Stability conditions on Fukaya categories of surfaces

Some new techniques and results

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Changing topics

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Sorry for the bait and switch!

Definition of stability conditions

T. Bridgeland, inspired by Douglas' work on Π-stability of branes in physics, defined a notion of stability conditions on any triangulated category $\mathcal{D}$.

Definition

A (Bridgeland) *stability* condition on $\mathcal{D}$ is a central charge function $Z : K_0(\mathcal{D}) \to \mathbb{C}$ and a slicing $\{\mathcal{P}_\phi\}$ (semistable objects of phase ϕ) such that:

- $Z(X) = m(X)e^{i\pi\phi}$ if $X \in \mathcal{P}_\phi$
- $\mathcal{P}_{\phi+1} = \mathcal{P}_\phi[1]$
- $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ if $X \in \mathcal{P}_\phi$ and $Y \in \mathcal{P}_\psi$, $\phi > \psi$
- Every object $X \in \mathcal{D}$ has a Harder-Narasimhan filtration

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \to \dots \to & X_{n-1} & \longrightarrow & X_n = X \\ & \nwarrow \!\!\!\cdot\cdot & \swarrow & & & \nwarrow \!\!\!\cdot\cdot & \swarrow \\ & & A_1 & & & & A_n \end{array}$$

where A_i is semistable of phase ϕ_i, $\phi_1 > \cdots > \phi_n$

Spaces of stability conditions

Bridgeland also proved that the set $\mathrm{Stab}(\mathcal{D})$ of stability conditions admits the structure of a complex manifold, with a map

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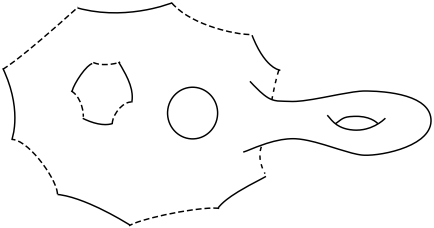
The space $\mathrm{Stab}(\mathcal{D})$ carries commuting actions of $\tilde{\mathrm{GL}}^+(2, \mathbb{R})$ and $\mathrm{Aut}(\mathcal{D})$.

A marked surface is a topological surface Σ with boundary $\partial\Sigma$ and a marked subset $M \subset \partial\Sigma$



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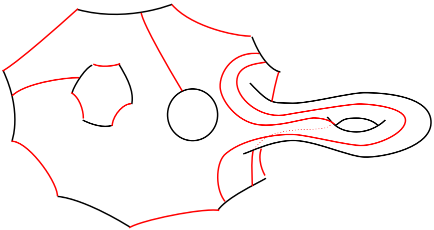

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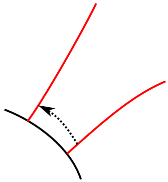
Fukaya categories of surfaces

As an A_∞ category

The A_∞ category associated with $\mathcal{A}$ has objects given by the arcs,

μ^2 (morphisms) given by boundary paths.



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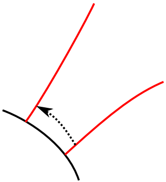
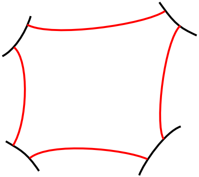


and higher μ^n given by polygons



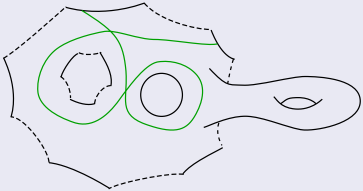
Definition

The Fukaya category $\mathcal{F}(\Sigma)$ is the category of twisted complexes over the above. This is independent of the choice of $\mathcal{A}$.

HKK's results

Theorem

(Geometricity) Every indecomposable object of $\mathcal{F}(\Sigma)$ can be rep'd by an immersed curve with local system



Their main theorem:

Theorem

A suitable choice of suitable quadratic differential φ on Σ gives a stability condition on $\mathcal{F}(\Sigma)$

Quadratic differentials

φ looks locally like $f(z)dz \otimes dz$. This defines a *horizontal foliation* (real directions).

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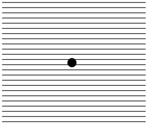

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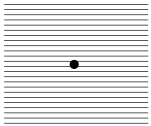
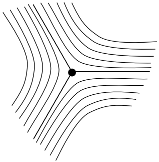


Around $f(z) \sim z^n, n = -1, 1, 2, \ldots,$
angle is $(n+2)\pi$



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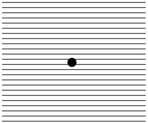
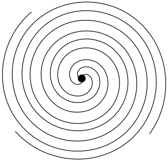


Around $f(z) \sim z^n, n \geq 2$



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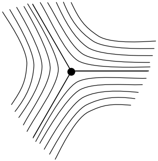

