The Wall-Chamber structures of the real Grothendieck groups

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0. Overview of results

A = finite dimensional algebra / k = field, $\dim A = n$

$\langle \, , \rangle : K_0(\operatorname{proj} A) \times K_0(\operatorname{mod} A) \longrightarrow \mathbb{Z}$ Euler pairing

$K_0(\operatorname{proj} A)_{\mathbb{R}} \ni \theta \underset{\text{(King)}}{\leadsto}$ notion of (semi)stability

$\leadsto$ subset $\mathrm{Wall} \subset K_0(\operatorname{proj} A)_{\mathbb{R}}$

$\mathrm{Chambers}(A) = \{$ components of $K(\operatorname{proj} A)_{\mathbb{R}} \setminus \mathrm{Wall} \}$

Main theorem I (thms. 1.1 & 1.4)

There are bijections

$$\mathrm{Chambers}(A) \longleftrightarrow TF_n(A) \longleftrightarrow 2\text{-silt}(A)$$

$TF_n(A) = \{$ TF equiv. classes of dim n $\}$

$2\text{-silt}(A) = \{$ basic 2-term silting objects in $K^b(\operatorname{proj} A) \}$

τ-tilting reduction

Given U 2-term presilting object, $\exists$ algebra B s.t.

$$\begin{array}{ccc} 2\text{-presilt}_U(A) & \longleftrightarrow & 2\text{-presilt}(B) \\ \parallel & & \\ \multicolumn{3}{l}{\{2\text{-term pre-silting objs w/ } U \text{ as direct summand}\}} \end{array}$$

Main theorem 2

$\exists$ nbhd N_U of $[U] \in K_0(\mathrm{proj}\,A)_{\mathbb{R}}$, $N_U = \bigcup$ chambers

s.t. there is a diagram

$$\begin{tikzcd} \left\{\begin{array}{c}\text{TF equiv. classes} \\ \text{in } N_U\end{array}\right\} \arrow[r, leftrightarrow, "\sim"] & \left\{\begin{array}{c}\text{TF equiv. classes} \\ \text{in } K_0(\mathrm{proj}\,B)_{\mathbb{R}}\end{array}\right\} \\ 2\text{-presilt}_U(A) \arrow[u] \arrow[r, leftrightarrow, "\sim"] & 2\text{-presilt}(B) \arrow[u] \end{tikzcd}$$

Application:

$A = kQ$, $\quad Q$ = acyclic quiver

dimension vector $\vec{d} \rightsquigarrow$ largest wall $4_{\vec{d}} = \max\limits_{|M| < \vec{d}} 4_M$

Main theorem 3

$4_{\vec{d}}$ can be constructed inductively on supp $\vec{d} = \{ i \mid d_i \neq 0 \}$

In this talk:

- Torsion pairs
- Tilting / Silting objects
- t-structures / simple-minded collections
- Main thm. I

1. Background

$\mathcal{A}$ = abelian category

Def a *torsion pair* is a pair $(\mathcal{T}, \mathcal{F})$ of additive full subcats.
s.t. i) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$

ii) $\forall\, 0 \neq X \in \mathrm{Ob}(\mathcal{A})$, $\exists$ exact seq $0 \to T \to X \to F \to 0$
with $T \in \mathcal{T}$, $F \in \mathcal{F}$

$\mathcal{C}$ = triangulated category w/ susp. functor Σ

(Beilinson, Bernstein, Deligne)

Def a *t-structure* on $\mathcal{C}$ is a pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of full subcats s.t.

i) $\Sigma \mathcal{C}^{\leq 0} \subseteq \mathcal{C}^{\leq 0}$ and $\Sigma^{-1}\mathcal{C}^{\geq 0} \subseteq \mathcal{C}^{\geq 0}$

ii) $\mathrm{Hom}(\mathcal{C}^{\leq 0}, \Sigma^{-1}\mathcal{C}^{\geq 0}) = 0$

iii) $\forall X \in \mathcal{C}$, $\exists$ dist. triangle $\underset{\mathcal{C}^{\leq 0}}{\underset{\cap}{A}} \longrightarrow X \longrightarrow \underset{\Sigma^{-1}\mathcal{C}^{\geq 0}}{\underset{\cap}{B}} \xrightarrow{+1}$

t-structure $(\mathcal{U}, \mathcal{V})$ $\quad \mathcal{U}$ = aisle, $\mathcal{V}$ = co-aisle

Heart $\mathcal{H} = \mathcal{U} \cap \mathcal{V}$ is abelian subcat. of $\mathcal{C}$ (BBD)

Bounded if $\mathcal{C} = \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{U} = \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{V}$

Length if heart $\mathcal{H}$ is a length abelian cat $\quad \forall X$ fin. length

std t-structure

Ex. On $\mathcal{C} = D(\text{mod } A)$, take $\mathcal{U} = \mathcal{C}^{\leq 0} = \{X \mid H^{>0}(X) = 0\}$

$\Sigma = [1]$ $\qquad \mathcal{V} = \mathcal{C}^{\geq 0} = \{X \mid H^{<0}(X) = 0\}$

Restricts to bounded t-structures on $D^b(\text{mod } A)$

From now on, all t-structures will be bounded, length.

When $\mathcal{C} = D^b(\text{mod } A)$

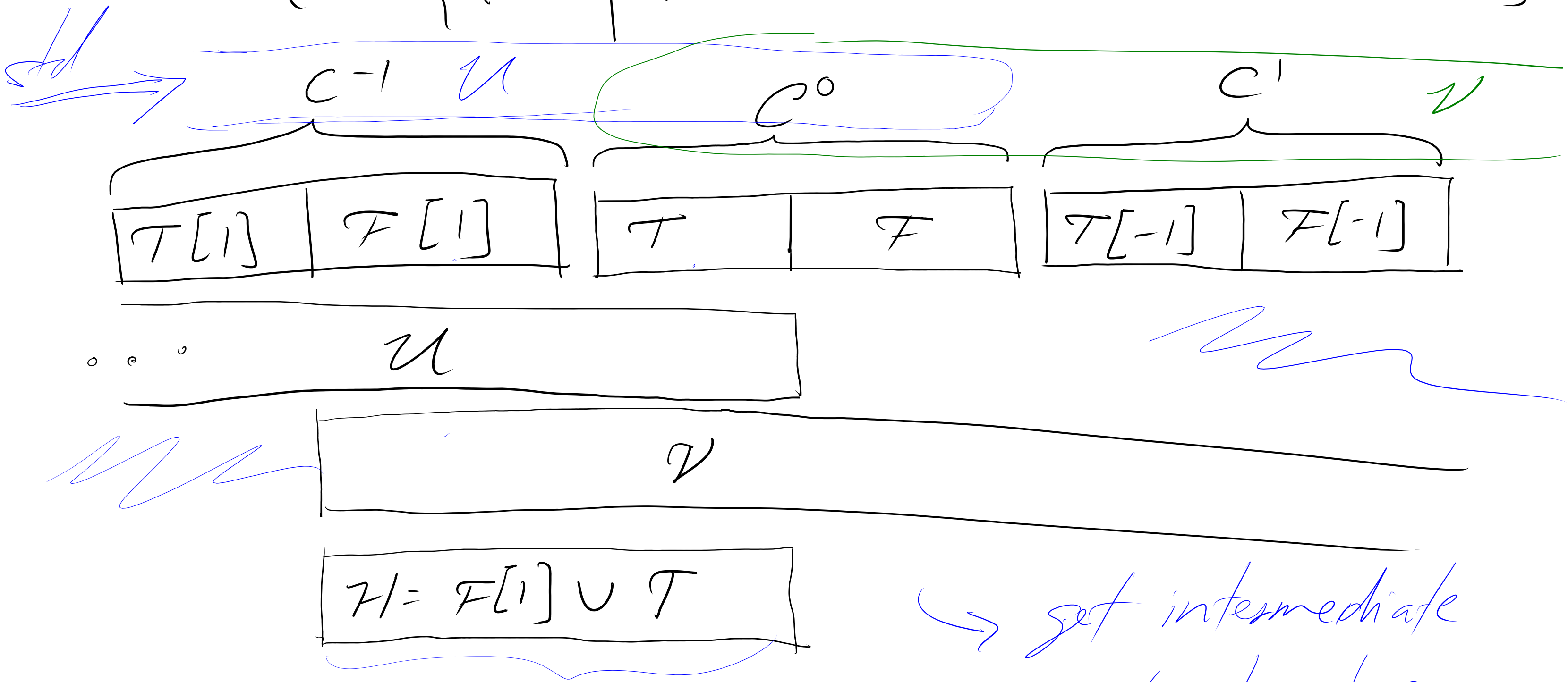
t-structure $(\mathcal{U}, \mathcal{V})$ is **intermediate** if $\mathcal{C}^{\leq -1} \subseteq \mathcal{U} \subseteq \mathcal{C}^{\leq 0}$

