The FRS construction

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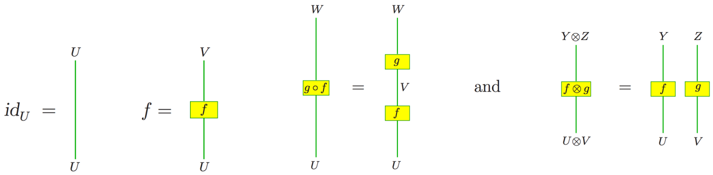
General idea

Main reference: Fuchs, Runkel, Schweigert, *TFT construction of RCFT correlators I: partition functions*, arXiv:hep-th/0204148

- Data of a full CFT = "chiral data" + Frobenius algebra object
- Chiral data is the collection of braiding and fusing matrices, which can be encapsulated in a modular tensor category.
- In the CFT this will be the category of representations of the chiral algebra $\mathfrak{V}$, and the braiding and fusing matrices correspond to tensor product and intertwiners of $\mathfrak{V}$-representations
- Chiral data doesn't determine the CFT; theories with the same chiral data have different partition functions on the torus

A MTC is a category $\mathcal{C}$ with several properties, which we can visualize in terms of ribbon diagrams.

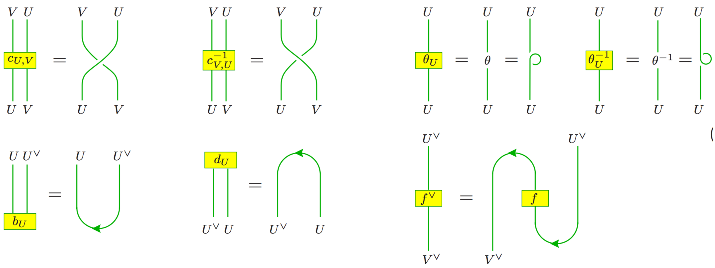
$\mathcal{C}$ is a semisimple abelian tensor category over $\mathbb{C}$. We can visualize the morphisms



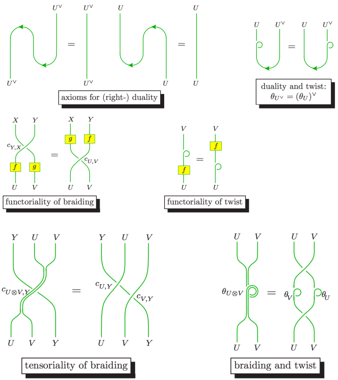
where morphisms are given by the yellow coupons, and the tensor product is strictly associative

$\mathcal{C}$ is also a ribbon category; it comes with the distinguished morphisms of duality, braiding and twist

- Duality: for every object $U \in \mathcal{C}$, there is a dual $U^\vee$ with distinguished morphisms $b_U \in \mathrm{Hom}(1, U \otimes U^\vee)$ and $d_U \in \mathrm{Hom}(U \otimes U^\vee, 1)$
- Braiding: family of isomorphisms $c_{U,V} \in \mathrm{Hom}\, U \otimes V, V \otimes U$
- Twist: for each object U there is isomorphism $\theta_U \in \mathrm{Hom}(U, U)$

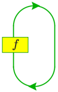


These have to satisfy several relations that can be seen in terms of ribbon diagrams



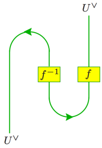
We also define the trace of a morphism; the trace of the identity will be the quantum dimension of the object $\dim(U) = \mathrm{tr}(id_U)$

$$\mathrm{tr_r}(f) =$$



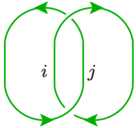
and for any self-dual object U with and isomorphism $f \in \mathrm{Hom}(U, U^\vee)$ we define the Frobenius-Schur indicator by $\mathcal{V}_U = \nu_U id_U$

