# Wild character varieties and surface operators

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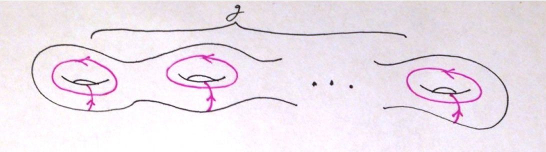
# Character varieties of surfaces

- Choose a Riemann surface $\Sigma$ and a group $G$. The character variety is roughly the space that parametrizes G-bundles on the surface, modulo some equivalence relation.
- This space often appears in physics as the classical phase space of some field theories, sometimes intuition from physics helps to calculate topological invariants, geometry etc. (citations)
- If $\Sigma$ has punctures $P$, we can allow fields to have singularities on $P$. The case where singularities are regular (*tame*) is well-known (citations)
- Irregular singularities (*wild*) are a more complicated story (Stokes phenomena). This is related to knot invariants.
- Attempting to elucidate the physical interpretation behind this relation

# Character varieties of surfaces




- An equivalence class of G-bundles given by "monodromy representation" $\pi_1(\Sigma) \to G$
- If $\Sigma$ has genus $g$

$$\mathrm{Hom}(\pi_1(\Sigma), G) = \{(A_1, B_1, \ldots, A_g, B_g) \big| [A_1, B_1] \ldots [A_g, B_g]\}$$

- Representation is the same if conjugate by $G$, so must quotient by the conjugation action

$$\mathcal{M} = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

# G finite

Take $G$ to be a finite group, $\Sigma$ has genus $g$. Character variety is the finite set $\mathrm{Hom}(\pi_1(\Sigma), G)$, and cardinality is given by counting formula

$$|\mathrm{Hom}(\pi_1(\Sigma), G)| = |G| \sum_{\mathrm{irreps}\, V} \left(\frac{\dim V}{|G|}\right)^{2-2g}$$

So invariants of character variety gives information on the representation theory of $G$.

# G compact Lie group

Next step is $G =$ compact Lie group, consider $G = U(n)$. Difficulty is that usual quotient has bad singularities, but we can restrict to irreducible representations

**Theorem (Narasimhan-Seshadri (1965))**

*There is a diffeomorphism*

$$Hom^{irr}(\pi_1(\Sigma), U(n))/U(n) \simeq \mathcal{N}(n, 0)$$

*where $\mathcal{N}(n, 0)$ is the moduli space of stable holomorphic vector bundles with rank $n$ and degree 0*

# Gauge theory description

Atiyah-Bott (1983) gives a gauge-theoretical description of this problem.

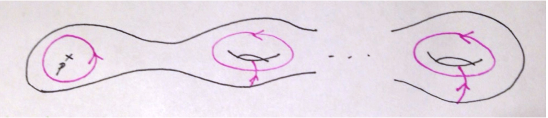
- $A \in \Omega^1(\Sigma, \mathfrak{g})$ is a connection one-form, $F_A$ its curvature
- Action $S = \int \mathrm{Tr}(F_A^2)$

Stationary points of the action satisfy $D_A * F_A = 0$, and we have the theorem

**Theorem (Atiyah, Bott)**

*The space of such connection modulo gauge transformations is isomorphic to the twisted character variety of irreducible representations*

$$Hom^{irr}(\hat{\pi}_1(\Sigma), U(n)) = \{(A_1, \ldots, B_g, \gamma) \big| [A_1, B_1] \ldots [A_g, B_g] = \gamma, \\ \gamma = \exp(2\pi i d/n)\}$$



# Symplectic structure

$\mathcal{M}$ carries a natural symplectic structure, which can be given by the Atiyah-Bott construction.

- $\mathcal{A} =$ inf. dimensional space of $\mathfrak{g}$-connections on $\Sigma$
- $\mathcal{A}$ has symplectic structure $\omega(A_1, A_2) = \int_M A_1 \wedge A_2$
- Action of gauge transformations $\mathcal{G}$, given by $g \cdot A = gAg^{-1} - (dg)g^{-1}$
- Action is Hamiltonian wrt $\omega$, moment map $\mu(A) = \mathrm{Tr}(F_A^2)$
- Taking symplectic reduction $\mathcal{M} = \mathcal{A} /\!\!/ \mathcal{G}$, $\mathcal{M}$ carries natural symplectic structure.