Denote $\text{int-t-str}(A) \subset \text{t-str}(A) = \left\{ \begin{array}{l} \text{bdd, length} \\ \text{t-structures} \\ \text{on } D^b(\text{mod } A) \end{array} \right\}$

from torsion pair to t-structure

$(\mathcal{T}, \mathcal{F})$ tors. pair on $\mathcal{A} = \operatorname{mod} A$ $\Rightarrow$ mutation of standard t-structure on $\mathcal{C} = D^b(\operatorname{mod} A)$

Mutation at $(\mathcal{T}, \mathcal{F})$ $= \begin{cases} \mathcal{U} = \{ X \in \mathcal{C} \mid H^{>0}(X) = 0 \text{ and } H^0(X) \in \mathcal{T} \} \\ \mathcal{V} = \{ X \in \mathcal{C} \mid H^{<-1}(X) = 0 \text{ and } H^{-1}(X) \in \mathcal{F} \} \end{cases}$

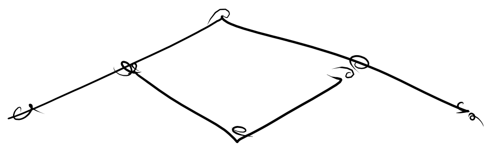


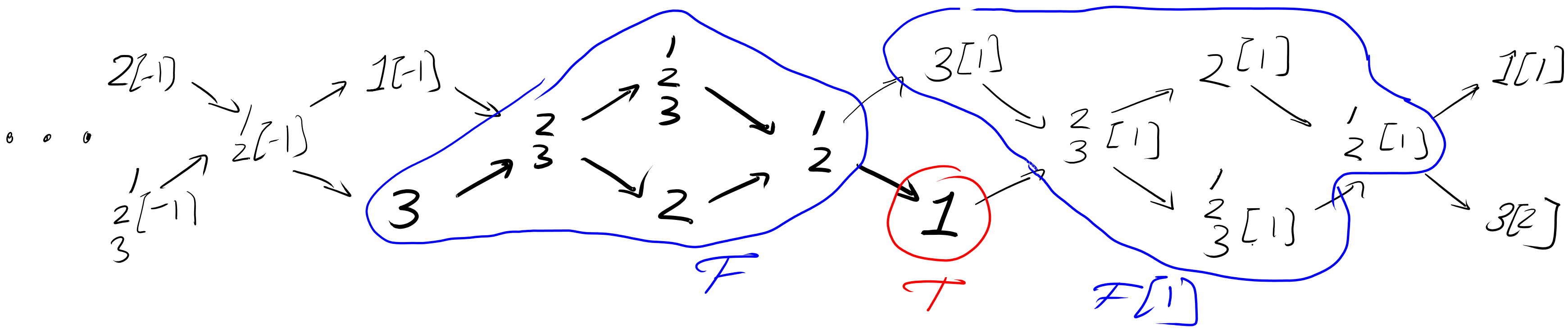
$\hookrightarrow$ get intermediate t-structure

Ex $A = kA_3$ $\quad A_3 = 1 \to 2 \to 3$

$\mathcal{H}^{std}$

$\mathcal{H} \simeq \operatorname{mod} A$



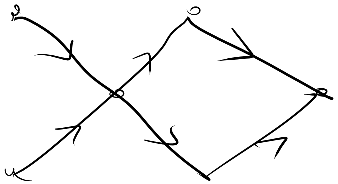


$$\operatorname{mod}(\underbrace{\cdot \to \cdot \leftarrow \cdot}_{}) \simeq \mathcal{H}$$

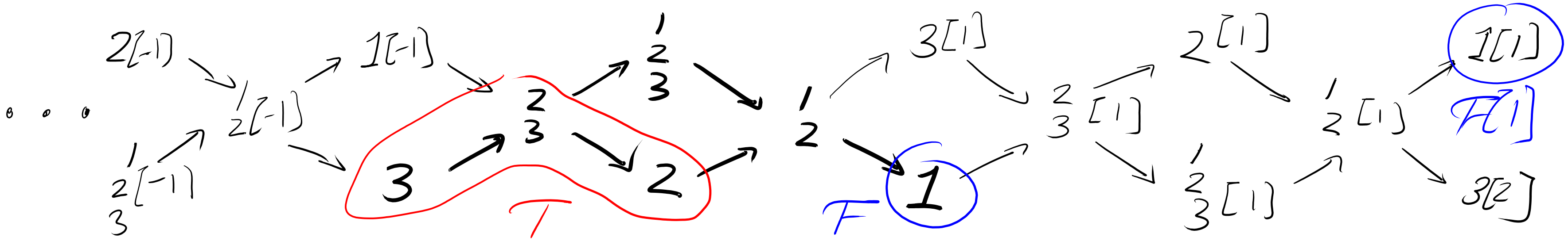
A'

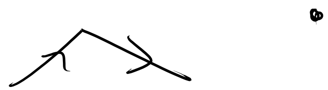
$\nparallel$

$\operatorname{mod}(A)$



$$D^b(\operatorname{mod} A) \simeq D^b(\operatorname{mod} A')$$



$\mathcal{H} =$  $\simeq \operatorname{mod}(A_2) \times \operatorname{vect}$

not derived equiv. to A_3

not faithful

Tilting, silting and simple-minded collections

Silting / tilting objects

$\mathcal{C}$ = triangulated category

Def (Keller-Vossieck, Aihara-Iyama)

- $\mathcal{D} \subset \mathcal{C}$ is a silting subcategory if
  i) $\mathcal{D}$ closed under direct summands
  ii) $\mathcal{C} = \mathrm{thick}(\mathcal{D})$
  iii) $\mathrm{Hom}(\mathcal{D}, \Sigma^m \mathcal{D}) = 0$ for all $m > 0$
- $X \in \mathrm{ob}(\mathcal{C})$ is a silting object if
  $\mathrm{add}(X)$ is a silting subcategory
- A silting object is tilting if moreover $\mathrm{Hom}(X, \Sigma^m X) = 0$
  for all $m \neq 0$

X silting/tilting $= \bigoplus_{\text{indecomp}} X_i \Rightarrow \{X_i\}$ silting/tilting set

In $C = K^b(\text{proj } A)$, often called silting complex

Def $M \in K^b(\text{proj } A)$ is pre-silting if $\text{Hom}(M, M[>0]) = 0$

thick$(M) \neq C$ maybe

M is 2-term if $\underset{q.\text{iso}}{\simeq} (P^{-1} \to P^0)$

2-(pre)silt(A) = $\{$2-term (pre)silting objects of $K^b(\text{proj } A)\}$

- From silting objects to t-structures: A gl.dim $< \infty$

M silting object in $K^b(\text{proj } A)$. Define subcats of $D^b(\text{mod} A)$

$$\left\{\begin{array}{l} \mathcal{U}_M = \{N \mid \text{Hom}(M, N[>0]) = 0\} \Leftrightarrow \text{Hom}(M[<0], N) \\ \mathcal{V}_M = \{N \mid \text{Hom}(M, N[<0]) = 0\} \Leftrightarrow \text{Hom}(M[>0], \underset{A}{N}) \end{array}\right.$$

Thm (Keller-Vossieck '88 for $A = k(\text{Dynkin})$, Assem, Souto, Trepode (hered.), König-Yang)

$(\mathcal{U}_M, \mathcal{V}_M)$ is bounded t-structure with heart $\simeq \text{mod}(\text{End}(M))$

Def a set of isom classes $\{X_i\}$ of $D^b(\mathrm{mod}\,A)$ is a **simple-minded collection (smc)** if

i) $\mathrm{End}(X_i)$ is a division ring R_i $\forall i$

ii) $\mathrm{Hom}(X_i, X_j) = 0$ for $i \neq j$

iii) $\mathrm{Hom}(X_i, X_j[<0]) = 0$ for any i, j

iv) $\mathrm{thick}(\oplus X_i) = D^b(\mathrm{mod}\,A)$

Denote $2\text{-smc}(A) \subset \mathrm{smc}(A)$

Given t-structure on $D^b(\mathrm{mod}\,A)$, take $\{X_i\} = \mathrm{simple}(\mathcal{H})$

Thm (Koenig-Yang 5.3 & 6.1)

i) The maps $\underset{M}{\mathrm{silt}(A)} \longrightarrow t\text{-str}(A) \longrightarrow \underset{\mathcal{X}}{\mathrm{smc}(A)}$ are bijections

ii) Let $M \in \mathrm{silt}(A)$ with image $\mathcal{X} \in \mathrm{smc}(A)$.