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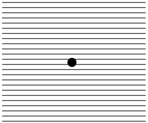

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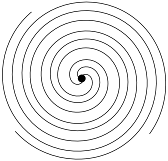

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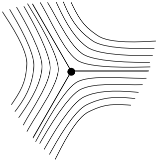
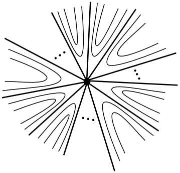


Around essential singularity $f(z) \sim \exp(1/z^n)$



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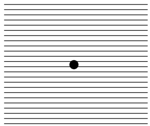

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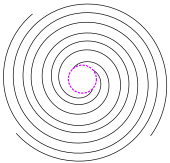

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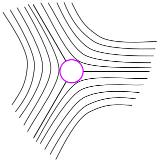

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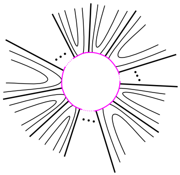
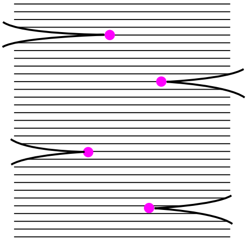


The stability condition

Quadratic differential φ gives the surface the structure of a *flat surface*.



This gives a stability condition:

- Central charge function is given a period map

$$K_0(\mathcal{F}(S)) \xrightarrow{\sim} H_1(\Sigma, M, \mathbb{Z}_\tau) \xrightarrow{\int \sqrt{\phi}} \mathbb{C}$$

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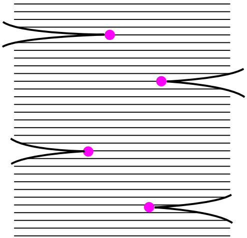

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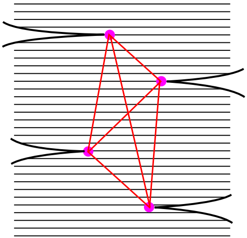

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Space of stability conditions

Theorem (Haiden, Katzarkov and Kontsevich)

The map from moduli space of quadratic differentials $\mathcal{M}(\Sigma) \to \mathrm{Stab}(\mathcal{F}(\Sigma))$ *is a homeomorphism to a union of connected components.*

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Theorem (Haiden, Katzarkov and Kontsevich)

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Example

Consider the disk Δ_{n+1} with $n+1$ marked intervals on the boundary. We have a family of quadratic differentials

$$\mathbb{C}^n \to \mathcal{M}(\Delta_n)$$
$$(a_0, \ldots, a_{n-1}) \mapsto \exp(z^n + a_{n-1}z^{n-1} + \ldots a_0)dz^{\otimes 2}$$

which turns out give the whole stability space.

New results

Let us call these HKK stability conditions. In upcoming work I prove that this is the whole story in some cases:

Theorem (T.)

Assume that Σ is such that each boundary circle has both marked and unmarked parts (i.e. fully stopped). Then all stability conditions are HKK stability conditions, and $\mathcal{M}(\Sigma) \to \mathrm{Stab}(\mathcal{F}(\Sigma))$ is a homeomorphism.*

This is proven by developing a notion of *relative stability condition*, that behaves analogously to compactly supported cohomology.

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* As of now we also need to assume at least one boundary component has ≥ 2 marked parts, i.e. ≥ 2 stops.

Important lemma

Fix a stability condition $\sigma \in \mathrm{Stab}(\mathcal{F}(\Sigma))$, not assuming that it is an HKK stability condition. We have the following general lemmas

Lemma

Any stable object in $\mathcal{F}(\Sigma)$ can be represented either by a embedded circle or by an embedded interval.

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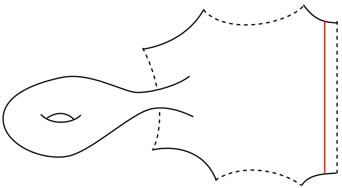
Lemma

The Harder-Narasimhan filtration of any indecomposable object under σ looks like a chain of intervals, isotopic to the original object.

Relative stability conditions

First let's give the definition. Let γ be an arc isotopic to an unmarked boundary component



Definition

A relative stability condition on $\mathcal{F}(\Sigma), \gamma$ is a stability condition on $\mathcal{F}(\tilde{\Sigma})$ where $\tilde{\Sigma} = \Sigma \cup_\gamma \Delta_n$ such that every marked boundary of Δ_n appears in the HN filtration of γ.