# Complex Lie group

Now take the case $G = \mathrm{GL}_n\mathbb{C}$

$$\mathcal{M}_{\mathbb{C}} = \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), \mathrm{GL}_n\mathbb{C})/\mathrm{GL}_n\mathbb{C}$$

Taking the quotient is more problematic because $G$ isn't compact, but we have an affine algebraic variety, so we get arithmetic methods etc. This quotient is singular, but if we take a twisted quotient it is smooth

$$\mathcal{M}_{\mathbb{C}}(n,d) = \{(A_1, \dots, B_g, \gamma) \big| [A_1, B_1] \dots [A_g, B_g] = \gamma, \\ \gamma = \exp(2\pi i d/n)\}/\mathrm{GL}_n\mathbb{C}$$

Real compact $U(n) \subset \mathrm{GL}_n\mathbb{C}$, and $\mathcal{N}(n,d) \subset \mathcal{M}_{\mathbb{C}}(n,d)$, but as real submanifolds. In fact,

$$T^*\mathcal{N}(n,d) \subset \mathcal{M}_{\mathbb{C}}(n,d)$$

but not holomorphically, though both are complex manifolds

# Hitchin moduli space

Another description is given by Hitchin (1986). Consider Yang-Mills on $\Sigma \times \mathbb{R}^2$, with gauge field $A$, and look at solutions pulled back from $\Sigma$. If we relabel $\phi = (A_3 + iA_4)/2, \bar{\phi} = (A_3 - iA_4)/2$, and imposing *self-duality* $F = *F$ we get the *Hitchin equations*

$$F_A + [\phi, \phi^*] = 0$$
$$D_A \phi = D_A * \phi = 0$$

The space of such solutions, modulo gauge transformations is the Hitchin moduli space $\mathcal{M}_H$. If we choose the underlying vector bundle to have degree $d$, $\mathcal{N}(n, d) \subset \mathcal{M}_H$

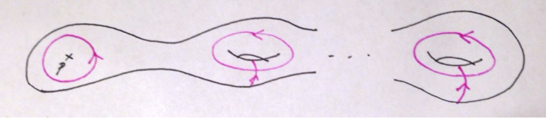
# Hyperkähler geometry

We've seen a lot of spaces, but they're all the same! There are three different descriptions of the same character variety

- $\mathcal{M}_B = \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), \mathrm{GL}_n\mathbb{C})/\mathrm{GL}_n\mathbb{C}$ (Betti moduli space)
- $\mathcal{M}_{dR} =$ ($\mathfrak{gl}_n\mathbb{C}$-valued connections)/(gauge transformations) (de Rham moduli space)
- $\mathcal{M}_H$ (Hitchin, or Dolbeault moduli space)

Monodromy map $\mathcal{M}_{dR} \simeq \mathcal{M}_B$ is complex analytic isomorphism (Riemann-Hilbert correspondence), and $\mathcal{M}_{dR} \simeq \mathcal{M}_H$ is diffeomorphism (Nonabelian Hodge correspondence)

# Regular singularities



- Mark a point $p$ on $\Sigma$, and consider connections with singularities at $p$
- Restrict to regular singularities $\rightarrow$ nice generalization of Riemann-Hilbert correspondence

## Regular singularities

Connection given by $A(z)dz$ is regular if flat sections grow polynomially, can find local coordinates such that $A(z)$ has a simple pole

$$A(z) = \frac{C}{z} + \text{holomorphic}$$

# Irregular singularities

We'd like to generalize to irregular singularities. Consider connections on disk $D$ with a meromorphic singularity on $p$. In local coordinates, gauge-equivalent to

$$\nabla = d - A(z)dz$$

$$A(z) = \frac{A_k}{z^k} + \cdots + \frac{A_2}{z^2} + \frac{A_1}{z^1}$$

A gauge transformation is given by a $G$-valued function $g(z)$, acting as

$$g(z) \cdot A(z) = gAg^{-1} - (\partial_z g)g^{-1}$$

# Irregular singularities

Accept arbitrary gauge transformations on $D \setminus p$, monodromy is the only invariant.