Then $\exists\, P_i, X_i$, $i = 1, \cdots, n$ s.t. $M = \oplus P_i$, $\mathcal{X} = \{X_i\}$

and $\mathrm{Hom}(P_i, X_j) = \begin{cases} R_j = \mathrm{End}(X_j) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

Wall-Chamber Structures of Real Grothendieck groups, part II

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0. Review

Main thm I (1.1 & 1.4) Let A = f.d. algebra / k. field $\exists$ bijections

$$\text{Chambers}(A) \longleftrightarrow TF_n(A) \longleftrightarrow 2\text{-silt}(A)$$

$\text{Chambers}(A)$ = complement of Walls

$TF_n(A)$ = {TF equiv. classes of dim n}

$2\text{-silt}(A)$ = {2-term basic silting objs. in $K^b(\text{proj }A)$} in $D^b(\text{mod }A)$

Torsion pairs

$(\mathcal{T}, \mathcal{F})$ torsion pair
add. full subcats of mod A
+ axioms

$\rightsquigarrow$ intermediate t-structure $(\mathcal{U}, \mathcal{V})$ on $D^b(\text{mod }A)$

$C^{\le -1} \subset \mathcal{U} \subset C^{\le 0}$

$$\mathcal{U} = \{X \mid H^{>0}(X) = 0,\ H^0(X) \in \mathcal{T}\}$$

$$\mathcal{V} = \{X \mid H^{<-1}(X) = 0,\ H^0(X) \in \mathcal{F}\}$$

Silting objects & simple-minded collections

$X \in K^b(\mathrm{proj}\, A)$ is **pre-silting** if $\mathrm{Hom}(X, X[n]) = 0 \quad \forall n > 0$

silting if moreover $\mathrm{thick}(X) = K^b(\mathrm{proj}\, A)$

2-term if $\underset{q.iso}{\simeq} \left(\underset{-1}{P_{-1}} \longrightarrow \underset{0}{P_0} \right)$

$\{X_i\}$ set of isom. classes of indec. objects of $D^b(\mathrm{mod}\, A)$ is **simple-minded collection**

if
$$\begin{cases} \mathrm{End}(X_i) = R_i \text{ is a div. ring } \forall i \\ \mathrm{Hom}(X_i, X_j) = 0 \quad \forall i \neq j \\ \mathrm{Hom}(X_i, X_j[<0]) = 0 \quad \forall i,j \\ \mathrm{thick}(\oplus X_i) = D^b(\mathrm{mod}\, A) \end{cases}$$

Let $n = \#$ isom. classes of simples

Thm (Koenig-Yang)

i $\#(\text{any smc on } D^b(\mathrm{mod}\, A)) = \#\left\{ \begin{matrix}\text{indec. direct summands} \\ \text{of silting object in } K^b(\mathrm{proj}\, A)\end{matrix} \right\} = n$

ii $\exists$ bijections $\quad \mathrm{silt}(A) \xleftrightarrow{\sim} t\text{-}\mathrm{str}(A)$

$$T \longmapsto (T[<0]^{\perp}, T[>0]^{\perp}) =: (\mathcal{U}, \mathcal{V})$$

where

$$T[<0]^{\perp} = \{X \in D^b(\mathrm{mod}\,A) \mid \mathrm{Hom}(T[m], X) = 0 \quad \forall\, m < 0\}$$

$$T[>0]^{\perp} = \{X \in D^b(\mathrm{mod}\,A) \mid \mathrm{Hom}(T[m], X) = 0 \quad \forall\, m > 0\}$$

and $\quad t\text{-}\mathrm{str}(A) \xleftrightarrow{\sim} \mathrm{smc}(A)$

$$(\mathcal{U}, \mathcal{V}) \longmapsto \{\text{isom. classes of simples in } \mathcal{H} = \mathcal{U} \cap \mathcal{V}\}$$

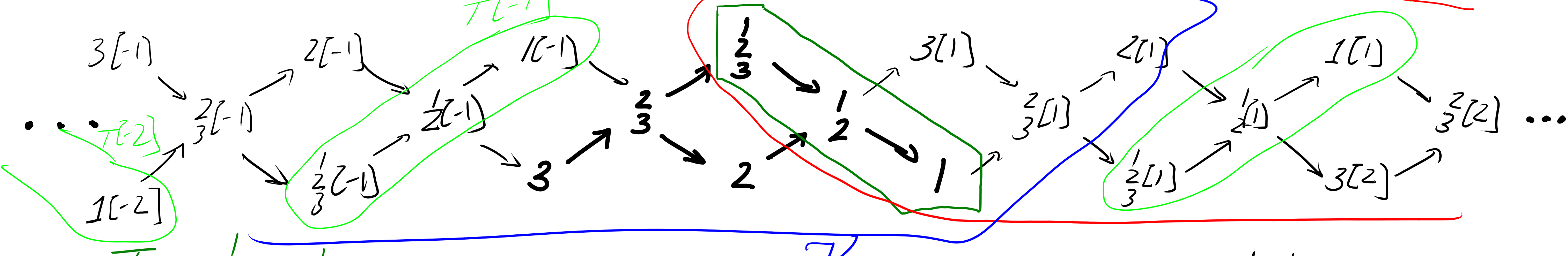
$\rightarrow$ Also restrict to $\quad 2\text{-}\mathrm{silt}(A) \xleftrightarrow{\sim} 2\text{-}\mathrm{smc}(A) \xleftrightarrow{\sim} \text{int-}t\text{-}\mathrm{str}(A)$

Thm (Adachi–Iyama–Reiten) $T \in 2\text{-}\mathrm{silt}(A) \Rightarrow (\mathcal{T}_T, \mathcal{F}_T)$ is torsion pair,

where $\quad \mathcal{T}_T = {}^{\perp}H^{-1}(\nu T), \quad \mathcal{F}_T = H^0(T)^{\perp}$

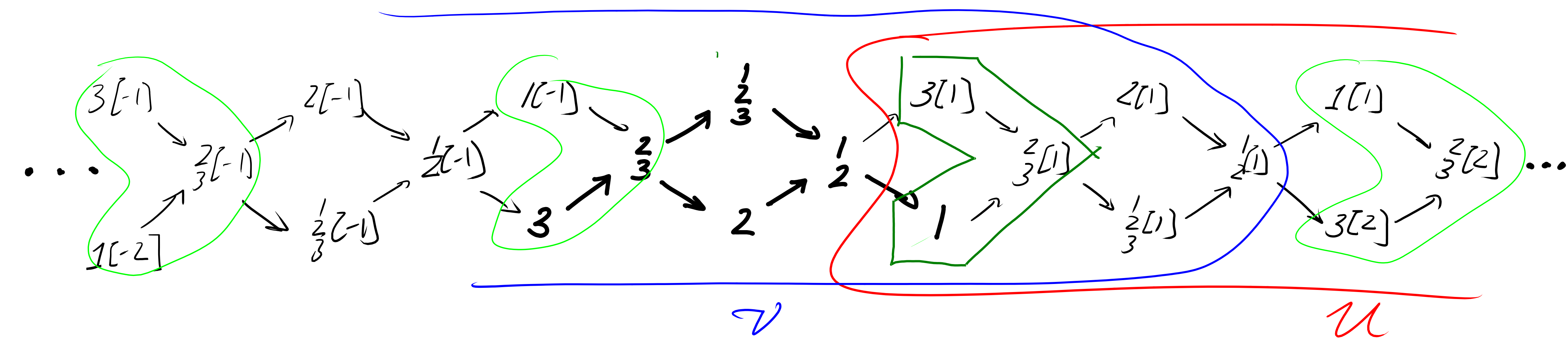
$\left(\begin{array}{l} \nu = \text{Nakayama functor} \\ D\mathrm{Hom}_A(X, Y) = \mathrm{Hom}_A(X, \nu Y) \end{array} \right)$