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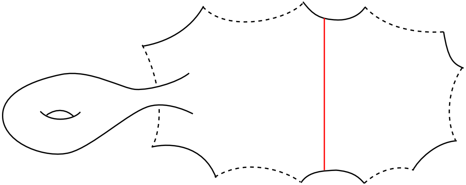

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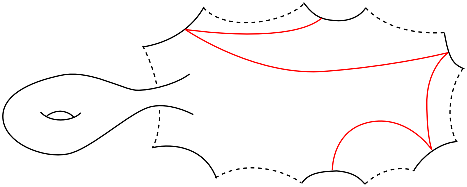

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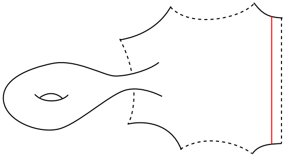
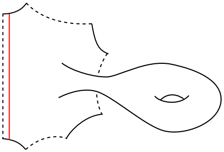
Compatibility

Suppose we have relative stability conditions σ_L on (Σ_L, γ) and σ_R on (Σ_R, γ). They are compatible if the HN filtration of γ agrees in phase and central charge:





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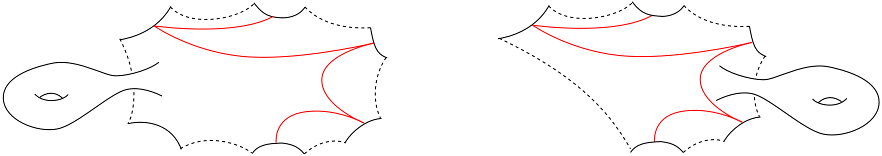


Cutting and gluing

Fix a stability condition $\sigma \in \mathcal{F}(\Sigma)$, and pick an embedded interval γ that cuts the surface into two components Σ_L and Σ_R.

Lemma (Cutting)

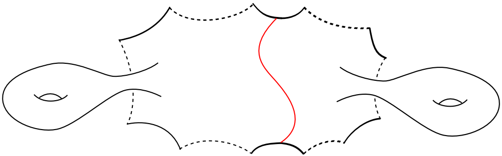
From σ one can construct relative stability conditions σ_L on (Σ_L, γ) and σ_R on (Σ_R, γ), compatible with each other.

Lemma (Gluing)

From compatible stability conditions σ_L on Σ_L, γ and σ_R on Σ_R, γ, one can construct a (usual) stability condition on $\Sigma_L \cup_\gamma \Sigma_R$.

Cutting

For the cutting procedure, one first uses σ to decompose γ into its stable components:



Let's say γ cuts the surface into Σ_L, Σ_R, From this data we construct $\tilde{\Sigma}_L, \tilde{\Sigma}_R$. The relative stability conditions are obtained by restriction (it's nontrivial to prove that this gives valid stability conditions on $\mathcal{F}(\tilde{\Sigma}_L)$ and $\mathcal{F}(\tilde{\Sigma}_R)$)

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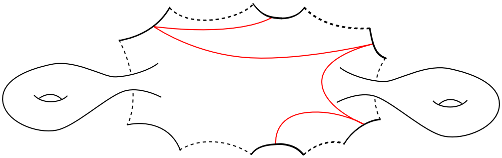

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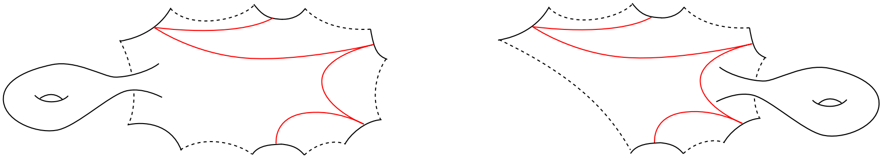

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Gluing

First we express $\mathcal{F}(\Sigma)$ as the following categorical pushouts, using the compatibility of the relative stability conditions:

$$\begin{array}{ccc} \mathrm{Mod}_k \cong \langle \gamma \rangle & \longrightarrow & \mathcal{F}(\Sigma_R) \\ \downarrow & & \downarrow \\ \mathcal{F}(\Sigma_L) & \longrightarrow & \mathcal{F}(\Sigma) \end{array}$$

$$\begin{array}{ccc} \mathrm{Mod}(\mathbb{A}_N) \cong \langle \{\gamma_i\} \rangle & \xrightarrow{j} & \mathcal{F}(\tilde{\Sigma}_R) \\ \downarrow{\scriptstyle i} & & \downarrow \\ \mathcal{F}(\tilde{\Sigma}_L) & \longrightarrow & \mathcal{F}(\Sigma) \end{array}$$

This is a pushout of fully faithful maps, so an object in $\mathcal{F}(\Sigma)$ can be expressed by a triple

$$(X_L, X_R, \phi), X_L \in \mathcal{F}(\tilde{\Sigma}_L), X_R \in \mathcal{F}(\tilde{\Sigma}_R), \phi : i^* X_L \xrightarrow{\sim} j^* X_R$$

where i^*, j^* are the right adjoints.
The central charge is defined by the following formula

$$\begin{aligned} Z(X) &:= Z_L(X_L) + Z_R(X_R) - Z_L(i \circ i^* X_L) \\ &= Z_L(X_L) + Z_R(X_R) - Z_R(j \circ j^* X_R) \end{aligned}$$

The slicing

Let us define the subcategories $\mathcal{P}_\phi$ of semistable objects of phase ϕ. This is defined constructively:

- First we include into $\mathcal{P}_\phi$ all the objects in the images of $\mathcal{P}^L_\phi$ and $\mathcal{P}^R_\phi$.

