Quiz 8 Solutions

1.

(a) We first use Gram-Schmidt:

$$v_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Normalizing gives

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{3/2}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Thus,

$$\operatorname{span} \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \right\}$$

is our orthonormal basis.

(b) From the above calculation, we have

$$a_1 = v_1 = \sqrt{2} u_1$$

$$a_2 = \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1 + v_2 = \frac{1}{2} \left(\sqrt{2} u_1 \right) + \sqrt{\frac{3}{2}} u_2 = \frac{\sqrt{2}}{2} u_1 + \frac{\sqrt{6}}{2} u_2$$

Thus, from these coefficients (can also compute $Q^T A$ is another way) R becomes the matrix

$$\begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} \end{bmatrix}$$

and we can verify (not needed)

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. We begin by setting up the system $A^T A\hat{x} = A^T b$:

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \end{bmatrix}$$

Thus, the system we are solving is $14\hat{x} = 18$ giving

$$\hat{x} = \frac{9}{7}$$

3. The linear system we are trying to solve is (we can list the two columns of the matrices on the left in either order, same idea)

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Similarly to Problem 2, we compute

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

Thus, our least squares solution is given by the augmented matrix

$$\begin{bmatrix} 5 & 3 & 14 \\ 3 & 3 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{14}{5} \\ 3 & 3 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{14}{5} \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{14}{5} \\ 0 & 1 & \frac{4}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{4}{3} \end{bmatrix}$$

Thus the line of best fit is given by

$$y = 2x + \frac{4}{3}$$