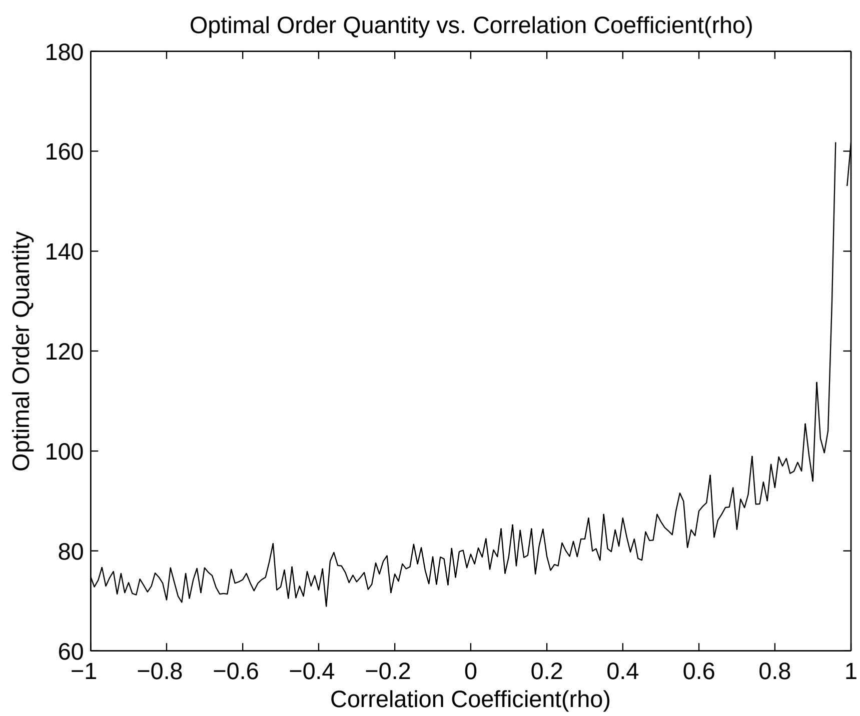

Figure 3: Optimal Order Quantity as a function of correlation between Demand and Supply

3 Improving Run Time of the Algorithm

The *unimodality* of the Expected Return function can be explored to a great extent whilst trying to improve the run time of the algorithm. The scheme described in section 2.2, evaluates the expected return as a function of order sizes that are enumerated with some interval length. This procedure is expensive in terms of time. The fact that the Expected return is unimodal with respect to the order size to come up with much better algorithm. An implementation of the *Improved Fibonacci Algorithm*[1] was used an the following difference in running times were observed:

Table 2: Running Times of Algorithms (seconds) against the population in $(0, 1)^2$ space

m	Enumeration in unit steps	Enumeration one decimal accuracy	Fibonacci search (10)	Fibonacci search (20)	Fibonacci search (30)	Fibonacci search (40)
10	0.048609	0.482151	0.005524	0.010811	0.016163	0.021511
50	0.200945	1.975357	0.021251	0.041721	0.062172	0.082839
100	0.389137	3.9345	0.049413	0.102105	0.156624	0.209019
200	0.762922	7.799991	0.080363	0.158154	0.235712	0.313847
500	1.885759	20.05759	0.170418	0.324809	0.485303	0.632095
1000	3.795459		0.389691	0.765713	1.140973	1.516799