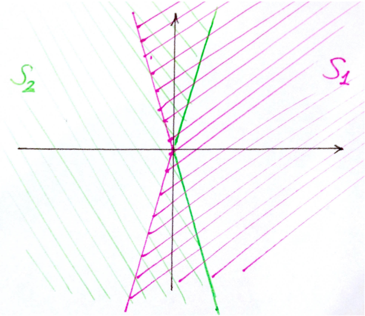
$$A(z) \sim \frac{A_1'}{z} dz, \text{monodromy} = e^{2\pi i A'}$$

Accept formal meromorphic gauge transformations, then connection can be put in canonical form

$$A(z) \sim \frac{T_n}{z^n} + \cdots + \frac{T_2}{z^2} + \frac{C}{z} + \text{holomorphic}$$

where $T_n, \ldots, T_2$ diagonal, and $C$ commutes with $T_i$.
This is the *formal irregular type* of the connection. Once we pick a formal type, our problem becomes:
What's the space of connections that are *formally equivalent* to this formal type, up to meromorphic gauge equivalence?

# An example



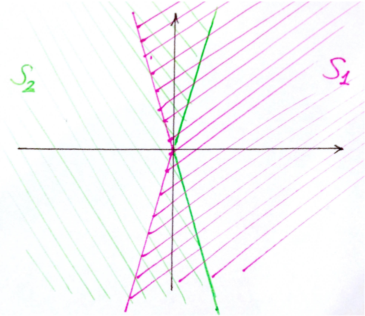

Consider a connection that is *formally* equivalent to

$$A(z) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{z^2}$$

From classical theory of irregular singularities we know

- There might not be meromorphic gauge transformation that takes $\nabla$ to $\nabla_0$
- But if we take small enough sectors $S_i$, on each sector there is $g_i(z)$. On overlaps consider $g_{ij} = g_i g_j^{-1}$

# An example




$\Psi = \begin{pmatrix} e^{1/z} & 0 \\ 0 & e^{-1/z} \end{pmatrix}$ is the matrix of formal solutions.

On overlaps $g_{ij} \cdot A = A \implies g_{ij} = \Psi C \Psi^{-1}$

$$g_{ij} \sim 1 \implies C_{12} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, C_{21} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

These are the *Stokes matrices* of the connection.

# An example

Space of connections is given by space of possible Stokes matrices, together with *formal monodromy*. This comes with two monodromy maps. In example above:

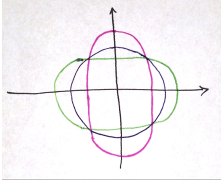
$$\begin{aligned}\text{(formal monodromy) } \mu_T :& U_+ \times U_- \times T \to T \\ & (u_+, u_-, t) \to t \\ \text{(global monodromy) } \mu_G :& U_+ \times U_- \times T \to G \\ & (u_+, u_-, t) \to u_+ u_- t\end{aligned}$$

Analogous to regular case, to get symplectic manifold need to pick conjugacy class of image of $\mu_G$, but also $\mu_F$. *Wild character variety* turns out to be (some sort of) symplectic reduction:

$$\mathcal{M} = (\mu_T^{-1}(\mathcal{C}_T) \cup \mu_G^{-1}(\mathcal{C}_G)) /\!\!/ (G \times T)$$

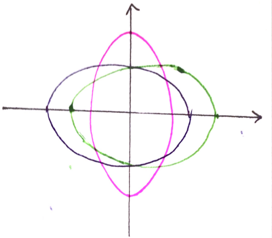
# Stokes multipliers

In general, to find correct unipotent subgroups, look at solutions and draw asymptotics



This corresponds to regular $T_3$

$$A(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \frac{1}{z^3}$$



This corresponds to $T_3$ with repeated eigenvalues, but $T_2$ regular

$$A(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \frac{1}{z^3} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \frac{1}{z^2}$$

# Unexpected isomorphisms

Considering higher order poles doesn't necessarily lead to different spaces, and many of the new spaces are isomorphic

**Example (Double pole)**

On the plane, consider the problems

- Two regular singularities
- One order 2 pole $A(z) \sim \frac{T_2}{z^2}$

The moduli spaces (tame character variety and wild character variety) are isomorphic

