My favourite problems

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Problems

1. Consider a simple graph where any pair of vertices is connected by at most one edge and no edge connects a vertex to itself. Each vertex is coloured black or white. *Switching* a vertex means changing its colour as well that of its immediate neighbours (vertices removed by one edge). Assume that all vertices are white to start with. Prove that we can always perform a series of switches to change all of them to black.
   German TST 2004, by Eric Muller

Solutions

1. A super solution by Janos Kramar, posted on Mathlinks by Michael Lipnowski.
   Use induction on the number of vertices. Assume it's true for n-1. Now, there are 2^n possible states of color, and each point can be either switched or not, giving 2^n possible sequences of operations. Therefore, if we cannot get them all black, there must be a set S of k points such that switching every point in S causes no change. Now, each point in S is connected to an even number of points in S, so each point in S has an odd number of neighbours in S. So $|S|$ is even. Now, as each point is connected to an even number of points in S, the number of white vertices in S never changes parity mod 2. Now, fix a vertex V in S. If we can get all points except for V black, then V must also be black due to our parity argument. But this can be done by induction. Hence, result.

   This one is by Peter Scholze, posted by Darij Grinberg. The last paragraph contains the unobvious step.

   Again we go with induction over the number n of the vertices of the graph. For n = 1, there is nothing to prove. Now turn to the step n $\rightarrow$ n+1:

Assume that the theorem holds for an n = k. Consider a graph G with k+1 vertices, all colored white. Choose an arbitrary vertex A of this graph, and let $G' = G\ A$ be the graph consisting of the remaining k vertices. Call O the operation ”switch a vertex and simultaneously switch all of its neighbours”. Now, since the theorem is assumed true for n = k, there exists an algorithm T which consists of multiple applications of the operation O and which, applied to the graph G', colors all of its vertices black. If we apply this algorithm T to the graph G' regarded as a subgraph of G, then we know that all vertices of G' become black, but we don't know what happens to the vertex A.

(In the following, ”applying the algorithm T to a vertex A of G” will mean ”applying the algorithm T to the graph $G' = G\ A$”.)

If after our algorithm T applied to A, the vertex A is colored black, too, then our graph G is completely black, and we are done. If the vertex A remains white after the algorithm, then start again with a white graph G and try applying the algorithm T to another vertex B of G instead of A. If this doesn't work, too, (i. e. if the vertex B remains white) then start again, choose another vertex C and apply T again, etc.. If our algorithm T doesn't work for any of our vertices, then we can draw the following conclusion: For any vertex V of the graph G, there exists an algorithm $T(V)$ which switches all vertices of G apart from V, while V retains its initial color.

Now, we will distinguish between two cases: if k+1 is even, and if k+1 is odd.

If k+1 is even, then start with the white graph G, apply the algorithm $T(V)$ to all vertices V of G one after the other, and finally each vertex gets switched k times, and, since k is odd, all vertices become black. Proof complete.

Now consider the case when k+1 is odd. Then there exists a vertex A of G with even degree (because the sum of the degrees of all vertices is even.) Also consider the vertices $B_1, B_2, \cdots B_{2m}$ connected with A. Since k+1 is odd, we have an even number of vertices which are neither coincident nor connected with A; call these vertices $C_1, C_2, \cdots C_{2m}$. Apply the algorithm T to each of these vertices $C_1, C_2, \cdots C_{2m}$. Then, all the C_i become black (since each of them gets switched an odd number of times). On the other hand, each of the vertices $A, B_1, B_2, \cdots B_{2m}$ was switched 2m times and hence remains white. Now finally apply the operation O to the vertex A (switch A and all neighbours of A), and all of the vertices $A, B_1, B_2, \cdots B_{2m}$ (and only they) become black. Again, we have colored G completely black. Proof complete.