Worksheet 11 Solutions

1. We find that

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

So therefore

$$A^T A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$$

meaning the determinant of this (characteristic polynomial of $A^T A$) is (expanding in first row)

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix} = (1-\lambda)^3 - (1-\lambda)$$

which factors into

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right]$$

And that becomes 0 at $\lambda = 1, 0, 2$ making those the eigenvalues of $A^T A$. Then we observe that

$$A^T A - 2 \cdot I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

so here again the middle equation forces $x_2 = 0$ with the top and bottom giving $x_1 = x_3$ so a normalized eigenvector for this is $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. After that

$$A^T A - 1 \cdot I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

so here the first and third equations force $x_1 = x_3 = 0$ while x_2 has no restriction making a normalized eigenvector of this to be $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Finally we then have

$$A^T A - 0 \cdot I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

so checking the null space of this, the middle equation forces $x_2 = 0$ and the top and bottom equations are the same, forcing $x_1 + x_3 = 0$. Therefore a normalized eigenvector for this is $v_3 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

So those three eigenvectors collectively will create V, that is

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We then see that

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and then

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

meaning that we have

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $\sum$ is just the "diagonal matrix with the singular values on the diagonal so that our final answer is

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sum = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

2. The characteristic equation of this differential equation is

$$r^4 + 2r^2 + 1 = 0$$

which factors into $(r^2+1)^2 = 0$ making the roots of this to be $r = i, i, -i, -i$. The fact that we have $\pm i$ as roots suggests the solutions $\sin t$ and $\cos t$. But of course since this is fourth order, there ought to be two others. Due to the repeated root we suspect those would be $t\sin t$, $t\cos t$ just as we did with repeated real root. Taking $y = t\sin t$ we note that

$$y' = \sin t + t\cos t$$

$$\begin{aligned} y'' &= 2\cos t - t\sin t \\ y^{(3)} &= -3\sin t - t\cos t \\ y^{(4)} &= -4\cos t + t\sin t \end{aligned}$$

So plugging these in, we see they do satisfy the given differential equation. Therefore our general solution is

$$y(t) = c_1 \sin t + c_2 \cos t + c_3 t\sin t + c_4 t\cos t$$

Now of course the issue of why multiplying by t works is still in play, but as this sort of shows, that's not just isolated to that one scenario, it's clearly a standard thing that is used in many situations, we just need to clarify how that works which is coming up a bit later.

3. Suppose to the contrary we had an n-th order differential equation which did have this as a solution, call it

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

where the a_i's are constants, and we assume $a_n \neq 0$ so that this is actually n-th order. Plugging in $y = \ln t$, we would have

$$a_0 \ln t = -a_n y^{(n)} - a_{n-1} y^{(n-1)} - ... - a_1 y$$

But in this, the right side is a rational function and the left side, if $a_0 \neq 0$, is a logarithm function which definitely cannot be identical functions.

Thus the only way this could possibly work is if $a_0 = 0$. Trying that, then recalling the pattern of the derivatives of $\ln t$ (using Taylor series style ideas) it would become

$$\frac{a_n(-1)^{n-1} n!}{t^n} + \frac{a_{n-1}(-1)^{n-2}(n-1)!}{t^{n-1}} + ... + \frac{a_1}{t} = 0$$

Multiplying by t^n this yields

$$a_1 t^{n-1} - 2a_2 t^{n-2} + ... + a_{n-1}(-1)^{n-2}(n-1)!t + a_n(-1)^{n-1}n! = 0$$

Thus this polynomial must be identically 0. However its constant term is nonzero ($a_n \neq 0$ was stated earlier) so that is impossible. Hence that equation cannot hold, proving that $y = \ln t$ cannot be a solution, as desired.

4. (a) Say our equation is

$$ay'' + by' + cy = 0$$

where a, b, c are constants, $a \neq 0$. Then plugging in each of our prospective solutions into this, we will get

$$ae^t + be^t + ce^t = 0$$

$$4ae^{2t} + 2be^{2t} + ce^{2t} = 0$$
$$9ae^{3t} + 3be^{3t} + ce^{3t} = 0$$

Dividing each equation by its exponential term (which we can do since it certainly is not identically 0, in fact never 0) yields the system

$$a + b + c = 0$$

$$4a + 2b + c = 0$$
$$9a + 3b + c = 0$$

or written with matrices,

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But the matrix on the left here has nonzero determinant since no row is a multiple of another and doing two times the second row minus the first matches the third row in the last two positions but not the first so there is no way to write the third row as a linear combination of the first two. Therefore this homogeneous system will only have the trivial solution, contradicting that $a \neq 0$. So that shows why this is not possible.

(b) In this case we proceed exactly the same way, just the coefficients, a, b, c become functions $a(t), b(t), c(t)$ so our system will become

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a(t) \\ b(t) \\ c(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But the exact same argument still works as this would force $a(t)$ to be identically the function 0 which is not allowed. So even when the coefficients are not constants, we still cannot have all three of those being solutions.

5. (a) Since A is orthogonal, it must be square matrix. So one potential option is to just take $U = A$ with $\sigma = V = I$. We could also do $U = \sum = I$ with $V = A^T$ (since A is orthogonal so too is its transpose). Both of these will create A as the needed product so those are two options.