**Example ("3" and "$3+1^m$")**

On the plane, consider the problems

- One order 3 pole and m simple poles
- One order 3 pole

The moduli spaces (wild character varieties) are isomorphic

# Description by twisted SYM

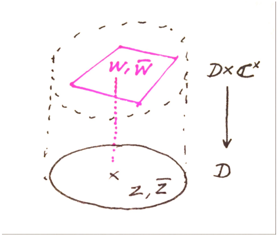
The Hitchin space $\mathcal{M}_H$ has a description in the context of twisted $N = 4$ 4-dimensional super Yang-Mills. Kapustin and Witten consider the "GL-twist" of $N = 4$ in the context of geometric Langlands. Twist depends on a twisting parameter $t \in \mathbb{C}$

- Fields: connection one-form $A$ on G-bundle $E \to M^4$, ad$(E)$-valued one-form $\phi$ + fermions etc.
- Supersymmetry equations $(F - \phi \wedge \phi - tD\phi)^{\text{self}} = (F - \phi \wedge \phi - t^{-1}D\phi)^{\text{anti}} = 0, D * \phi = 0$

Consider the theory on $\Sigma \times \mathbb{R}^2$. Take limit where $\Sigma$ is small.
Low energy states = solutions pulled back from $\Sigma$, so
$D\phi = 0 \implies F - \phi \wedge \phi = 0$.
Low energy theory is sigma-model on $\mathbb{R}^2$ with target $\mathcal{M}_H$.

# Gauge theory description

In the gauge theory description, adding regular singularities corresponds to adding 't Hooft (disorder) operators.




In local coordinates $z, \bar{z}, w, \bar{w}$, we require fields to have the following behavior

$$A = \alpha d\theta$$

$$\phi = (\beta + i\gamma)\frac{dz}{2z} + (\beta - i\gamma)\frac{d\bar{z}}{2\bar{z}}$$

Monodromy of $\mathcal{A} = \alpha - i\gamma$

Behavior of $\phi$ can be generated by source term

$$I = \int_{\mathbb{C}\times 0} dw d\bar{w} \left[(\beta + i\gamma)\partial_{\bar{z}}\phi_z + (\beta - i\gamma)\partial_z \phi_{\bar{z}}\right]$$

# Gauge theory description

Witten also found analogues of such operators for the case of irregular singularities

$$\begin{aligned} A =& \alpha d\theta \\ \phi =& \left(\frac{T_k}{z^k} + \cdots + \frac{T_2}{z^2} + \frac{\beta + i\gamma}{z}\right)\frac{dz}{2} + \left(\frac{T_k}{\bar{z}^k} + \cdots + \frac{T_2}{\bar{z}^2} + \frac{\beta + i\gamma}{\bar{z}}\right)\frac{d\bar{z}}{2} \\ & \text{Monodromy of } \mathcal{A} = \alpha - i\gamma \end{aligned}$$

Behavior of $\phi$ can be generated by source terms

$$I_k = \frac{\pi}{e^2(k-1)!}\int_{\mathbb{C}\times 0} dw d\bar{w}\, T_i \partial_z^i \phi_{\bar{z}}$$

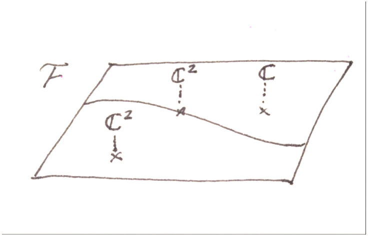
Witten shows these operators don't depend on the entries of the matrices $T_2, \ldots, T_k$ (also $\beta$) as long as these stay regular. Finds fermionic operator $V_k$ with $\{Q, V_k\} = I_k$

# Constructible sheaves

The same spaces appear in the work of Shende, Treumann and Zaslow, in the context of constructible sheaves. This is roughly:

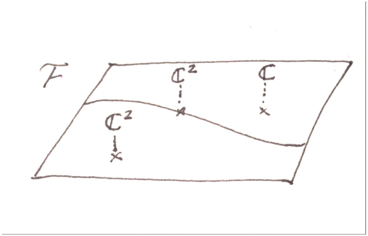
- A stratification of the space $M = \bigcup M_i$
- A sheaf $\mathcal{F}$ is constructible if $\mathcal{F}|_{M_i}$ is locally constant

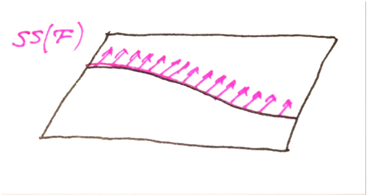
In our case, can see this as a vector space $\mathbb{C}^r$ on each point, $r$ can jump as you change strata



