Hence also $\quad \Xi(z) = O(e^{A|z|\log|z|}) \quad (|z| > A),$

and $\Xi(z)$ is of order 1. But $\Xi(z)$ is an even function. Hence $\Xi(\sqrt{z})$ is also an integral function, and is of order $\frac{1}{2}$. It therefore has an infinity of zeros, whose exponent of convergence is $\frac{1}{2}$. Hence $\Xi(z)$ has an infinity of zeros, whose exponent of convergence is 1. The same is therefore true of $\xi(s)$. Let $\rho_1, \rho_2,\ldots$ be the zeros of $\xi(s)$.

We have already seen that $\zeta(s)$ has no zeros for $\sigma > 1$. It then follows from the functional equation (2.1.1) that $\zeta(s)$ has no zeros for $\sigma < 0$ except for simple zeros at $s = -2, -4, -6,\ldots$; for, in (2.1.1), $\zeta(1-s)$ has no zeros for $\sigma < 0$, $\sin\frac{1}{2}s\pi$ has simple zeros at $s = -2, -4,\ldots$ only, and $\Gamma(1-s)$ has no zeros.

The zeros of $\zeta(s)$ at $-2, -4,\ldots$, are known as the 'trivial zeros'. They do not correspond to zeros of $\xi(s)$, since in (2.1.12) they are cancelled by poles of $\Gamma(\frac{1}{2}s)$. It therefore follows from (2.1.12) that $\xi(s)$ has no zeros for $\sigma > 1$ or for $\sigma < 0$. Its zeros $\rho_1, \rho_2,\ldots$ therefore all lie in the strip $0 \leqslant \sigma \leqslant 1$; and they are also zeros of $\zeta(s)$, since $s(s-1)\Gamma(\frac{1}{2}s)$ has no zeros in the strip except that at $s = 1$, which is cancelled by the pole of $\zeta(s)$.

We have thus proved that $\zeta(s)$ has an infinity of zeros $\rho_1, \rho_2,\ldots$ in the strip $0 \leqslant \sigma \leqslant 1$. Since

$$(1-2^{1-s})\zeta(s) = 1-\frac{1}{2^s}+\frac{1}{3^s}-\ldots > 0 \quad (0 < s < 1) \qquad (2.12.4)$$

and $\zeta(0) \neq 0$, $\zeta(s)$ has no zeros on the real axis between 0 and 1. The zeros $\rho_1, \rho_2,\ldots$ are therefore all complex.

The remainder of the theory is largely concerned with questions about the position of these zeros. At this point we shall merely observe that they are in conjugate pairs, since $\zeta(s)$ is real on the real axis; and that, if ρ is a zero, so is $1-\rho$, by the functional equation, and hence so is $1-\bar{\rho}$. If $\rho = \beta+i\gamma$, then $1-\bar{\rho} = 1-\beta+i\gamma$. Hence the zeros either lie on $\sigma = \frac{1}{2}$, or occur in pairs symmetrical about this line.

Since $\xi(s)$ is an integral function of order 1, and $\xi(0) = -\zeta(0) = \frac{1}{2}$, Hadamard's factorization theorem gives, for all values of s,

$$\xi(s) = \tfrac{1}{2}e^{b_0 s}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \qquad (2.12.5)$$

where b_0 is a constant. Hence

$$\zeta(s) = \frac{e^{bs}}{2(s-1)\Gamma(\frac{1}{2}s+1)}\prod\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \qquad (2.12.6)$$

where $b = b_0+\frac{1}{2}\log\pi$. Hence also

$$\frac{\zeta'(s)}{\zeta(s)} = b-\frac{1}{s-1}-\frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)}+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \qquad (2.12.7)$$

Making $s \to 0$, this gives

$$\frac{\zeta'(0)}{\zeta(0)} = b+1-\frac{1}{2}\frac{\Gamma'(1)}{\Gamma(1)}.$$

Since $\zeta'(0)/\zeta(0) = \log 2\pi$ and $\Gamma'(1) = -\gamma$, it follows that

$$b = \log 2\pi-1-\tfrac{1}{2}\gamma. \qquad (2.12.8)$$

2.13. In this section† we shall show that the only function which satisfies the functional equation (2.1.1), and has the same general characteristics as $\zeta(s)$, is $\zeta(s)$ itself.

Let $G(s)$ be an integral function of finite order, $P(s)$ a polynomial, and $f(s) = G(s)/P(s)$, and let

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s} \qquad (2.13.1)$$

be absolutely convergent for $\sigma > 1$. Let

$$f(s)\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s} = g(1-s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\pi^{-\frac{1}{2}(1-s)}, \qquad (2.13.2)$$

where

$$g(1-s) = \sum_{n=1}^{\infty}\frac{b_n}{n^{1-s}},$$

the series being absolutely convergent for $\sigma < -\alpha < 0$. Then $f(s) = C\zeta(s)$, where C is a constant.

We have, for $x > 0$,

$$\phi(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}x^{-\frac{1}{2}s}\,ds$$

$$= \sum_{n=1}^{\infty}\frac{a_n}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(\tfrac{1}{2}s)(\pi n^2 x)^{-\frac{1}{2}s}\,ds$$

$$= 2\sum_{n=1}^{\infty}a_n e^{-\pi n^2 x}.$$

Also, by (2.13.2),

$$\phi(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} g(1-s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\pi^{-\frac{1}{2}(1-s)}x^{-\frac{1}{2}s}\,ds.$$

We move the line of integration from $\sigma = 2$ to $\sigma = -1-\alpha$. We observe

† Hamburger (1)–(4), Siegel (1).