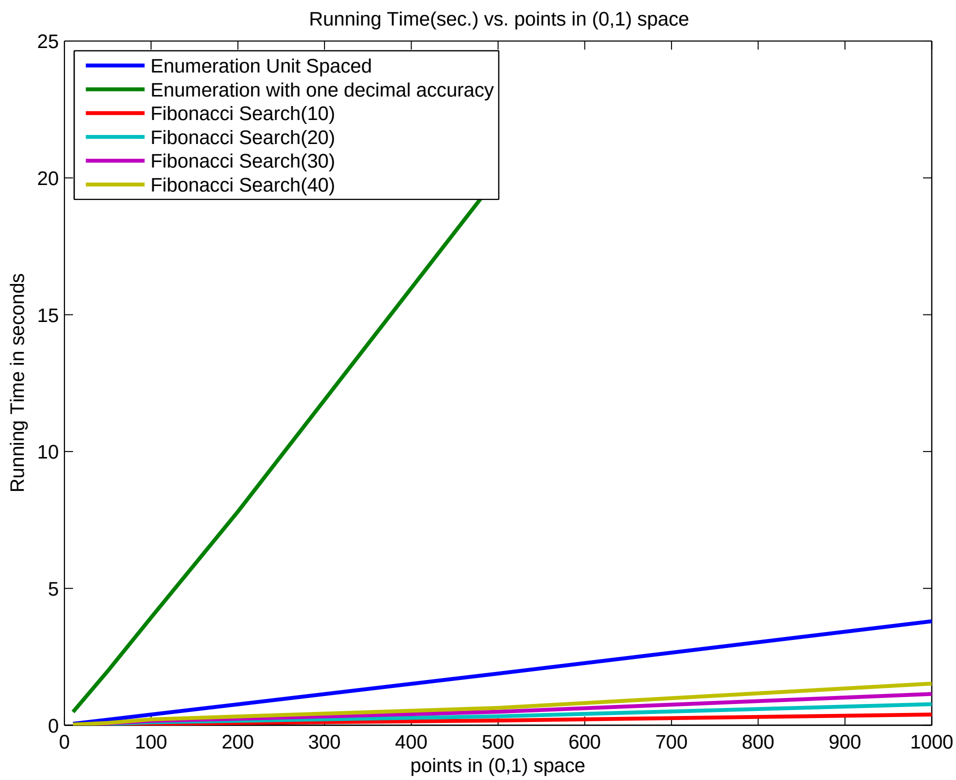

Figure 4: Running Times of Algorithms (seconds) against the population in $(0, 1)^2$ space

[1] Subasi M., Yildirim N., Yildiz B., *An improvement on Fibonacci search method in optimization theory*, Applied Mathematics and Computation, 147 (2004) pp – 893–901

4 Possible further analysis and future work

There are lot of challenges that can provide a great deal of insight with respect to decision making in context of the newsvendor problem. I would like to suggest a few and would like to pursue it further in future,

Price Sensitivity and Decision Making This analysis can be of great importance for decision making. We assume that price and demand are inversely proportional and thus try to estimate if and by how much reducing price can be advantageous to have greater demands and probable quantity discounts on large orders. However the key here is that this comes at cost of higher risk as the demand variance blows up with decrease in price. That is,

$$\begin{aligned} P * D = & \quad P_0 * D_0 \\ \Rightarrow \quad var(D) = & \quad \frac{{P_0}^2}{P^2} * var(D_0) \\ \Rightarrow \quad P < P_0 \Leftrightarrow & \quad var(D) > var(D_0) \end{aligned}$$

Analysis of Different Supply and Demand Distributions The Supply and Demand not necessarily, in particular, are distributed normally. It would be of interest to see if the results and observations here would hold for various other distributions as well. A few interesting cases would involve analyzing mixed distributions and bimodal distributions in which a further analysis of correlated case can yield significant insights.

Improving Efficiency of Simulation Schemes and Algorithms This is again an important issue when it comes to simulating the aforementioned scenarios especially in case of multimodal distributions in which case the Expected return can have multiple local optimum. It would be required to have robust global optimization algorithms for 1-D multimodal optimization[2].

Analysis of Semi-perishable (Shelf life ≈ 2-4 days) inventory models An extension of this analysis would be to the semi-perishables with a slightly longer shelf lives (≈ 2-4 days).

[2] Refer to the works of A. Zilinskas, *Institute of Mathematics and Cybernetics of the Lithuanian SSR Academy of Sciences*, Vilnius, U.S.S.R