Differential graded categories

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1 Motivation

In past talks at this conference we've seen the definition of a triangulated category and some examples of familiar triangulated categories, including the homotopy category $\mathcal{H}(R)$ and the derived category $\mathcal{D}(R)$ of modules over some ring R. These categories are defined by applying certain constructions to the category $\mathcal{C}(R)$ of complexes of R-modules; the derived category is usually defined by some localization construction. Both these categories get their triangulated structures from the abelian structure of $\mathrm{Mod}(R)$ and the shift operation on complexes.

However, there is a number of ways in which the derived category is insufficient or problematic; one could say that in passing to this localization one forgets too much data. For example, the derived category $\mathcal{D}(R)$ is not abelian: it does not have limits or colimits, and the existence of the kernel or cokernel of a morphism is not guaranteed. In fact one can show the existence of the weaker notion of homotopy limits or colimits, but the derived category with only the triangulated structure does not give a prescription for how to construct them.

Example. Here's an example from [To] of how the derived category fails to have kernels. Consider the derived category $\mathcal{D}(\mathbb{Z})$ of $\mathbb{Z}$-modules. For two ordinary $\mathbb{Z}$-modules M, N, seen as objects of the derived category in degree zero, the maps between M and a shift of N in the derived category are calculated by the Ext functors:

$$\mathcal{D}(\mathbb{Z})(M, N[i]) = \mathrm{Ext}^i(M, N)$$

Let's take $M = N = \mathbb{Z}/2$. There's one nontrivial element in $\mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/2)$, represented by the $\mathbb{Z}/4$ extension

$$\mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \to \mathbb{Z}/2$$

which represents the nontrivial morphism $f \in \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2[1])$. Now suppose this map had a kernel K i.e. there was an exact sequence

$$0 \to K \to \mathbb{Z}/2 \to \mathbb{Z}/2[1]$$

in $\mathcal{D}(\mathbb{Z})$. For any i, applying the left exact functor $\mathrm{Hom}^0(\mathbb{Z}[-i], -) = \mathrm{Hom}^i(\mathbb{Z}, -)$ we would get an exact sequence of (ordinary) $\mathbb{Z}$-modules

$$0 \to \mathrm{Hom}^i(\mathbb{Z}, K) \to \mathrm{Hom}^i(\mathbb{Z}, \mathbb{Z}/2) \to \mathrm{Hom}^{i+1}(\mathbb{Z}, \mathbb{Z}/2)$$

Taking $i \neq 0$ we see that $\mathrm{Hom}^i(\mathbb{Z}, K) = 0$ and taking $i = 0$ we get an isomorphism $\mathrm{Hom}^0(\mathbb{Z}, K) \cong \mathrm{Hom}^0(\mathbb{Z}, \mathbb{Z}/2) = \mathbb{Z}/2$. So the morphism $K \to \mathbb{Z}/2$ is an isomorphism in $\mathcal{D}(\mathbb{Z})$ since it induces isomorphisms on all cohomology groups, implying the extension $\mathbb{Z}/2 \to \mathbb{Z}/2[1]$ is trivial, which gives a contradiction.

Another issue is that the triangulated structure of $\mathcal{D}(R)$ alone *almost* determines some of the invariants of R, but not quite. Later in the talk we will define the Hochschild homology and cohomology of a ring, which are almost determined by $\mathcal{D}(R)$, in the precise sense that if there exists a triangulated equivalence $\mathcal{D}(R) \to \mathcal{D}(R')$ coming from a complex of (R, R')-bimodules, that complex of bimodules also induces an isomorphism of the Hochschild complex. However, not every triangulated functor $\mathcal{D}(R) \to \mathcal{D}(R')$ comes from such a complex of bimodules and worse, there are examples where there one can write down an equivalence of the derived categories of modules but the invariants are different.

Differential graded (dg) categories provide enhancements of triangulated categories that allow us to overcome such problems. We'll first define dg categories and describe some constructions that can be performed with them, and then we'll see how to compute some invariants using a dg enhancement of a derived category. Finally we'll present some theorems about such invariants, and how to derive them using properties of the dg enhancement.

This talk is purely expository and does not contain original material; mostly it is based on B. Keller's excellent survey on dg categories [Kel], and whenever possible I have used notation compatible with that source. I also included material and examples from the other sources listed as references as well.

2 Definitions

2.1 dg categories

Let k be a commutative ring. A differential graded (dg) module over k is a $\mathbb{Z}$-graded complex of k-modules $V = \oplus_n V^n$ endowed with a differential $d_V : V^n \to V^{n+1}$. A morphism $f : V \to W$ of dg k-modules is a (degree zero) morphism of the chain complexes, i.e. a family of morphisms $f_n : V^n \to W^n$ intertwining the differentials. The category $\mathcal{C}(k)$ of dg k-modules admits a monoidal structure given by the graded tensor product

$$(V \otimes W)^n = \bigoplus_{i+j=n} V^i \otimes W^j$$

whose differential acts on homogeneous objects by a graded version of the Leibniz rule

$$d_{V\otimes W}(a \otimes b) = d_V(a) \otimes b + (-1)^{\deg a} a \otimes d_W(b)$$

The unit of this monoidal structure is the dg-module given by k in degree zero.