# Singular support

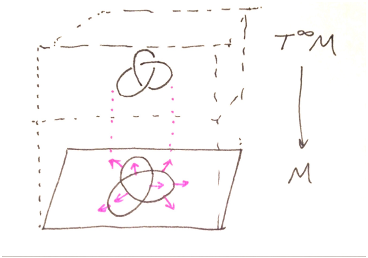
The singular support of a sheaf lives in $T^*M$, consists of the points and directions where the sheaf "jumps rank"




Singular support is a conical Lagrangian in $T^*M$

# STZ construction




Consider the plane $M$ with a knot $\Lambda$. Given a choice of perpendicular direction, can lift this knot to a circle bundle over $M$ (naturally $T^\infty M$). $T^\infty M$ has a natural contact structure (coming from symplectic structure of $T^*M$)

STZ constructs the categories

- $\mathrm{Sh}_\Lambda(M)$ = constructible sheaves on M that jump one rank across $\Lambda$
- $\mathrm{Fuk}_\Lambda(M)$ = Lagrangians in $T^*M$ ending on $\Lambda$

and prove these are equivalent.

# STZ construction

Some features

- *Microlocal monodromy $\mu mon$*: $\mathrm{Sh}_\Lambda(M) \to \mathrm{Loc}(\Lambda)$, takes a constructible sheaf and gives a locally constant sheaf on $\Lambda$
- Moving the knot projection on $M$ induces a Legendrian isotopy in $T^*M$ ($\Lambda \to \Lambda'$) and equivalence of categories $\mathrm{Sh}_\Lambda(M) \simeq \mathrm{Sh}_{\Lambda'}(M)$ Objects in this category can be represented by a moduli space $\mathcal{M}_\Lambda$. Moving the knot projection induces an algebraic isomorphism $\mathcal{M}_\Lambda \simeq \mathcal{M}_{\Lambda'}$

# Relation to character varieties

If we use the drawings we made for the irregular singularities, this moduli space of constructible sheaves agrees with the wild character variety. The example studied by Witten (regular, semisimple) corresponds to a torus knot.

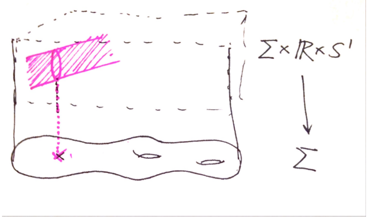
The monodromy maps all match as expected:

$\mu mon : \mathrm{Sh}_\Lambda(M) \to \mathrm{Loc}(\Lambda)$ induces map $\mathcal{M}_\Lambda \to T/_{\mathrm{ad}}T$, agrees with formal monodromy.

Monodromy map $\mathrm{Sh}_\Lambda(M) \to \mathrm{Loc}(S^1)$ induces map $\mathcal{M}_\Lambda \to G/_{\mathrm{ad}}G$, agrees with global monodromy.

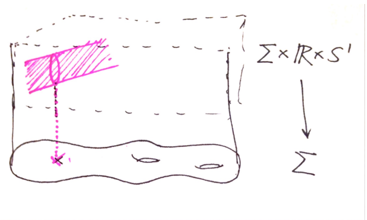
# Relation to Witten's construction

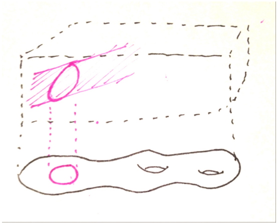
Witten considers the GL-theory on $\Sigma \times S^1 \times \mathbb{R}$ We propose to identify the cylinder $S^1 \times \mathbb{R}$ with the (punctured) cotangent space over a point.



# Relation to Witten's construction

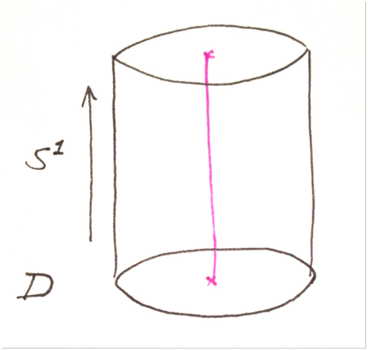
Look at a regular singularity first. For a regular singularity, operator is
$\int dwd\bar{w}\,[(\beta + i\gamma)\partial_{\bar{z}}\phi_z + (\beta - i\gamma)\partial_z\phi_{\bar{z}}]$
Supported at a point $\times S^1 \times \mathbb{R}$. if we assume there is a deformation of the support inducing an equivalence of surface operators

