Quiz 8 Solutions

1.

(a) The normal equation is given by $A^{\mathsf{T}}A\mathbf{x} = A^{\mathsf{T}}\mathbf{b}$. Plugging in A and $\mathbf{b}$ gives us

$$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

(b) Solving the equation above gives us $\mathbf{x} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$

2. Since it's easy to see that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, and their third entry is zero, we know that $\text{span}(\mathbf{u}_1, \mathbf{u}_2) = \text{span}(\mathbf{e}_1, \mathbf{e}_1)$. Thus the projection can be obtained by setting the last entry of $\mathbf{y}$ to zero.
If we don't see this trick, we can proceed as follows: We can first use Gram-Schmidt to obtain an orthogonal basis $\mathbf{u}_1, \mathbf{u}_2'$ for the span of $\mathbf{u}_1$ and $\mathbf{u}_2$.

$$\mathbf{u}_2' = \mathbf{u}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{u}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \mathbf{u}_2 - \frac{25}{25}\mathbf{u}_1 = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$$

Then the projection $\hat{\mathbf{y}}$ is given by

$$\hat{\mathbf{y}} = \frac{\mathbf{u}_1 \cdot \mathbf{y}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2' \cdot \mathbf{y}}{\mathbf{u}_2' \cdot \mathbf{u}_2'} = \frac{30}{25}\mathbf{u}_1 + \frac{-15}{25}\mathbf{u}_2' = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$$

3. The important thing to notice here is that computing the product of A^T with some vector boils down to adding all entries in the vector. So for exmample

$$A^T\mathbf{b} = \begin{pmatrix} 1 + 2 + 3 + \cdots + 99 + 100 \\ 1 + 2 + 3 + \cdots + 99 + 100 \end{pmatrix}$$

Instead of computing that sum by hand we can just observe that the average of numbers from 1 to 100 is 50.5 and since there are 100 of those numbers their sum is $50.5 \cdot 100 = 5050$. Since we know know $A^T\mathbf{b}$, getting the normal equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ is easy:

$$\begin{pmatrix} 100 & 100 \\ 100 & 100 \end{pmatrix} \hat{\mathbf{x}} = \begin{pmatrix} 5050 \\ 5050 \end{pmatrix}$$

Writing out the system of equations we just obtain the equation

$$100(\hat{x}_1 + \hat{x}_2) = 5050$$

twice, or equivalently

$$\hat{x}_1 + \hat{x}_2 = 50.5$$

Or, in parametric vector form

$$\hat{\mathbf{x}} = \begin{pmatrix} 50.5 \\ 0 \end{pmatrix} + \hat{x}_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$