Quiz 9 Solutions

1. We have

$$S = A^T A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The characteristic polynomial is $(\lambda - 1)^2 - 1 = \lambda(\lambda - 2) = 0$. So we get eigenvalues of S are $\lambda_1 = 2$ and $\lambda_2 = 0$ with corresponding orthonormal eigenvectors

$$v_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad , \quad v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

So $\sigma_1 = \sqrt{2}$ and

$$u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Choosing

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get

$$U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = A$$

2. The auxiliary equation is $r^2 + 2r + 3 = 0$, so $r = -1 \pm i\sqrt{2}$ and so

$$y(t) = c_1 e^{-t} \cos\left(t\sqrt{2}\right) + c_2 e^{-t} \sin\left(t\sqrt{2}\right)$$

is the general solution.

3. The auxiliary equation is $r^3 - 4r^2 + 5r - 2 = (r-1)^2(r-2) = 0$. Since 1 is a double root, the general solution is

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t}.$$

Differentiating, we get

$$y'(t) = (c_1 + c_2)e^t + c_2 te^t + 2c_3 e^{2t}$$

and

$$y''(t) = (c_1 + 2c_2)e^t + c_2 te^t + 4c_3 e^{2t}$$

So

$$\begin{cases} 1 = y(0) = c_1 + c_3 \\ 2 = y'(0) = c_1 + c_2 + 2c_3 \\ 3 = y''(0) = c_1 + 2c_2 + 4c_3 \end{cases}$$

Solving the system of linear equations, we get $c_1 = 1$, $c_2 = 1$, and $c_3 = 0$. So

$$y(t) = e^t + te^t$$