$$\mathcal{V}_U =$$



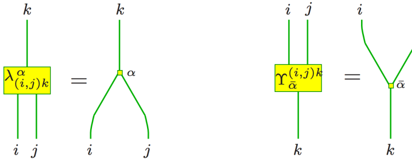
$\mathcal{C}$ is modular, i.e. the s-matrix is nondegenerate. The s-matrix is defined from the braiding: for two simple objects U_i and U_j we define $s_{ij} = \mathrm{tr}(c_{U_i,U_j}, c_{U_j,U_i})$, or pictorially

$$s_{i,j} =$$

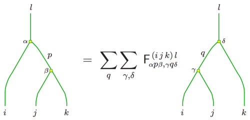


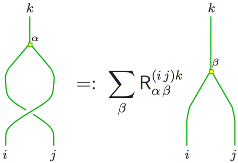
The simplest modular tensor category is the category of vector spaces over $\mathbb{C}$ itself, which has trivial braiding and twist.

These matrices are related to the usual fusing and braiding matrices of the CFT. To see this let's interpret what the MTC concepts mean in terms of the CFT. The collection $\{U_i\}$ of simple objects corresponds to irreducible representations of the chiral algebra $\mathfrak{V}$, and we fix basis elements in the coupling spaces $\lambda^{\alpha}_{i,j,k} \in \mathrm{Hom}(U_i \otimes U_j, U_k)$ and $\Upsilon^{i,j,k}_{\alpha} \in \mathrm{Hom}(U_k, U_i \otimes U_j)$ which we can draw as



The fusing (or 6j-symbol) matrix F and the braiding matrix R are defined by comparing compositions of these basis elements



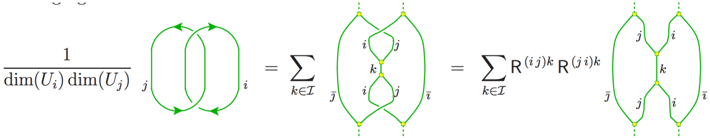


And the S-matrix of the CFT also agrees with the s-matrix of the MTC, up to normalization

$$s_{i,j} = S_{i,j}/S_{0,0}$$

which also lets us compute the quantum dimensionsby $\dim(U_i) = s_{i,0}$ Here 0 labels the vacuum representation of $\mathfrak{V}$.

These matrices are not all independent; for example the S-matrix can be calculated from twists and the F, R matrices



which gives

$$S_{i,j} = S_{0,0}\dim(U_i)\dim(U_j)\sum_{k\in\mathcal{I}}\frac{\theta_k}{\theta_i\,\theta_j}\,\mathsf{G}^{(\bar{\jmath}\,j\,i)\,i}_{0\,k}\,\mathsf{F}^{(\bar{\jmath}\,j\,i)\,i}_{k\,0}$$

Example: free boson compactified on a circle of rational radius squared $R^2 = p/q$. The MTC for this has a collection of $2N$ isomorphism classes of objects $[k]$, where $N = pq, 1 \leq k \leq 2N$. The conformal weights are given by

$$\Delta(k) = \begin{cases} \frac{1}{4N}\, k^2 & \text{for } k \leq N\,, \\ \frac{1}{4N}\,(2N-k)^2 & \text{for } k > N\,. \end{cases}$$

The fusion product is $[i] * [j] = [k]$ so we calculate the fusing and braiding matrices to be

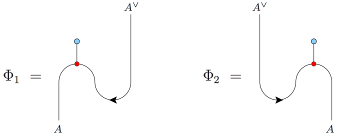
$$\mathsf{F}^{\,(i\,j\,k)\,[i+j+k]}_{\,[j+k]\,[i+j]} = (-1)^{(i+k+1)\{j\sigma(i+j+k)\,+\,(j+k)(\sigma(i+j)+\sigma(j+k))\}}\,,$$

$$\mathsf{R}^{\,(k\,\ell)[k+\ell]} = (-1)^{(k+\ell)\sigma(k+\ell)}\,\mathrm{e}^{-\pi\mathrm{i}k\ell/(2N)}\,.$$