Definition 1. A dg category $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$, i.e. a category where the morphism spaces $\mathcal{A}(X, Y)$ are dg k-modules and the compositions $\mathcal{A}(X, Y) \otimes \mathcal{A}(Y, Z) \to \mathcal{A}(X, Z)$ are morphisms of dg k-modules.

A dg category with only one object is the same as a differential graded algebra, i.e. a k-algebra with a k-linear differential satisfying $d^2 = 0$ and the graded Leibniz rule.

Given any dg category $\mathcal{A}$ we can define the closed category $Z^0(\mathcal{A})$ with the same objects but morphisms spaces given by closed morphisms of degree 0, i.e.

$$Z^0(\mathcal{A})(X, Y) = Z^0(\mathcal{A}(X, Y)) = ker(d^0 : \mathcal{A}(X, Y)^0 \to \mathcal{A}(X, Y)^1)$$

This forms a category since the composition of two closed morphisms is closed by the Leibniz rule. More importantly, we can form the cohomology category $H^0(\mathcal{A})$ with morphism spaces

$$H^0(\mathcal{A})(X,Y) = H^0(\mathcal{A}(X,Y)) = ker(d^0)/im(d^{-1})$$

This also gives a category; one can show that any choices of representatives for two classes in $H^0(\mathcal{A}(X,Y))$ leads to the same class under composition.

Remark. The category $\mathcal{C}(k)$ of dg k-modules is not itself a dg category, as the morphism spaces are just usual k-modules without any extra structure. One can enrich this into a dg category as in the next example.

2.2 The dg category of R-modules

Definition 2. For any k-algebra R, the dg category of right (left) R-modules $\mathcal{C}_{\mathrm{dg}}(R)$ has as objects chain complexes M of right (left) R-modules. The morphisms are first defined as graded k-modules: an element of $\mathcal{C}_{\mathrm{dg}}(R)(M,N)^n$ is a family of morphisms of left (right) R-modules $f^n : M^p \to N^{p+n}$. These graded morphism spaces are then given the structure of dg k-modules by the differential

$$d_f = d_N \circ f - (-1)^n f \circ d_M$$

which endows them with the structure of dg k-modules.

Remark. From now one we will use the right module structure by default, noting explicitly when we want left modules.

It's easy to check from the definitions that $Z^0(\mathcal{C}_{\mathrm{dg}}(R))$ is just the category $\mathcal{C}(R)$ of chain complexes of R-modules, with morphisms given by degree zero maps intertwining the differentials. Taking the zeroth cohomology category $H^0(\mathcal{C}_{\mathrm{dg}}(R))$ one gets the homotopy category $\mathcal{H}(R)$, whose morphism spaces given by degree zero maps modulo maps homotopic to zero. We say that $\mathcal{C}_{\mathrm{dg}}(R)$ is a dg enhancement of $\mathcal{H}(R)$; it is in a similar way that we will construct dg enhancements of derived categories.

3 Triangulated structures and dg categories

The dg category of modules $\mathcal{C}_{\mathrm{dg}}(R)$ is somewhat special in the sense that its zeroth cohomology category $\mathcal{C}(R)$ is triangulated. Note that in general this is not the case; in a dg category the Hom spaces are graded but not the objects, so it's unclear what taking cones and shifts of objects means. We can enforce this condition for a general dg category by looking at representability of the cone and shift functors acting on the Hom spaces.

Definition 3. A dg category $\mathcal{A}$ is (strongly) pretriangulated if for every object X and integer n, the functor $\mathcal{A}^{op} \to \mathcal{C}(k)$ given by

$$Z \mapsto \mathcal{A}(Z,X)[n]$$

is representable, and for every morphism $f : X \to Y$, the functor $\mathcal{A}^{op} \to \mathcal{C}(k)$ given by

$$Z \to Cone(f : \mathcal{A}(Z,X) \to \mathcal{A}(Z,Y))$$

is representable. We will call the objects representing these functors $X[n]$ and Cf, respectively.

It's easy to see that if a dg category $\mathcal{A}$ is pretriangulated, then its zeroth cohomology category $H^0(\mathcal{A})$ is a triangulated (ordinary) category, with shifts and distinguished triangles inherited from the corresponding representing objects in $\mathcal{A}$.

Example. For any k-algebra R, the dg category $\mathcal{C}_{\mathrm{dg}}(R)$ of modules over R is a pretriangulated dg category, with the shift and cone objects naturally just given by the shift and cone of chain complexes. Naturally its zeroth cohomology category is the homotopy category $\mathcal{H}(R)$ with the usual triangulated structure.