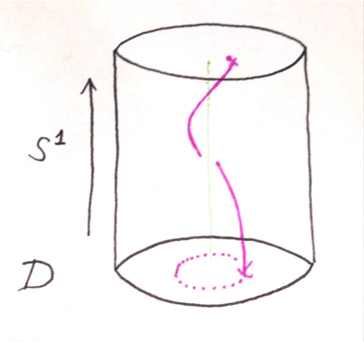


# Relation to Witten's construction

To see this better, omit the $\mathbb{R}$ direction

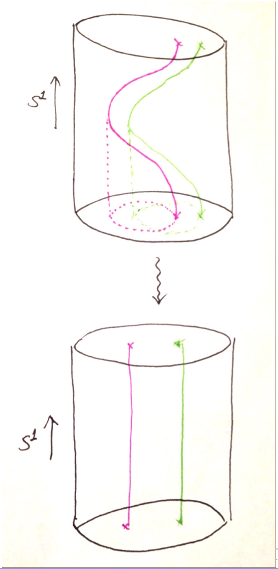


# Double pole in STZ construction

Now consider the simplest irregular singularity, $A(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 1/z^2$. Note that strands are actually unknotted, so pulling them into two regular singularities doesn't change $\mathcal{M}_\Lambda$

Question: what if we interpret these drawings as the support of operators in the twisted theory?



# Speculation about wild singularities

So we have the following speculation:

**Speculation**

The "wild ramification" operators are equivalent (?) to tame ramification operators, but wrapped around torus knots (for $T_k$ regular). This leads to a natural generalization for more general knots, some of which correspond to more general meromorphic singularities ($T_k$ with conincident eigenvalues)

This could explain some phenomena:

- Operator changes when eigenvalues coincide $\Leftrightarrow$ knot changes (strands fuse)
- Operator independent of (generic) eigenvalues $\Leftrightarrow$ entries in $T_i$ give position, size of knot strands

# Relation to knot invariants

The HOMFLY polynomial is a two-variable generalization of the Jones polynomial, given by the following skein relation

## HOMFLY polynomial

$$P(\text{unknot}) = \frac{a - a^{-1}}{q^{1/2} - q^{-1/2}}$$

$$aP(L_-) - a^{-1}P(L_+) = (q^{1/2} - q^{-1/2})P(L_0)$$

The lowest order coefficient (in $a$) is a polynomial $P_{\text{lowest},\Lambda}$ in $(q^{1/2} - q^{-1/2})$ One of the results of STZ is that the point count of $\mathcal{M}_\Lambda$ over finite fields is related to this coefficient

$$\#\mathcal{M}_{\Lambda+\text{full twist}}(\mathbb{F}_q) = P_{\text{lowest},\Lambda} \times (\text{Some factor } q^a(q-1)^b)$$

# Questions

## Math

- Wild character varieties (torus knot) have nice hyperkähler geometry, does this work for a general knot?
- Extend arithmetic methods of Hausel and Rodriguez-Villegas to this case; what sort of representation theory appears in the wild case?
- Relation to cluster charts and spectral networks?

## Physics

- Is speculation above true? If yes, we have a way of describing even more general surface operators in the Kapustin-Witten theory
- Can other knot invariants (other coefficients on HOMFLY etc.) be recovered from the theory? What happens if we consider the theory on higher-genus Riemann surfaces?
- Looks suspiciously related to constructions in "Fivebranes and Knots", elucidate if there's a relation

# References


M. Atiyah, R. Bott, *The Yang-Mills equations over Riemann surfaces* (1983)

N. Hitchin, *The self-duality equations on a Riemann surface* (1986)

M. Mulase, *Geometry of character varieties of surface groups* (2007)

P. Boalch, *Global Weyl groups and a new theory of multiplicative quiver varieties* (2014)

A. Kapustin, E. Witten, *Electric-Magnetic Duality And The Geometric Langlands Program* (2006)

S. Gukov, E. Witten, *Gauge Theory, Ramification, And The Geometric Langlands Program* (2007)

E. Witten, *Gauge theory and wild ramification* (2007)

V. Shende, D. Treumann, E. Zaslow, *Legendrian knots and constructible sheaves* (2014)