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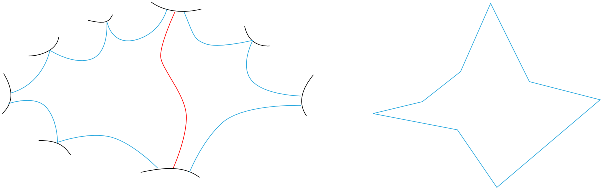
- First we include into $\mathcal{P}_\phi$ all the objects in the images of $\mathcal{P}_\phi^L$ and $\mathcal{P}_\phi^R$.
- Then we include objects coming from "unobstructed lozenges" of stable objects, ie. the following arrangements

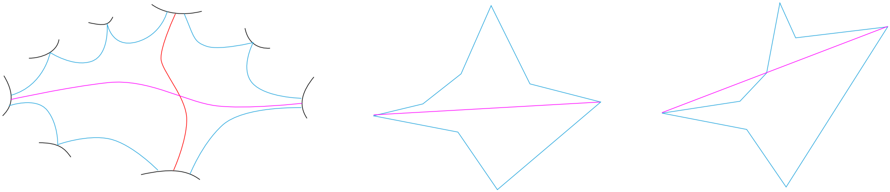


The unobstructed condition is an inequality of the central charges; if it's satisfied we include the diagonal of the lozenge as a stable object.

The slicing

Let us define the subcategories $\mathcal{P}_\phi$ of semistable objects of phase ϕ. This is defined constructively:

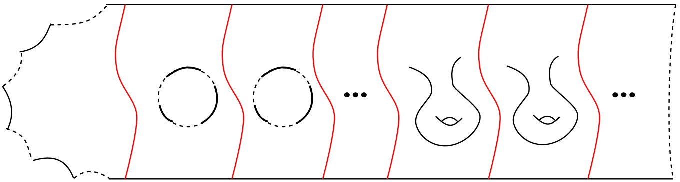
- First we include into $\mathcal{P}_\phi$ all the objects in the images of $\mathcal{P}^L_\phi$ and $\mathcal{P}^R_\phi$.
- Then we include objects coming from "unobstructed lozenges" of stable objects, ie. the following arrangements



The unobstructed condition is an inequality of the central charges; if it's satisfied we include the diagonal of the lozenge as a stable object.

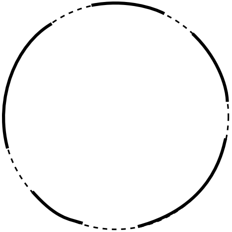
Reducing by cutting the surface Σ

We can prove that the cutting and gluing procedure are inverse to each other. Therefore we can analyze all of the stability conditions by cutting it down to pieces we understand. Can cut only along arcs dividing the surface into two.

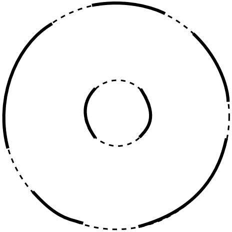


The pieces

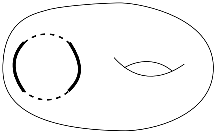
We end up with three kinds of pieces



Disk with n marked boundaries, $\mathcal{F} \cong \text{Mod}(\mathbb{A}_n)$, and $\text{Stab} \cong \mathbb{C}^n$



Annulus with $2, p$ marked boundaries, $\mathcal{F} \cong \text{Mod}(\tilde{\mathbb{A}}_{p+1})$ and $\text{Stab} \cong \mathbb{C}^{p+1}$



Punctured torus with 2 marked boundaries, $\mathcal{F} = \text{Mod}(\cdot \rightrightarrows \cdot \rightrightarrows \cdot)$, can prove by similar argument that $\text{Stab} \cong \mathcal{M}$

In all these base cases we only have HKK stability conditions, so by cutting/gluing relations this is also the case for Σ.

Open questions

- Do we need three base cases? Can we cut it down to disks?
- Can we drop the fully stopped condition? HKK do this using some sort of localization, can we do the same?
- Main technical question What can be said in those cases about the category spanned by the stable pieces of an object? This is the key to defining relative stability conditions in more general cases.

For Further Reading I

[1] T. Bridgeland, *Stability conditions on triangulated categories*, Annals of Mathematics **166** 2007, 317–345

[2] F. Haiden, L. Katzarkov and M. Kontsevich, *Flat surfaces and stability structures*, Publications Mathématiques de l'IHÉS **126** 2017, 247–318

$+$ others