In general, every dg category has a *pretriangulated envelope* $\mathrm{pretr}(\mathcal{A})$ with a fully faithful embedding

$$\mathcal{A} \hookrightarrow \mathrm{pretr}(\mathcal{A})$$

satisfying the universal property that any functor $\mathcal{A} \to \mathcal{B}$ to a pretriangulated category $\mathcal{B}$ factors through it. The pretriangulated envelope can be constructed explicitly with the use of twisted complexes [BonKap], which we will not describe in detail here.

4 Functor categories and modules over dg categories

We have seen that given an (ordinary) k-algebra R one can construct two different dg categories from it: a category with only one object and self-homs given by R in degree zero, or the dg category of R-modules $\mathcal{C}_{\mathrm{dg}}(R)$. We would like to do the same and define a dg category of modules over an arbitrary dg k-algebra $\mathcal{A}$.

However, if we try to naively generalize the definition of $\mathcal{C}_{\mathrm{dg}}$ above to a case where $\mathcal{A}$ has nonzero elements in multiple degrees, it would be necessary to keep track of a lot of different degrees by hand, which is very inconvenient. The correct way to do this is to formalize module categories as functor categories, and once we do so it is not any more work to define modules over arbitrary dg categories.

4.1 The category of dg categories

A dg functor F between two dg categories $\mathcal{A}, \mathcal{B}$ is a functor respecting the dg structure of the morphism spaces, i.e. such that $\mathcal{A}(X,Y) \to \mathcal{B}(FX,FY)$ is a morphism in $\mathcal{C}(k)$ for every pair of objects. This allows us to consider the category of dg categories. For set-theoretic reasons it is wise to restrict to (essentially) small categories, i.e. such that the isomorphism classes of objects form a set.

Definition 4. The category dg-Cat_k of small dg categories over k has as objects small dg categories over k and as morphisms dg functors between them.

Theorem 5. dg-Cat_k *is a symmetric monoidal category with a tensor product* $\otimes$ *and an internal Hom functor* $\mathcal{H}om$*, with an internal adjunctions*

$$\mathcal{H}om(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathcal{H}om(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C}))$$

The monoidal structure is given by the following tensor product of categories: $\mathcal{A} \otimes \mathcal{B}$ has objects given by pairs of objects (X_A, X_B) in $\mathcal{A}, \mathcal{B}$ and morphism spaces given by tensor of morphism spaces in $\mathcal{C}(k)$:

$$\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}((X_A,X_B),(Y_A,Y_B)) = \mathrm{Hom}_{\mathcal{A}}(X_A,Y_A) \otimes \mathrm{Hom}_{\mathcal{B}}(X_B,Y_B)$$

The internal hom category $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ has as objects dg functors $\mathcal{A} \to \mathcal{B}$, with the degree n piece $\mathcal{H}om(\mathcal{A}, \mathcal{B})(F,G)^n$ of a morphism space given by a family of degree n morphisms

$$\phi_X \in (\mathcal{B}(FX,GX))^n, (Gf)(\phi_X) = (\phi_Y)(Ff)$$

for all $f \in \mathcal{A}(X, Y)$. This graded k-module inherits a differential induced from the differential in $\mathcal{B}(FX, GX)$, giving $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ the structure of a dg category.

4.2 Modules over dg categories

The internal hom in dg-Cat_k lets us construct new dg categories as categories of functors; consider an arbitrary small dg category $\mathcal{A}$ over k, possibly with multiple objects and hom spaces in many degrees. We can define categories of modules over it as functor categories using $\mathcal{H}om$:

Definition 6. The dg category of right modules over $\mathcal{A}$ is defined by the internal Hom from the opposite category:

$$\mathcal{C}_{\mathrm{dg}}(\mathcal{A}) = \mathcal{H}om(\mathcal{A}^{op}, \mathcal{C}_{\mathrm{dg}}(k))$$

and the category of left modules over $\mathcal{A}$ is analogously defined as $\mathcal{H}om(\mathcal{A}, \mathcal{C}_{\mathrm{dg}}(k))$.

We can get ordinary categories from these by taking the closed and cohomology categories: we define the category of $\mathcal{A}$-modules $\mathcal{C}(\mathcal{A}) = Z^0(\mathcal{C}_{\mathrm{dg}}(\mathcal{A}))$ and the homotopy category of $\mathcal{A}$-modules $\mathcal{H}(\mathcal{A}) = H^0(\mathcal{C}_{\mathrm{dg}}(\mathcal{A}))$. It's easy to check that when $\mathcal{A}$ is the dg category with one object and self-homs given by an ordinary k-algebra R, these notions agree with our previous definitions of $\mathcal{C}_{\mathrm{dg}}(R)$, $\mathcal{C}(R)$ and $\mathcal{H}(R)$.

Fact. For any dg category $\mathcal{A}$ the dg category of $\mathcal{A}$-modules $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ is pretriangulated, with shifts and cones inherited from the target category $\mathcal{C}_{\mathrm{dg}}(k)$.