Ex $A = kA_3 \qquad A_3 = 1 \to 2 \to 3$



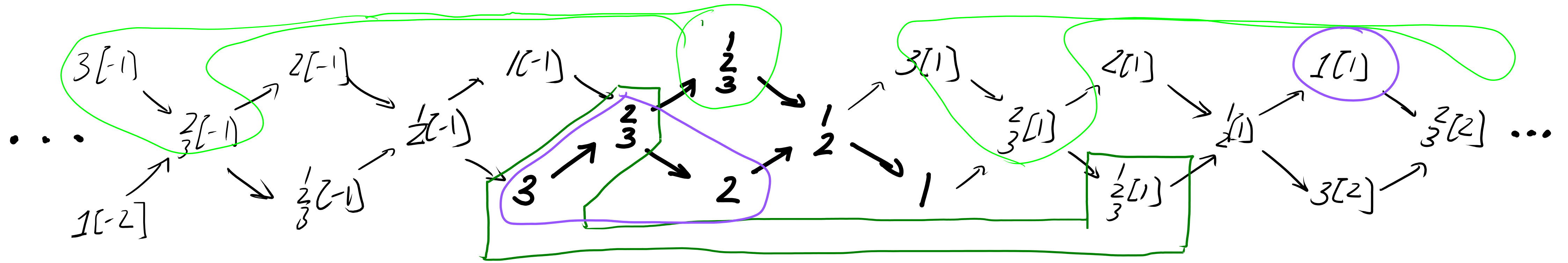
$T = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} + 1$

$\mathcal{H} \cong \operatorname{mod} kA_3$
$(1 \to 2 \to 3)$



$T = 1 \oplus 3[1] + \begin{smallmatrix}2\\3\end{smallmatrix}[1]$

$\mathcal{H} \cong \operatorname{mod}(1 \to 2 \leftarrow 3)$



$T = 3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$

$\to$ silting but not tilting $\Rightarrow$ $\mathcal{H}$ not faithful

$\mathcal{H} = \mathcal{U} \cap \mathcal{V}$

$\simeq \operatorname{mod} A_2 \times \operatorname{vect}$

1. Wall-chamber structure

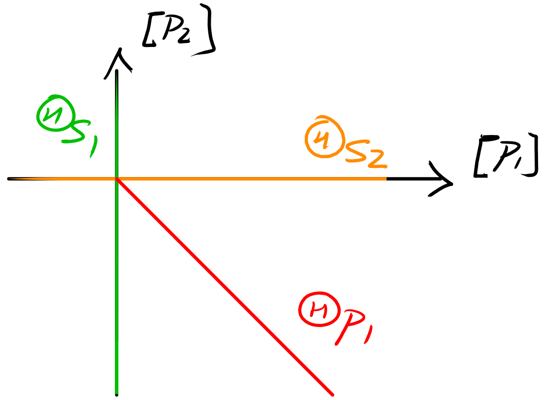
$\langle -,- \rangle : K_0(\mathrm{proj}\,A) \otimes K_0(\mathrm{mod}\,A) \to \mathbb{Z}$ Euler form

Recall (from King): $\theta \in K_0(\mathrm{proj}\,A)_{\mathbb{R}} \rightsquigarrow \mathcal{W}_\theta \subset \mathrm{mod}(A)$ subcat. of θ-semistable modules

Def For any $0 \neq M \in \mathrm{mod}\,A$, $\Theta_M = \{ \theta \in K_0(\mathrm{proj}\,A)_{\mathbb{R}} \mid M \in \mathcal{W}_\theta \}$ = wall assoc. to M

Ex $A = kA_2$
$A_2 = (1 \to 2)$

$K_0(\mathrm{proj}\,A)_{\mathbb{R}}$



$\forall M$ Θ_M is rational polyhedral cone

Def Θ_M is strongly convex if $\Theta_M \cap (-\Theta_M) = 0$

Q. When is Θ_M str. convex

e.g. Θ_{P_1} is strongly convex, Θ_{S_1}, Θ_{S_2} are not

Pick isom. classes of simples S_i, $i = 1, \dots, n$

Def $\mathrm{supp}(M) = \{ i \mid S_i \text{ is a factor of } M \} \subseteq \{1, \dots, n\}$.

M is sincere if $\mathrm{supp}(M) = \{1, \dots, n\}$

Lma For any $M \neq 0$, set $H_1 = \bigoplus_{i \in \operatorname{supp} M} \mathbb{R}[P_i]$, $H_2 = \bigoplus_{i \notin \operatorname{supp} M} \mathbb{R}[P_i]$

i $\textcircled{n}_M \cap (-\textcircled{n}_M) = H_2$ $\quad$ ($\Rightarrow$ $\textcircled{n}_M$ str. convex iff M is sincere)

ii $\Theta_M = (\textcircled{n}_M \cap H_1) \cup H_2$, $\quad$ $\textcircled{n}_M \cap H_1$ is strongly convex

* NOTE
typo in the paper

Pf i $H_2 \subset \pm \textcircled{n}_M$ $\quad$ $[P_i] \in H_2 \to S_i \notin$ comp. servs of M

$\theta = \pm [P_i] \Rightarrow \theta(M) = 0 = \theta(\text{quot } M) \Rightarrow M \in \mathcal{W}_{\pm\theta}$

$\Rightarrow \theta \in \pm \textcircled{n}_M$

$\textcircled{n}_M \cap -\textcircled{n}_M \subset H_2$ $\quad$ $0 \subset M_1 \subset \cdots \subset M_\ell = M$

$\Rightarrow \theta(M_i) \underset{\leq 0}{\geq 0} \Rightarrow \theta(M_i) = 0 \Rightarrow \theta(M_i/M_{i-1}) = 0$

$\Rightarrow \theta \in H_2$

ii $e \in A$ s.t. $S_i e = 0 \Leftrightarrow i \notin \operatorname{supp} M$ $\quad$ $K_0(\operatorname{mod} A/\langle e \rangle)_{\mathbb{R}} \hookrightarrow K_0(\operatorname{mod} A)_{\mathbb{R}}$

M is sincere $A/\langle e \rangle$-module $\quad$ $\textcircled{n}^{A/\langle e \rangle}_M \longmapsto \textcircled{n}_M \cap H_1$

Properties of Θ_M

Def Pick $M \neq 0$, $\theta \in \Theta_M$. $\mathrm{supp}_\theta M = \{ S \in \mathrm{brick}\, A \mid S \in \text{comp. series for } M \text{ in } \mathcal{W}_\theta \}$

$\mathcal{W}_\theta$: θ-semistable

We can describe faces of Θ_M

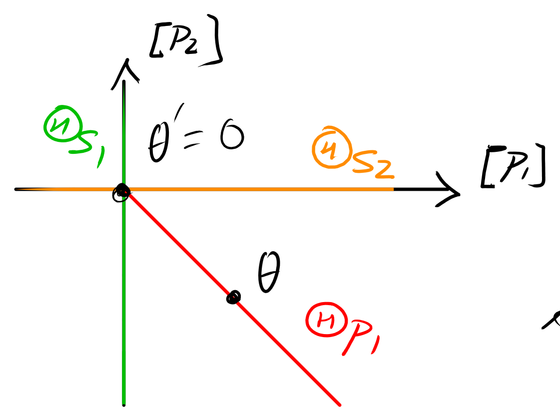
Lma For fixed M, θ as above, set $H = \ker \langle -, \mathrm{supp}_\theta M \rangle$. Then

i. $\Theta_M \cap H$ is the smallest face containing θ

ii. $\theta' \in \Theta_M \cap H \iff \mathrm{supp}_\theta M \subset \mathcal{W}_{\theta'}$

iii. $H = H' \iff \mathrm{supp}_\theta M = \mathrm{supp}_{\theta'} M$ (faces labelled by $\mathrm{supp}_\theta M$)

Ex



$M = P_1$

At $\theta \neq 0$ $\quad \mathcal{W}_\theta = \langle P_1 \rangle \quad \mathrm{supp}_\theta M = \{P_1\} \quad \Theta_M \cap H = \Theta_M$

$H = \ker\langle -, [P_1] \rangle = \ker\langle -, [S_1] + [S_2] \rangle$

$= \mathbb{R} \cdot ([P_1] - [P_2])$

At $\theta' = 0 \quad \mathcal{W}_{\theta'} = \langle P_1, S_1, S_2 \rangle \quad \mathrm{supp}_\theta M = \{S_1, S_2, P_1\}$

$H = 0 \qquad \Theta_M \cap H = \{0\}$

Cor $\dim \textcircled{n}_M = n-1 \iff \exists\, \theta \in \textcircled{n}_M$ s.t. $\forall\, X, X' \in \operatorname{supp}_\theta M$, $[X] \propto [X']$