   (b) Assuming A is $m \times n$ with $m > n$ (if it has orthonormal columns it must have more rows than columns), we have $A^T A = I_n$ so that the eigenvalues of $A^T A$ are all 1 (repeated n times). And then $A^T A - 1 \cdot I_n = 0$ so we can just take the standard basis to be the set of eigenvectors for this eigenspace.

   Doing that we simply find that $V = I_n$. Then using $u_i = \dfrac{1}{\sigma_i} A v_i$, since each $\sigma_i = 1$ and $v_i = e_i$, that means u_i is just the i-th column of A for $1 \leq i \leq n$. After that we just extend the columns of A to becoming an orthonormal basis for $\mathbb{R}^m$.

And $\sum$ by definition must become the matrix whose every diagonal entry is 1 with all other entries are 0.

Hence this will be as such: U has its first m columns as A, its remaining m columns filling out an orthonormal basis for $\mathbb{R}^m$, $\sum$ the $m \times n$ matrix with every diagonal entry 1 and every nondiagonal entry 0, and $V = I_n$. This will work since when we do $U \sum$, all those "new" columns of A will disappear, only the original ones of A will remain due to the nature of $\sum$. That is, $U \sum = A$ and since $V = I_n$, we definitely have $U \sum V^T = A$, as desired.

(c) In this case since $A^T A = 0$, all its eigenvalues are 0 which would suggest we would need $\sum = 0$ matrix. But that would then imply that $A = 0$ is the only possibility.

And indeed that is true! Since the diagonal entries of $A^T A$ are sums of squares of various entries in A, for those diagonal entries all to be 0, it's quite clear that every entry of A is 0 anyway. Using singular value decomposition is just another indirect way of showing that that is the case.

6. (a) Say the equation is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

where the a_i's are constants. Plugging in $y = \sin t$, this becomes

$$\sin t(a_0 - a_2 + a_4 - ...) = 0$$

$$\cos t(a_1 - a_3 + a_5 - ...) = 0$$

which forces

$$a_0 - a_2 + a_4 - ... = 0$$

$$a_1 - a_3 + a_5 - ... = 0$$

because if not both are 0, this can't be identically 0 (via taking values of t where one of them is 0, that becomes apparent). Realizing this, if we plug in $\cos t$ instead the coefficient of $\cos t$ in the overall expression will be

$$a_0 - a_2 + a_4 - ...$$

while the coefficient of $\sin t$ will be

$$-a_1 + a_3 - a_5 + ...$$

both of which are 0 from what we said above. So indeed $\cos t$ is forced to be a solution as well.

(b) This is basically the same idea as if we have the same equation and $y = e^t \sin t$ becomes a solution then we have

$$e^t \sin t(a_0 + a_1 - 2a_3 - 4a_4 + ...) = 0$$

$$e^t \cos t(a_1 + 2a_2 + 2a_3 - ...) = 0$$

Now this seems a bit less clear since it's not as obvious what the coefficients in the higher derivatives are exactly. But that isn't really relevant, what is relevant is that when we plug in $e^t \cos t$, its coefficient of $e^t \cos t$ will be identical to what it was for the coefficient $e^t \sin t$ when we plugged in $y = e^t \sin t$.

And likewise its coefficient for $e^t \sin t$ will be the negative of what the coefficient of $e^t \cos t$ was above. So once the above two are 0 so must be those for when we plug in $e^t \cos t$. Thus it is forced to be a solution here too.

Many might be tempted to try to prove this based on how the solutions work for second order equations and the seeming extensions as both of these seem like they ought to be true based on how those solutions break down when we have nonreal roots for the characteristic equation.

That's a bit of a jump though as it's not really been made totally clear what happens in higher order equations, especially if we say had repeated nonreal roots (did an example of that in Problem 2, but ok that wasn't clear already so have to be careful about making assumptions). But this shows it more concretely.

7. Let us think about how the matrix U is created. The matrix V is said to be fixed (we are trying to create two decompositions with $\sum$ and V the same, only U can change). So then we have for all nonzero singular values,

$$u_i = \frac{1}{\sigma_i} A v_i$$

which means that those particular u_i seem like they might be fairly forced (not totally clear of course, just because that's our default doesn't mean that's only choice). But regardless, what if these u_i don't create a full basis, i.e. we have to extend them, likely with the help of Gram-Schmidt, to an orthonormal basis? Well there might be multiple ways of doing that which gives us a hint of how to prove we can actually have multiple distinct U which do accomplish this.

If we have 0 as an eigenvalue of multiplicity higher than one, then we need to extend the u_i's by at least two vectors that would likely be an easy way as we could even extend it by the same two vectors, just in a different order to create the same end matrix. To that end, take

$$\sum = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies our $A^T A$ had eigenvalues $1, 0, 0$. Realizing this, we would likely have the first u_i as something like $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, but then the other two columns we would have various options of filling in the orthonormal basis. So let us take

$$U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Due to what $\sum$ is, $U_1 \sum = U_2 \sum$ so whatever V might be, we have created two decompositions of the end matrix A where $\sum$ and V are the same, but the U differs. So yes it is possible.