4.3 The Yoneda embedding

For any ring R, there is a distinguished object in $\mathcal{C}_{\mathrm{dg}}(R)$, the unit of the monoidal structure, given by R placed in degree zero, with self-homs given by R itself. Looking at R as a dg category concentrated in degree zero, we see this is just the image of the obvious embedding of dg categories $R \to \mathcal{C}_{\mathrm{dg}}(R)$.

This can be generalized to an arbitrary dg category $\mathcal{A}$ over k in the setting described above: for any object X of $\mathcal{A}$ we define the object $\hat{X}$ in $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ given by the functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$. This is the *Yoneda embedding*

$$\mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$$

which one can easily check is a fully faithful dg functor.

As we remark above, the dg category $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ is automatically triangulated, even if $\mathcal{A}$ itself isn't: for any functor $M : \mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(k)$ one can compose it with the shift $[n]$ in $\mathcal{C}_{\mathrm{dg}}(k)$ to get $M[n]$.

Let's take now the triangulated hull of the collection $\{\hat{X}[n]\}$ of all the shifts of the images of all the objects X under the Yoneda embedding. Remember that the triangulated hull of a collection of objects is the smallest triangulated subcategory containing those objects. In our case we will denote this triangulated hull by $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$, the dg category of *perfect complexes* over $\mathcal{A}$. From this we can also get the ordinary category of perfect complexes by $\mathrm{per}(\mathcal{A}) = Z^0(\mathrm{per}_{\mathrm{dg}}(\mathcal{A}))$. Note that definition, the Yoneda embedding factors through $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$, and we will also call this map the Yoneda embedding.

4.4 The dg derived category

As we stated in the beginning of the talk, one main objective of defining dg categories is to come up with enhancements of triangulated categories that contain more structure, that is, to find pretriangulated dg categories whose H^0 category recovers some triangulated category we want to study.

It's not clear that we should be able to find a meaningful dg enhancement of an arbitrary triangulated category, but in specific cases, when the triangulated category is given in some algebraic or geometric context, we can often find natural dg enhancements. We have seen an example of this already: for any ring R we defined the dg category $\mathcal{C}_{\mathrm{dg}}(R)$ so that it gives an enhancement of the homotopy category $H^0(\mathcal{C}_{\mathrm{dg}}(R)) = \mathcal{H}(R)$. It's then a natural question to ask whether the derived category $\mathcal{D}(R)$ (and also $\mathcal{D}(\mathcal{A})$ for some general dg category $\mathcal{A}$) also has a similar dg enhancement. More generally, we can ask whether other derived categories of interest, such as derived categories of quasicoherent or coherent sheaves on a scheme X, possess similar dg enhancements

The answer turns out to be that all these examples do have dg enhancements, and some even have several different dg enhancements: for example the derived category of quasicoherent sheaves $\mathcal{D}(\mathrm{qcoh}X)$ on a separated noetherian scheme X has at least three different constructions of a dg enhancement [Or]. One of these constructions involves a familiar construction of quotients of categories.

Proposition 7. *[Or, Dr] For any dg category $\mathcal{A}$ with a full dg subcategory $\mathcal{B}$ there is a dg category denoted $\mathcal{A}/\mathcal{B}$ with an universal morphism (up to quasiequivalence) in* dg-Cat$_k$

$$\mathcal{A} \to \mathcal{A}/\mathcal{B}$$

such that any dg functor $\mathcal{A} \to \mathcal{C}$ with the property that the corresponding map on homotopy categories $H^0(\mathcal{A}) \to H^0(\mathcal{C})$ sends all elements of $\mathcal{B}$ to zero factors through $\mathcal{A} \to \mathcal{A}/\mathcal{B}$.

This quotient dg category can be constructed easily when e.g. the ground ring k is a field; in general the construction is more involved. We can apply this to construct dg enhancements of our familiar derived categories.

Example. Consider the dg category $\mathcal{C}_{\mathrm{dg}}(\mathrm{qcoh}X)$ of unbounded complexes of quasicoherent sheaves on a separated noetherian scheme X, and the full dg subcategory $\mathcal{A}c_{\mathrm{dg}}(\mathrm{qcoh}X)$ spanned by all the acyclic complexes. The quotient

$$\mathcal{D}_{\mathrm{dg}}(\mathrm{qcoh}X) = \mathcal{C}_{\mathrm{dg}}(\mathrm{qcoh}X)/\mathcal{A}c_{\mathrm{dg}}(\mathrm{qcoh}X)$$

is an enhancement of the category $\mathcal{D}(\mathrm{qcoh}X)$.

We can apply this same construction to any abelian category C in place of $\mathrm{qcoh}(X)$: applied to $C = \mathrm{Mod}(R)$ we get a dg category $\mathcal{D}_{\mathrm{dg}}(R)$ which is an enhancement of the derived category $\mathcal{D}(R)$. These dg enhancements are referred to in the literature as the *dg derived category* of an abelian category.