Prf pick θ generic $\Rightarrow$ $\textcircled{n}_M = \textcircled{n}_M \cap H$ $\Rightarrow$ $\dim H \geq n-1$ $\Leftarrow$ $\dim[\operatorname{supp}_\theta M] = 1$

Cor For any $M \neq 0$, $\exists\, S \in \operatorname{brick} A$ s.t. $\textcircled{n}_M \subset \textcircled{n}_S$, $\underline{\dim \textcircled{n}_S} = n-1$

Prf pick θ generic $\Rightarrow H = \textcircled{n}_M$ $\quad \operatorname{supp}_\theta M \subset \mathcal{W}_\theta$, take $S \in \operatorname{supp}_\theta M$

$$\underline{\Rightarrow \textcircled{n}_M \subset \textcircled{n}_S}$$

Building $\textcircled{n}_M$ from smaller modules:

for $M_1, M_2 \in \operatorname{mod} A$, define $M_1 * M_2 = \left\{ M \;\middle|\; \overset{\exists \text{ ses}}{0 \to M_1 \to M \to M_2 \to 0} \right\}$

Lem Let $M \in \operatorname{mod} A$ s.t. $|\operatorname{supp}(M)| \geq 3$. Then $\textcircled{n}_M$ is the smallest polyhedral cone containing $\displaystyle\bigcup_{\substack{M_1, M_2 \\ M \in M_1 * M_2}} \textcircled{n}_{M_1} \cap \textcircled{n}_{M_2}$

2. Numerical torsion classes + TF equivalence

Each $\theta \in K_0(\text{proj } A)$ defines two torsion pairs in $\text{mod}(A)$

Def Numerical torsion classes

$\overline{\mathcal{T}_\theta} = \{ M \mid \forall\, M \twoheadrightarrow X,\ \theta(X) \geq 0 \}$ $\qquad$ $\overline{\mathcal{F}_\theta} = \{ M \mid \forall\, X \hookrightarrow M,\ \theta(X) \leq 0 \}$

$\quad \cup\!\mid \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup\!\mid$

$\mathcal{T}_\theta = \{ M \mid \forall\, M \twoheadrightarrow X \neq 0,\ \theta(X) > 0 \}$ $\qquad$ $\mathcal{F}_\theta = \{ M \mid \forall\, 0 \neq X \hookrightarrow M,\ \theta(X) < 0 \}$

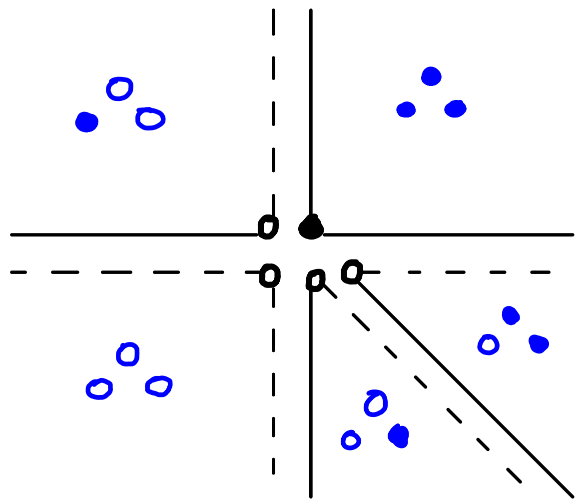
thm (Baumann - Kamnitzer - Tingley) $(\mathcal{T}_\theta, \overline{\mathcal{F}_\theta})$ and $(\overline{\mathcal{T}_\theta}, \mathcal{F}_\theta)$ are torsion pairs

Ex $A = kA_2$

e.g. $S_2 \to P_1 \to S_1$

$\bullet^{\circ}\circ = \langle S_2 \rangle$

$\overline{\mathcal{T}_\theta}$

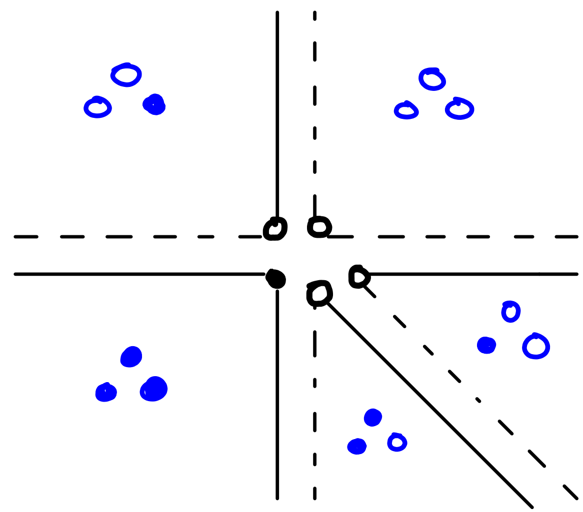


in interior

$\mathcal{T} = \overline{\mathcal{T}}$

$\mathcal{F} = \overline{\mathcal{F}}$

$\overline{\mathcal{F}_\theta}$



Def $\theta, \theta' \in K_0(\operatorname{proj} A)$ are *TF-equivalent* if $\overline{\mathcal{T}}_\theta = \overline{\mathcal{T}}_{\theta'}$ & $\overline{\mathcal{F}}_\theta = \overline{\mathcal{F}}_{\theta'}$

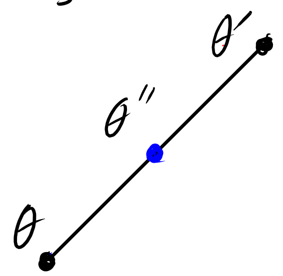
In example above, get 11 TF-equivalence classes

Let us denote by $[\theta]$ the class of θ

Lma $[\theta]$ is convex

$[\theta] = [\theta']$ $\qquad (\mathcal{T}, \mathcal{F})$

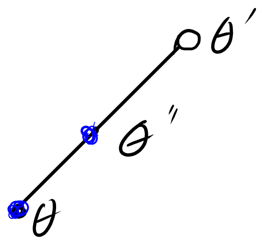
Prf



$$\begin{cases} \overline{\mathcal{T}}_\theta = \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{T}}_{\theta'} \subseteq \overline{\mathcal{T}}_{\theta''} \\ \overline{\mathcal{F}}_\theta = \overline{\mathcal{F}}_\theta \cap \overline{\mathcal{F}}_{\theta'} \subseteq \overline{\mathcal{F}}_{\theta''} \end{cases} \Rightarrow \quad \overline{\mathcal{T}}_\theta = \overline{\mathcal{T}}_{\theta''} \qquad \overline{\mathcal{F}}_\theta = \overline{\mathcal{F}}_{\theta''}$$

Lma $\theta' \in \overline{[\theta]} \iff \begin{cases} \overline{\mathcal{F}}_\theta \subset \overline{\mathcal{F}}_{\theta'} \\ \overline{\mathcal{T}}_\theta \subset \overline{\mathcal{T}}_{\theta'} \end{cases}$ $\qquad \boxed{\Leftarrow}$

Prf



$\theta'' \in [\theta, \theta'[$

$$\overline{\mathcal{T}}_\theta = \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{T}}_\theta \subset \overline{\mathcal{T}}_{\theta''}$$

$$\overline{\mathcal{F}}_\theta = \overline{\mathcal{F}}_\theta \cap \overline{\mathcal{F}}_\theta \subseteq \overline{\mathcal{F}}_\theta \cap \overline{\mathcal{F}}_{\theta'} \subseteq \overline{\mathcal{F}}_{\theta''}$$

$$\Rightarrow (\overline{\mathcal{T}}_{\theta''}, \overline{\mathcal{F}}_{\theta''}) = (\overline{\mathcal{T}}_\theta, \overline{\mathcal{F}}_\theta)$$

$$\Rightarrow [\theta''] = [\theta]$$

Relation between TF & $④_M$

Thm (2.16) Let $\theta \neq \theta'$. TFAE

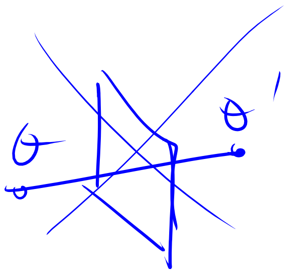
a $[\theta] = [\theta']$

b $[\theta''] = [\theta]$ for all $\theta'' \in [\theta, \theta']$

c $W_{\theta''}$ constant " " "

d $\forall M$, either $[\theta, \theta'] \subset ④_M$ or $[\theta, \theta'] \cap ④_M = \emptyset$

e $\nexists$ brick S s.t. $[\theta, \theta'] \cap ④_S$ has exactly one element



Prf See p. 12

Thus, if we set $\text{Chambers}(A) = \{\text{conn. comp. of } K_0(\text{proj } A) \setminus \overline{\text{Wall}}\}$ then get a map

$$\text{Chambers}(A) \longrightarrow TF_n(A) = \left\{ \begin{array}{l} TF \text{ equiv. classes w/} \\ \text{non-empty interior} \end{array} \right\}$$

dim n

$$C \mapsto TF \text{ equiv. class containing } C$$

Thm This is a bijection, with inverse given by $E \to E^\circ$

Pf $E \subset K_0(\mathrm{proj}\, A)_{\mathbb{R}} \setminus \mathrm{Wall} \Rightarrow E^\circ \subset K_0(\mathrm{proj}\, A)_{\mathbb{R}} \setminus \overline{\mathrm{Wall}}$