The full CFT should also specify what kind of boundary condition it has, i.e. which fields can live in the boundary. In the MTC language this turns out to be the choice of a *symmetric special Frobenius algebra* object in $\mathcal{C}$. This is an object in $\mathcal{C}$ with

- A product $m \in \text{Hom}(A \otimes A, A)$ and an unit $\eta \in \text{Hom}(1, A)$.
- A co-product $\Delta \in \text{Hom}(A, A \otimes A)$ and a counit $\epsilon \in \text{Hom}(A, 1)$

These are required to follow some relations



As an object of $\mathcal{C}$, A is a sum of simple objects; the objects appearing in A are the primary fields that are allowed to live on a boundary of the worldsheet, and the Frobenius structure has to do with the OPE of boundary fields.

Example: free boson

There is such an algebra A_{2r} for each divisor r of $N = pq$. As an object of $\mathcal{C}$ it is

$$A_{2r} = \bigoplus_{n=0}^{N/r-1} [2nr]$$

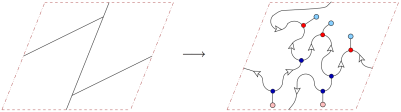
with the simple multiplication rule $m_{[a][b]}^{[a+b]} = 1$ and zero otherwise.

FRS construction of the partition function

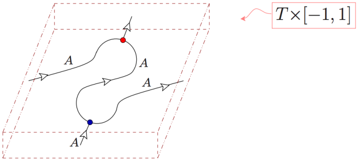
- We start with an orientable worldsheet Σ with boundary $\partial\Sigma$, and the choice of an A-module for each component of $\partial\Sigma$
- We make the complex double $\hat{\Sigma}$ to account for the two chiral halves of the CFT
- We make the connecting manifold M whose boundary is $\hat{\Sigma}$, with a copy of Σ sitting in the middle
- We pick a dual triangulation of the surface, and assign to it a ribbon graph sitting inside of M
- The partition function is calculated in terms of the invariant associated to this ribbon graph by the MTC matrices

The torus partition function

Starting with the following dual triangulation of the torus and applying simplifications we end up with the following diagram



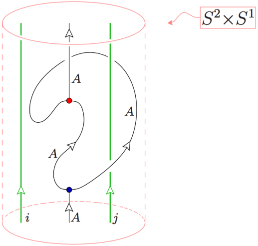
$Z =$



And if we want to compute Z in terms of the standard basis of conformal blocks, we need to add solid toruses with lines to the front and back of this diagram, i.e. if we have

$$Z = \sum_{i,j} Z_{ij}\chi_i(\tau)\chi_j(\tau)^*$$

$Z_{ij} :=$



Torus partition function for the free boson

In the particular case that we have only one-dimensional fusion spaces $N_{ij}^{k} = 0, 1$, we can calculate this element to be:

$$Z_{ij} = \sum_{a,b,c \prec A} {m_{bc}}^{a} \Delta_{a}^{cb} \sum_{k \in \mathcal{I}} \mathsf{G}_{a\,k}^{(c\,b\,j)\,\bar{\imath}} \, \mathsf{R}^{(c\,b)a} \, \frac{\theta_k}{\theta_j} \, \mathsf{F}_{k\,a}^{(c\,b\,j)\,\bar{\imath}}$$

and further specializing to the case of the free boson with algebra object A_{2r} we get

$$Z_{[x][y]}(A_{2r}) = \delta_{[x+y] \prec A} \cdot \frac{r}{N} \sum_{a \prec A} \exp(2\pi \mathrm{i} \, \tfrac{x-y}{2} \, \tfrac{a}{2N}) = \delta_{x+y,0 \bmod 2r} \, \delta_{x-y,0 \bmod 2N/r}$$

which makes it quite evident that the theories with T-dual compactification radii are dual

$$Z_{[x][2N-y]}(A_{2r}) = Z_{[x][y]}(A_{2N/r})$$