When actually computing morphisms in the derived category, it is often more useful to use the formalism of fibrant and cofibrant replacements, which are generalizations of projective and injective resolutions. A more rigorous and thorough treatment of these techniques goes through the discussion of Quillen model structures but we will avoid that and refer to more competent sources [DwSp]. In our specific case we can define the cofibrant and fibrant objects of the category of dg modules over some arbitrary dg category $\mathcal{A}$ as follows:

Definition 8. An object P of $\mathcal{C}(\mathcal{A})$ is cofibrant if for every surjective quasi-equivalence $L \to M$, every $P \to M$ factors through L. An object I of $\mathcal{C}(\mathcal{A})$ if for every injective quasi-equivalence $L \to M$, every $L \to I$ extends to M.

Fact. The category $\mathcal{C}(\mathcal{A})$ admits cofibrant and fibrant replacements; i.e. for any object M there are quasi-isomorphisms $P \to M$ and $M \to I$ where P is cofibrant and I is fibrant. Moreover all objects $\hat{M}$ in the image of the Yoneda embedding $\mathcal{A} \to \mathcal{C}(\mathcal{A})$ are cofibrant.

So we can also define the derived category and compute its morphisms by using e.g. the fibrant replacement and computing in the homotopy category:

$$\mathcal{D}(\mathcal{A})(X,Y) = \mathcal{H}(\mathcal{A})(P,Y) = H^0(\mathcal{C}_{\mathrm{dg}}(P,Y))$$

5 Additive invariants

5.1 Hochschild homology of associative algebras

Hochschild homology and cohomology were initially defined as invariants of associative algebras, but the definition can be extended to dg algebras and dg categories, and we can use the dg enhancements we constructed above to define invariants of e.g. derived categories of coherent sheaves.

Definition 9. Given an associative k-algebra A and an A-bimodule M, the *Hochschild chain complex* of A with coefficients in the bimodule M is concentrated in non-positive degrees and is defined by [Ar, Hap]

$$C^{-n}(A,M) = M \otimes A^{\otimes n}$$

for $n \geq 0$ with a differential $d : C^{-n}(A,M) \to C^{-n+1}(A,M)$ given by

$$d(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

The Hochschild homology of A with coefficients in M is defined as the cohomology of this complex: $HH_n(A,M) = H^{-n}(C^*(A,M))$. Hochschild cohomology is defined using a dual complex concentrated in non-negative degrees

$$C^n(A,M) = \mathrm{Hom}(A^{\otimes n}, M)$$

with a differential $d : C^n(A,M) \to C^{n+1}(A,M)$ given by

$$df(a_1,\ldots,a_{n+1}) = a_1 f(a_2,\ldots,a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1,\ldots,a_i a_{i+1},\ldots,a_{n+1}) + (-1)^{n+1} f(a_1,\ldots,a_n)a_{n+1}$$

Remark. Here we reverse the degrees and direction of the differential in the Hochschild homology complex from the usual conventions just so that it is also cohomologically graded, since it will simplify our notation in the future.

To get an invariant of associative algebras, we can take M to be the diagonal bimodule A_Δ, i.e. just A as a bimodule over itself with the left and right algebra actions, and get the Hochschild (co)homology of A: $HH_n(A) = HH_n(A, A_\Delta)$ and $HH^n(A) = HH^n(A, A_\Delta)$. We can also stop before taking the (co)homology of the complex and define the Hochschild complex as an object of $\mathcal{C}(k)$.

Besides the dg structure Hochschild homology and cohomology of A carry extra structures; for instance $HH_*(A)$ automatically carries an S^1 action which allows us to also define further invariants such as cyclic homology and negative cyclic homology, which we refer to other sources [Lod]

Example. Let A be an associative algebra over k. Then its first two Hochschild homologies are

$$HH_0(A) = A/[A,A], \qquad HH_1(A) = \Omega^1(A)$$

where $\Omega^1(A)$ is the vector space of Kähler differentials on A, i.e. spanned over A by symbols da for $a \in A$, modulo the following relations

$$dx = 0, \quad d(a+b) = da + db, \quad d(ab) = da\ b + adb$$

for every $x \in k,\ a, b \in A$. Note that if A is the algebra of functions on some manifold then the Kähler differentials is an algebraic versio of the space of one-forms. The fact that the first Hochschild homology captures the space of one-forms is our first example of a more general fact we'll get to later, the Hochschild-Kostant-Rosenberg theorem.

The first two Hochschild cohomologies are

$$HH^0(A) = Z(A), \quad HH^1(A) = Der(A)/Inn(A)$$

where $Der(A)$ is the space of derivations of A and $Inn(A) \subseteq Der(A)$ are the derivations given by commutators with some element in A. More generally for some A-bimodule M

$$HH_0(A, M) = M/[M, A], \quad HH^0(A, M) = Z_A(M)$$

i.e. respectively the coinvariants and the invariants under the adjoint A action.