$\cap$

$TF_n(A)$ $\quad E$ is convex $\Rightarrow E^\circ$ is connected $\rightsquigarrow C = E^\circ$ is a single chamber

---

Next: want to prove that actually $\forall\, E \in TF_n(A)$, E is open
$\Rightarrow E^\circ = E$ and
Chambers$(A) = TF_n(A)$ (as sets of subsets of $K_0(\mathrm{proj}\, A)$)

To prove this, will need to use Koenig-Yang correspondences

Recall that we have bijections

$$\mathrm{silt}(A) \longrightarrow t\text{-}\mathrm{str}(A) \longrightarrow \mathrm{smc}(A)$$

restricting to $\quad 2\text{-}\mathrm{silt}(A) \longrightarrow \mathrm{int}\text{-}t\text{-}\mathrm{str}(A) \longrightarrow 2\text{-}\mathrm{smc}(A)$

Moreover, if $\{X_i\}$ = image of $T \in 2\text{-}\mathrm{silt}(A)$, can find decomposition
$T = \bigoplus T_i$ s.t. $\langle T_i, X_j \rangle = \begin{cases} \dim_K \mathrm{End}(X_j) & i = j \\ 0 & i \neq j \end{cases}$

Ex. Let's calculate $(\bar{T}, \mathcal{F})$, $(T, \bar{\mathcal{F}})$ at points 1, 2, 3

1. $\bar{T}$ $\quad\quad$ $\bar{\mathcal{F}}$

   T $\quad\quad$ $\mathcal{F}$

2. $\bar{T}$ $\quad\quad$ $\bar{\mathcal{F}}$

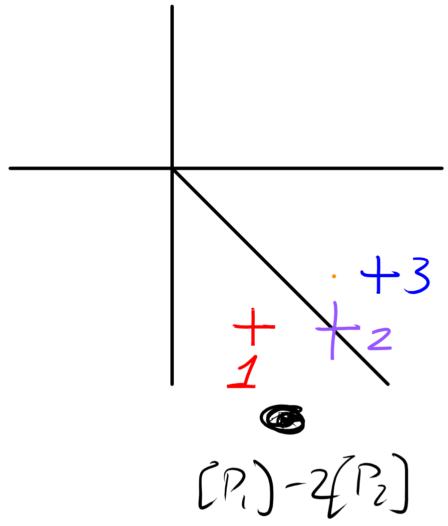
   T $\quad\quad$ $\mathcal{F}$

3. $\bar{T} = T$ $\quad\quad$ $\bar{\mathcal{F}} = \mathcal{F} =$

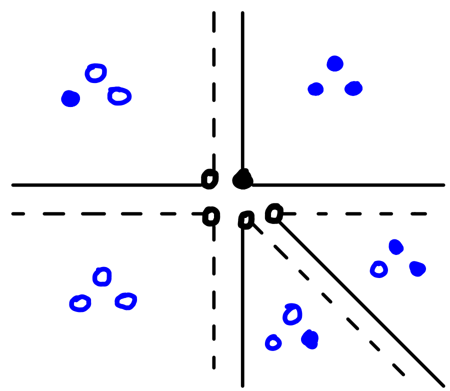
4. $\theta = t_1$ $\quad (\bar{T}, \mathcal{F}) = (T, \bar{\mathcal{F}}) \Longleftrightarrow U = S_2[1] \oplus S_1$ tilting object

$$[U] = -[P_2] + [P_1] - [P_2] = [P_1] - 2[P_2]$$

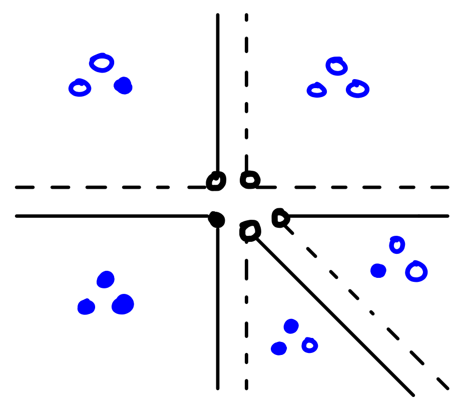
5. $\theta = t_2$ $\quad$ Can find



$\bar{T}_\theta$



$\bar{\mathcal{F}}_\theta$



We can then formalize these observations in a general case

Def Given $U \in K^b(\operatorname{proj} A)$, $U = \bigoplus_i U_i$, U_i indecomp., define **cones** $\subset K_0(\operatorname{proj} A)_{\mathbb{R}}$

$$C(U) = \left\{ \textstyle\sum_i \alpha_i [U_i] \;\middle|\; \alpha_i \geq 0 \right\}, \qquad C_+(U) = \left\{ \textstyle\sum_i \alpha_i [U_i] \;\middle|\; \alpha_i > 0 \right\}$$

Will look at the cases $U \in$ 2-(pre)silt(A)

Properties (using 2-term condition)

- $U \in$ 2-presilt(A) $\Rightarrow$ $[U_i]$ all lin. indep $\Rightarrow$ $C_+(U) \subset C(U)$ is relative interior
- $U \in$ 2-silt(A) $\Rightarrow$ exactly n summands $\Rightarrow$ $\dim C(U) = n$, n walls $C(T/T_i)$
- $U, U' \in$ 2-silt(A), $U \not\cong U'$ $\Rightarrow$ $C_+(U) \cap C(U') = C(U) \cap C_+(U') = \emptyset$

$U \in$ 2-presilt(A), let ν denote the Nakayama functor.

$$\overline{\mathcal{T}}_U = {}^{\perp}H^{-1}(\nu U), \qquad \overline{\mathcal{F}}_U = H^0(U)^{\perp}$$

Thm (follows from Adachi-Iyama-Reiten)

i. $\exists\ \mathcal{T}_U, \mathcal{F}_U$ s.t. $(\overline{\mathcal{T}}_U, \mathcal{F}_U)$ & $(\mathcal{T}_U, \overline{\mathcal{F}}_U)$ are torsion pairs

ii. $\exists!\ T \in$ 2-silt(A) s.t. $U \in \operatorname{add} T$, $\overline{\mathcal{T}}_U = \overline{\mathcal{T}}_T$ (T = "Bongartz completion")

$\exists!\ T' \in$ 2-silt(A) s.t. $U \in \operatorname{add} T'$, $\overline{\mathcal{F}}_U = \overline{\mathcal{F}}_{T'}$ (T' = "~~co~~ Bongartz co-completion")

Thm (3.11) $U \in$ 2-presilt$(A) \Rightarrow C_+(U)$ is a TF equivalence class.

Prf suffices to show $\theta \in C_+(U) \Longleftrightarrow$ $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_U, \mathcal{F}_U)$, $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_U, \overline{\mathcal{F}}_U)$

$\boxed{\Rightarrow}$ see Yurikusa

$\boxed{\Leftarrow}$ take Bongartz completion $T = \bigoplus T_i$ of U

First $\theta \in C(T)$ $\quad \theta = \sum a_i [T_i] \quad a_i \in \mathbb{R}$

Pick $\{X_i\} \in$ 2-smc(A) dual $\Rightarrow \theta(X_i) = a_i \dim_k \operatorname{End}(X_i)$

(Brüstle-Yang) $X_i \in \mathcal{X}$, $\mathcal{X} \in$ 2-smc$(A) \Rightarrow$ either $X_i \in \operatorname{mod}(A)$ or $X_i \in \operatorname{mod}(A)[1]$

$$\left.\begin{array}{l} X_i \in \operatorname{mod} A \Rightarrow X_i \in \overline{\mathcal{T}}_T \Rightarrow \theta(X_i) \geq 0 \\ \in \operatorname{mod} A[1] \Rightarrow X_i \in \mathcal{F}_T[-1] \Rightarrow \theta(X_i) > 0 \end{array}\right\} \Rightarrow a_i \geq 0 \quad \theta \in C(T)$$

$\Rightarrow \theta \in C_+(U)$

It remains to prove that every chamber is a TF equiv. class of dim n

Suffices to find, for every chamber C, a corresponding 2-term silting obj

Lma Let $\theta \in K_0(\operatorname{proj} A)_{\mathbb{Q}}$ s.t. $\mathcal{W}_\theta = \{0\}$ (i.e. $\theta \notin$ Wall)

Then $\exists$ $T \in 2\text{-silt}(A)$ s.t. $\theta \in C_+(T)$

Prf proving corresponding t-structure is finite-heart

(Bridgeland)

Any $X \in \mathcal{H}$ $\quad X_F \to X \to X_T$ where $X_F \in \mathcal{F}_\theta[1]$

$X_T \in \mathcal{T}_\theta$

$\Rightarrow \theta(X) > 0$

But $\theta \in K_0(\operatorname{proj} A)_{\mathbb{Q}}$ so can find $q \in \mathbb{Q}$ s.t. $\theta(X) \geq q$

$\Rightarrow$ every comp. series terminates $\Rightarrow \mathcal{H}$ length

$\Rightarrow$ t-str. is fin-heart $\overset{KY}{\Rightarrow} \exists\, U \in 2\text{-silt}(A)$

Wall-Chamber Structure on Real Grothendieck groups, part II

arXiv: 1905.02180

0. Recap

A = f.d. algebra/k, $n = rk(K_0(\mathrm{mod} A))$

$\langle , \rangle : K_0(\mathrm{proj} A) \times K_0(\mathrm{mod} A) \longrightarrow \mathbb{Z}$

Dual bases $\{P_i\}$, $\{S_i\}$, $P_i \in \mathrm{proj} A$, $S_i \in \mathrm{mod} A$

$\theta \in K_0(\mathrm{proj} A)_{\mathbb{R}} \rightsquigarrow \mathcal{W}_\theta \subseteq \mathrm{mod} A$ subcat. of θ-semistable modules

$M \in \mathrm{mod} A \rightsquigarrow$ wall $\Theta_M = \{\theta \mid M \in \mathcal{W}_\theta\} \subset K_0(\mathrm{proj} A)$ polyhedral cone.

Wall = $\bigcup_M \Theta_M$

Chambers(A) = $\{$conn. comp's of $K_0(\mathrm{proj} A)_{\mathbb{R}} \setminus \mathrm{Wall}\}$

Thm Chambers(A) $\overset{I, II}{=}$ $TF_n(A) = \{TF$ equivalence classes of dim $n\}$

$\simeq \updownarrow I$

$2\text{-silt}(A) = \{2\text{-term silting objects in } K^b(\mathrm{proj} A)\}$

TF equivalence $\quad \theta \rightsquigarrow$ two torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$, $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$

$\theta \underset{TF}{\sim} \theta'$ if $\overline{\mathcal{T}}_\theta = \overline{\mathcal{T}}_{\theta'}$ and $\overline{\mathcal{F}}_\theta = \overline{\mathcal{F}}_{\theta'}$

dim $n \Rightarrow \mathcal{T} = \overline{\mathcal{T}}$, $\mathcal{F} = \overline{\mathcal{F}}$

2-term silting objs $\quad$ 2-term $U \in 2\text{-presilt}(A)$ if $\mathrm{Hom}(U, U[>0]) = 0$

$T \in 2\text{-silt}(A)$ if $\begin{cases} \mathrm{Hom}(T, T[>0]) = 0 \\ \mathrm{thick}(T) = K^b(\mathrm{proj} A) \end{cases}$

$\Leftrightarrow |T| = n$

$2\text{-presilt}(A) \longrightarrow TF(A)$

$U \in 2\text{-presilt}(A)$, $U = \bigoplus_i U_i$

$C(U) = \{\sum a_i [U_i] \mid a_i \geq 0\} \qquad C^+(U) = \{\sum a_i [U_i] \mid a_i > 0\}$

conical subsets of $K_0(\mathrm{proj} A)_{\mathbb{R}}$ $\qquad$ $C^+(U)$ rel. interior of $C(U)$

Thm Let $U \in$ 2-presilt(A)

a. $C^+(U)$ is a TF equiv. class

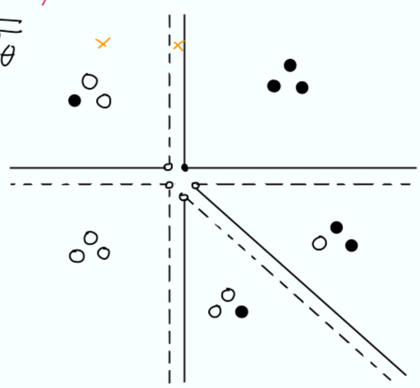
b. U is silting $\iff \dim C^+(U) = n \iff \mathcal{T}_U = \overline{\mathcal{T}}_U,\ \mathcal{F}_U = \overline{\mathcal{F}}_U$

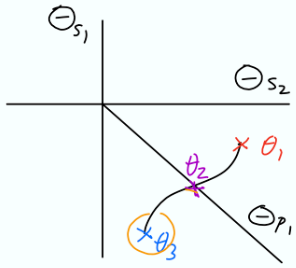
c. Restricts to bijection 2-silt$(A) \xleftrightarrow{\simeq} TF_n(A)$

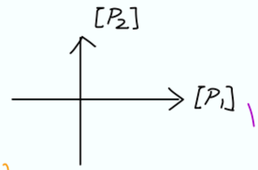
d. $\exists$ unique $\underset{\text{2-silt}(A)}{T, T'}$, $\begin{cases} \overline{\mathcal{T}}_T = \overline{\mathcal{T}}_U \\ \overline{\mathcal{F}}_{T'} = \overline{\mathcal{F}}_U \end{cases}$, $\begin{matrix} C^+(U) \subseteq C(T) \\ C^+(U) \subseteq C(T') \end{matrix}$ Bongartz completion / co-completion

Example $A: kA_2 = k(1 \to 2)$

$\overline{\mathcal{T}}_\theta$





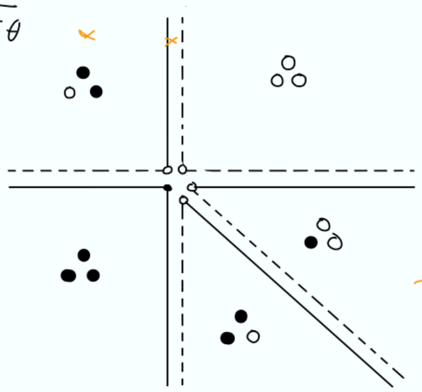


$N(\theta_2) \simeq$

AR quiver

$\cdots \to P_1[-1] \to S_1[-1] \to S_2 \to P_1 \to S_1 \to S_2[1] \to P_1[1] \to \cdots$

$\overline{\mathcal{F}}_\theta$



$\theta_1 \longleftrightarrow (\overset{-1}{P_2} \longrightarrow \overset{0}{P_1 \oplus P_1}) \simeq S_1 \oplus P_1$ — $T =$ compl.

$\theta_2 \longleftrightarrow (P_2 \longrightarrow P_1) \simeq S_1$ — U

$\theta_3 \longleftrightarrow (P_2 \oplus P_2 \longrightarrow P_1) \simeq S_1 \oplus S_2[1]$ — $T' =$ co-compl.

$\uparrow$ TF equiv. $\quad \uparrow$ 2-presilt$_A$

1. Reduction (§4 in paper)

Def $U \in$ 2-presilt(A), define subsets

$$2\text{-(pre)silt}_U(A) = \{ T \in 2\text{-(pre)silt}(A) \mid U \in \operatorname{add} T \} \subset 2\text{-presilt}(A)$$

2-silt$_U(A)$ is silting objects "in the vicinity" of U

Fix $U \in$ 2-presilt(A), $T =$ Bongartz co-completion * TYPO "T= completion" ??