Hochschild homology can be given an interpretation in terms of derived functors in the category of (A, A)-bimodules, i.e. $A \otimes A^{op}$-modules. Given any bimodule N in $\mathrm{Mod}(A \otimes A^{op})$, there are two functors $\mathrm{Mod}(A \otimes A^{op}) \to \mathrm{Mod}(k)$ given by $N \otimes_{A \otimes A^{op}} -$ and $\mathrm{Hom}_{A \otimes A^{op}}(N, -)$. It's easy to check from the definitions that by taking $N = A_\Delta$, this calculates the zeroth degree Hochschild homology and cohomology

$$HH_0(A, M) = A_\Delta \otimes_{A \otimes A^{op}} M, \qquad HH^0(A, M) = \mathrm{Hom}_{A \otimes A^{op}}(A_\Delta, M)$$

As you might expect the higher Hochschild homologies and cohomologies are the derived functors of the tensor and hom of bimodules: there is a left derived tensor $A_\Delta \otimes^L_{A \otimes A^{op}} -$ and a right derived hom $R\mathrm{Hom}_{A \otimes A^{op}}(A_\Delta, -)$, both functors from the derived category $\mathcal{D}(A \otimes A^{op}) \to \mathcal{D}(k)$, which calculate the Hochschild homology and cohomology

$$HH_\bullet(A, M) = A \otimes^L_{A \otimes A^{op}} M, \qquad HH^\bullet(A, M) = R\mathrm{Hom}_{A \otimes A^{op}}(A, M)$$

The connection between this more abstract definition and the explicit definition above is due to the fact that there is a standard free resolution of the diagonal bimodule $\mathcal{A}_\Delta$ given by the *bar complex* $\bar{C}_n(A)$ given in non-positive degrees by

$$\bar{C}^{-n}(A) = A \otimes A^{\otimes n} \otimes A$$

with the differential $d : \bar{C}^{-n}(A) \to \bar{C}^{-n+1}(A)$ given by

$$d(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = \sum_{i=0}^{i=n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

and one can check that for any (A, A)-bimodule M there is a quasi-isomorphism of complexes

$$\bar{C}^*(A) \otimes_{A \otimes A^{op}} M \cong C^*(A, M)$$

with the complex we initially used to define Hochschild homology.

Example. Consider a finite quiver Q with n vertices and no oriented cycles, and take $A = kQ$ to be its path algebra. Remember that A as a vector space over k is spanned by all the paths in Q, so as an algebra it is generated by the idempotents e_i (one for each vertex) and the elements f_{ij} (one for each edge $i \to j$), subject to the composition rules given by concatenation of paths. Note that since Q is finite, there is an identity element $\mathbf{1} = \sum_i e_i$, and up to scaling it is the only central element, so $Z(A) = k\mathbf{1}$. Note also that every path x with length ≥ 1 is in the commutator ideal; i.e. if x starts at the vertex i, then $x = [e_i, x]$. Thus we have that

$$HH^0(A) \simeq k \qquad HH_0(A) \simeq kQ/\langle f_{ij} \rangle \simeq k^{\otimes n}$$

where $k^{\otimes n}$ is spanned by the basis $\{e_i\}$.

It can also be shown [Cib] that in this case where Q has no oriented cycles, there's no higher Hochschild homology: $HH_i(kQ) = 0$ for $i \geq 1$. The Hochschild cohomology groups are more complicated [Hap]: $HH^i(kQ) = 0$ for $i \geq 2$ but

$$\dim HH^1(kQ) = 1 - n + \sum_{\text{edge } ij} (\text{number of paths } i \to j)$$

we can check that this is zero if and only if the underlying graph is a tree (even if the quiver has no oriented cycles the underlying graph might have cycles). This examples shows how Hochschild homology and cohomology can have quite different behaviors.

5.2 Hochschild homology of dg algebras and dg categories

The definition above can be easily extended to dg algebras and dg categories. For a dg category $\mathcal{A}$ over k, remember that we defined the dg category $\mathcal{C}_{\text{dg}}(\mathcal{A})$ of modules over $\mathcal{A}$. We can also define the dg category $\mathcal{C}_{\text{dg}}(\mathcal{A} \otimes \mathcal{A}^{op})$ of $(\mathcal{A}, \mathcal{A})$-bimodules using the tensor and internal hom structure of dg-Cat$_k$.

There is also a diagonal bimodule $\mathcal{A}_\Delta$ defined by the Hom functor $\mathcal{A} \otimes \mathcal{A}^{op} \to \mathcal{C}_{\text{dg}}(k)$ of $\mathcal{A}$:

$$\mathcal{A}_\Delta : (Y, X) \mapsto \mathcal{A}(X, Y)$$

Similarly to what we saw above, for any bimodule $\mathcal{M}$ there is a bimodule tensor functor $\mathcal{M} \otimes_{\mathcal{A} \otimes \mathcal{A}^{op}} -$ and a bimodule Hom functor $\text{Hom}_{\mathcal{A} \otimes \mathcal{A}^{op}}(\mathcal{M}, -)$ both mapping $\mathcal{C}_{\text{dg}}(\mathcal{A} \otimes \mathcal{A}^{op}) \to \mathcal{C}_{\text{dg}}(k)$. They give rise to derived functors $\mathcal{D}_{\text{dg}}(\mathcal{A} \otimes \mathcal{A}^{op}) \to \mathcal{D}_{\text{dg}}(k)$ so just as above we can define the Hochschild homology and cohomology of a dg category by

$$HH_*(\mathcal{A}) = \mathcal{A}_\Delta \otimes^L_{\mathcal{A} \otimes \mathcal{A}^{op}} \mathcal{A}_\Delta, \qquad HH^*(\mathcal{A}) = R\text{Hom}_{\mathcal{A} \otimes \mathcal{A}^{op}}(\mathcal{A}_\Delta, \mathcal{A}_\Delta)$$

This definition agrees with the earlier definitions when $\mathcal{A}$ is the dg category with one object and with self-homs given by some (dg) algebra.

5.3 Hochschild homology of dg enhancements of triangulated categories

Suppose now that we have a triangulated category $\mathcal{T}$ with a dg enhancement $\mathcal{D}(\mathcal{C})$, i.e. we have some dg algebra $\mathcal{C}$ and a triangulated equivalence $\mathcal{T} \cong \mathcal{D}(\mathcal{C})$. This allows us to define additive invariants for the triangulated category $\mathcal{T}$: in particular we can define the Hochschild (co)homology of $\mathcal{T}$ as the Hochschild (co)homology of the dg algebra $\mathcal{C}$. In general these invariants will depend on the particular dg enhancement we pick, but in some useful contexts the choice of dg enhancement doesn't matter for any of the additive invariants: e.g. when $\mathcal{T} = \mathcal{D}^b(\text{Coh}X)$ for a smooth proper scheme X [Yus].

Let X be a smooth projective variety, and consider the derived category $\mathcal{D}^b(X) = \mathcal{D}^b(\text{Coh } X)$. Here we follow [Kuz]. It's known that $\mathcal{D}^b(X)$ has a strong generator E, i.e. $\mathcal{D}^b(X)$ can be generated from E by taking sequences of cones and finite shifts. Consider the dg algebra of endomorphisms $\mathcal{A} = R\text{Hom}(E, E)$. Then we have a triangulated equivalence $\mathcal{D}^b(X) \cong \mathcal{D}^b(\mathcal{A})$, and we can define the Hochschild (co)homology of the derived category of coherent sheaves on X as the Hochschild (co)homology of the dg algebra $\mathcal{A}$:

$$HH^*(X) = HH^*(\mathcal{A}) \qquad HH_*(X) = HH_*(\mathcal{A})$$

Another way of defining Hochschild (co)homology for a scheme X is to just adapt to a geometric setting the notion of Hochschild homology as the self-tensor of the diagonal and Hochschild cohomology as the self-homs of the diagonal, by defining

$$HH_*(X) = \mathbb{H}^*(X \times X, \Delta_*\mathcal{O}_X \otimes^L \Delta_*\mathcal{O}_X), \qquad HH^*(X) = \text{Hom}^*_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$$

Here $\Delta : X \to X \times X$ is the diagonal embedding, $\mathbb{H}$ calculates the hypercohomology of a complex of sheaves and all the functors are implicitly understood as the corresponding derived functors on the categories of coherent sheaves.

One can prove that these two definitions of Hochschild (co)homology agree regardless of the particular choice of strongly generating object E. Besides being a calculational tool, the definition using the dg enhancement also shows that Hochschild homology satisfies a very nice property under semiorthogonal decomposition. Suppose we have a triangulated category $\mathcal{T}$ with a strong generator E, and a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{T}_1, \dots \mathcal{T}_n \rangle$$

into pieces $\mathcal{T}_i$. Then we can look at the projection E_i of E onto each piece $\mathcal{T}_i$. One can show that this gives strongly generating objects, so using the dg algebras $\mathcal{A}_i = R\text{Hom}(E_i, E_i)$, we get a direct sum decomposition of Hochschild *homology*

$$HH_*(\mathcal{T}) = \bigoplus HH_*(\mathcal{T}_i)$$

Note that we do *not* get a similar decomposition for Hochschild cohomology.

Example. Consider the derived category $\mathcal{D}(kQ)$ of representations of an acyclic quiver Q with n vertices. Here we already have a dg enhancement given by $\mathcal{D}_{\text{dg}}(kQ)$. This corresponds as picking as generating object the algebra kQ itself. Remember that we have a decomposition of (right) kQ-modules

$$kQ = \bigoplus_{\alpha \in Q} P_\alpha$$

into the projective modules P_α given by all the paths starting at the vertex α.

Let's choose a total ordering of the vertices of Q compatible with the partial ordering given by the quiver structure, i.e. we require $\alpha < \beta$ if there is a non-zero path going from α to β. Then we see that we have a semiorthogonal orthogonal decomposition of the category $\mathcal{D}(kQ)$ with n pieces $\mathcal{T}_\alpha$, each one with a single object given by the projective P_α. Each piece is equivalent to the category $\mathcal{D}(k)$ and has Hochschild homology given by $HH_0(\mathcal{T}_i) = \text{Hom}(P_\alpha, P_\alpha) = k$ and zero in higher degrees. So from the direct sum decompositionwe recover the result

$$HH_0(\mathcal{D}(kQ)) \simeq k^n, \qquad HH_i(\mathcal{D}(kQ)) = 0, i \geq 1$$

5.4 The Hochschild-Kostant-Rosenberg theorem

The classical statement of the Hochschild-Kostant-Rosenberg theorem [HKR] is a generalization of the fact that for a commutative k-algebra R, the first Hochschild homology gives the space of Kähler differentials.

Theorem 10. *Let R be a finitely presented k-algebra, where k has characteristic zero. Suppose also that R is smooth i.e. the space of Kähler differentials Ω^1_R is a projective R-module. Then we have an isomorphism*

$$HH_n(R) \cong \Omega^n_R = \wedge^n \Omega^1_R$$

There's also a version of the HKR theorem for the category of coherent sheaves on a smooth projective variety X. Let's again denote the diagonal inclusion by $\Delta : X \to X \times X$ and define two complexes of sheaves in $\mathcal{D}^b(\mathrm{Coh}\ X)$:

$$\mathcal{HH}_\bullet = \Delta^* \Delta_* \mathcal{O}_X, \qquad \mathcal{HH}^\bullet = \Delta^! \Delta_* \mathcal{O}_X$$

These are sheafy versions of the Hochschild homology and cohomology: taking global sections one can show that

$$HH_*(X) = \mathbb{H}^*(X, \mathcal{HH}_\bullet), \qquad HH^*(X) = \mathbb{H}^*(X, \mathcal{HH}^\bullet)$$

The HKR theorem then also holds at the sheaf level:

Theorem 11. *Let X be a smooth projective variety of dimension n. Then there are quasi-isomorphisms*

$$\mathcal{HH}_\bullet \simeq \bigoplus_{p=0}^{n} \Omega^p_X[p], \qquad \mathcal{HH}^\bullet \simeq \bigoplus_{p=0}^{n} T^p_X[p]$$

where $\Omega^p_X = \wedge^p \Omega^1_X$ is the sheaf of p-forms on X and $T^p_X = \wedge^p T^1_X$ where T^1_X is the tangent sheaf of X.

So in the case of a smooth projective variety X over $\mathbb{C}$, taking global sections of this sheaf calculates the Hochschild homology of X in terms of the Hodge groups of X

$$HH_k(X) \cong \bigoplus_{p-q=k} H^p(X, \Omega^q_X) = \bigoplus_{p-q=k} H^{p,q}(X)$$

In particular, since derived equivalences preserves the Hochschild homology we conclude that any derived equivalence $\mathcal{D}(X) \cong \mathcal{D}(X')$ preserves the sum $\sum_{p-q=k} h^{p,q}$, i.e. the column sums of the Hodge diamond.

References

[Ar] D. Aranha, *On Hochschild Cohomology, Koszul Duality and DG Categories*, available at `www.math.uni-bonn.de/ag/stroppel/MA_DhyanAranha.pdf`

[BonKap] A.I. Bondal, M.M. Kapranov, *Enhanced Triangulated Categories*, Mat. Sb., 181:5 (1990), 669?683

[Cib] C. Cibils, *Hochschild homology of an algebra whose quiver has no oriented cycles*, in Representation Theory: Finite Dimensional Algebras, LNM 1177 (Springer 1986), 55–59.

[Dr] V. Drinfeld, *Dg quotients of dg categories*, arXiv:math/0210114.

[DwSp] W. Dwyer and L. Spaliński, *Homotopy theories and model categories*, Handbook of Algebraic Topology, North-Holland, Amsterdam 1995, 73?126

[Hap] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Lecture Notes in Math., 1404, Springer, Berlin, 1989

[HKR] G. Hochschild, B. Kostant, A. Rosenberg, *Differential forms on regular affine algebras*, Transactions AMS 102 (1962), No.3, 383?408

[Kel] B. Keller, *On differential graded categories*, arXiv:math/0601185

[Kuz] A. Kuznetsov, *Hochschild homology and semiorthogonal decompositions*, arXiv:0904.4330

[Lod] J.-L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften, Volume 301, 1998

[Or] D. Orlov, *Smooth and proper noncommutative schemes and gluing of dg categories*, arXiv:1402.7364

[Sh] E. Shinder, *Derived categories*, lecture notes available at `http://e-shinder.staff.shef.ac.uk/Mainz-Lecture-5.pdf`

[To] B. Toën, *Lectures on DG-categories*, Topics in Algebraic and Topological K-Theory (2011) 243-301

[Yus] A.K. Yusufzai *Perfect correspondences and Chow motives*, arXiv:1310.0249