Define algebra $B := \operatorname{End}_{K^b(\text{proj} A)}(T) / [T \twoheadrightarrow U \hookrightarrow T]$

Define functor $\text{red}: K^b(\text{proj } A) \longrightarrow K^b(\text{proj } B)$

$$\text{red}(X) = \operatorname{Hom}_{K^b(\text{proj} A)}(T, X) / [U]$$

where $[U]$ = ideal of morphisms factoring thru $T \twoheadrightarrow U$

Thm (Jasso) (4.2)

θ-ss modules for corresponding $\theta \leftrightarrow U$

a red restricts to an equivalence $\varphi: \mathcal{W}_U \longrightarrow \text{mod } B$

Let $U = \oplus_{i=m+1}^{n} T_i$, $T = \oplus_{i=1}^{n} T_i$ and $\{X_i\}$ the corresponding smc

then $\{\varphi(X_i)\}_{i=1}^{m}$ is the set of simples in mod B

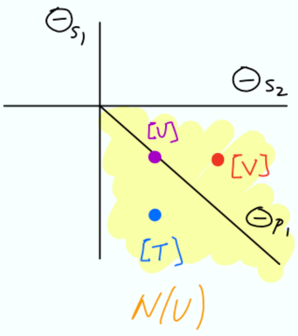
b red restricts to bijections $2\text{-(pre)silt}_U(A) \xrightarrow{\simeq} 2\text{-(pre)silt}(B)$

i.e. all 2-term pre-silting objs "in the vicinity of U" come from B

Def $N(U) = \{ \theta \mid \overline{\mathcal{T}}_\theta \subset \overline{\mathcal{T}}_U \text{ and } \overline{\mathcal{F}}_\theta \subset \overline{\mathcal{F}}_U \ (\Rightarrow \mathcal{W}_\theta \subset \mathcal{W}_U) \}$

Lem $N(U)$ open in $K_0(\text{proj} A)_{\mathbb{R}}$, union of TF equiv. classes, neighborhood of $C^+(U)$

Example $A = kA_2$



$U = (P_2 \to P_1) \simeq S_1$

$T = (P_2 \oplus P_2 \to P_1) \simeq S_1 \oplus S_2[1]$

$V = (P_2 \to P_1 \oplus P_1) \simeq S_1 \oplus P_1$

$red = \mathrm{Hom}_{K^b(\mathrm{proj} A)}(T, -)/[U]$

B

$red(T) = \mathrm{Hom}(S_1 \oplus S_2[1], S_1 \oplus S_2[1])/[S_1] = k \ni id_{S_2[1]}$

$red(U) = \mathrm{Hom}(S_1 \oplus S_2[1], S_1)/[S_1] = 0$

$red(V) = \mathrm{Hom}(S_1 \oplus S_2[1], S_1 \oplus P_1)/[S_1] = k[1]$

$S_2[1] \longrightarrow P_1[1]$

$K_0(\mathrm{proj}\, B)_{\mathbb{R}}$ gives a model for wall-chamber in nbhd $N(U)$. More precisely:

$U = \bigoplus_{i=m+1}^{n} T_i$ $\quad \{T_i\} \longleftrightarrow \{X_i\}$, define $S_i^B = \varphi(X_i)$, $i = 1, \dots, m$

$T = \bigoplus_{i=1}^{n} T_i$ $\quad$ smc $\quad$ w/ proj cover $P_i^B \longrightarrow S_i^B$

Define a linear projection $\pi : K_0(\mathrm{proj}\, A)_{\mathbb{R}} \longrightarrow K_0(\mathrm{proj}\, B)_{\mathbb{R}}$

by $\pi(\theta) = \sum_{i=1}^{m} \left(\dfrac{\theta(X_i)}{\dim \mathrm{End}(X_i)}\right)[P_i^B]$ $\quad$ Restrict to surjective

$\pi|_{N(U)} : N_U \twoheadrightarrow K_0(\mathrm{proj}\, B)_{\mathbb{R}}$

Thm (4.5) (Wall & chamber in $N(U)$ $\longleftrightarrow$ W&C for B)

a $\quad \theta, \theta' \in N_U$. $\quad \theta \underset{TF_A}{\sim} \theta' \iff \varphi(\theta) \underset{TF_B}{\sim} \varphi(\theta')$

b $\quad \theta \in N_U$, $M \in \mathcal{W}_U \Rightarrow \Theta_{\varphi(M)} = \pi(\Theta_M \cap N_U)$

c

$$\begin{array}{ccc} 2\text{-presilt}_U A & \xrightarrow{red} & 2\text{-presilt } B \\ \simeq \downarrow & & \downarrow \simeq \\ TF(A)|_{N_U} & \xrightarrow{\pi} & TF(B) \end{array}$$

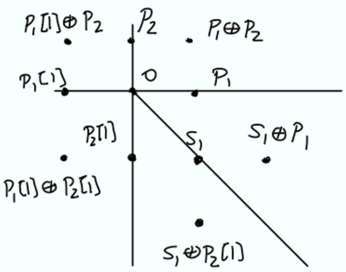
2. τ-tilting finiteness

Thm (Demonet–Iyama–Jasso) TFAE

- $|\{\text{isom. classes of basic } \tau\text{-tilting } A\text{-modules}\}| < \infty$
- $|\{\text{" " " indec. } \tau\text{-rigid "}\}| < \infty$
- $|\{\text{" " " indec 2-term pre-silting objs in } K^b(\text{proj} A)\}| < \infty$
- $|2\text{-silt}(A)| < \infty$

Example / counterexample

$A = kA_2$



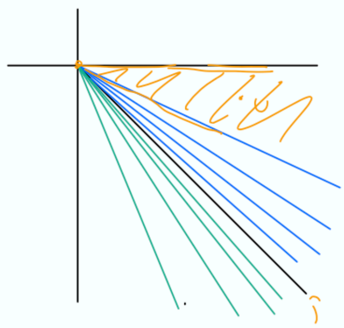
2-presilt(A)

$|2\text{-presilt}(A)| = 11$

$|2\text{-silt}(A)| = 5$

$\bigcup_{U \in 2\text{-silt} A} C(U) = K_0(\text{proj} A)_{\mathbb{R}}$

$A = k(1 \rightrightarrows 2)$



Wall =

$\mathbb{R}[P_1]$

$\mathbb{R}[P_2]$

$\mathbb{R}_{\geq 0}([P_1] - [P_2])$

$\mathbb{R}_{\geq 0}((i+1)[P_1] - i[P_2])$

$\mathbb{R}_{\geq 0}(i[P_1] - (i+1)[P_2])$

$|2\text{-silt}(A)| = \infty$

$\bigcup_{U \in 2\text{-silt}(A)} C(U) \neq K_0(\text{proj} A)_{\mathbb{R}}$

Thm A is τ-tilting finite $\iff K_0(\text{proj} A)_{\mathbb{R}} = \bigcup_{T \in 2\text{-silt}(A)} C(T)$

Prf $\boxed{\Rightarrow}$ (Demonet–Iyama–Jasso)

Let $F = \bigcup_{T \in 2\text{-silt}(A)} C(T)$, $G = \bigcup_{\substack{U \in 2\text{-presilt}(A) \\ |U| = n-2}} C(U) \subset F$ codim 2

$(F \setminus G) \subset K_0(\text{proj} A)_{\mathbb{R}} \setminus G$

- open by general facts
- closed b/c $|2\text{-silt}(A)| < \infty$
- $\neq \emptyset$

$\Rightarrow F = K_0(\text{proj} A)_{\mathbb{R}}$

$\boxed{\Leftarrow}$ (conj. Demonet, proven also by Zimmermann–Zvonareva)

Let $I = \{$isom. classes of indecomposable 2-term presilting objects$\}$

$K_0(\mathrm{proj}\, A)_{\mathbb{R}} = \bigcup_{T \in 2\text{-silt}(A)} C(T) \implies$ also $= \bigcup_{U \in I} N_U$ ← open

Looking at unit sphere $\Sigma \Rightarrow \Sigma = \bigcup_{U \in I} (N_U \cap \Sigma)$

Compact $\Rightarrow$ finite $J \subset I$ suffices ← N_U open!

But $V, U \in I$, $[V] \in N_U \Rightarrow V \simeq U \Rightarrow I$ is finite

---

3. Path algebras

$A = kQ$, $|Q_0| = n$

Recall $\mathrm{Wall} = \bigcup_{\vec{d} \in \mathbb{N}^n} \Theta_{\vec{d}}$, $\quad \Theta_{\vec{d}} = \bigcup_{\dim M = \vec{d}} \Theta_M$

Facts: $\forall \vec{d}$, $\exists M$ s.t. $\Theta_{\vec{d}} = \Theta_M \Rightarrow \Theta_{\vec{d}}$ is rat'l polyhedral cone

---

Recall from part I

Thm (2.8) $M \in \mathrm{mod}\, A$, $|\mathrm{supp}(M)| \geq 3 \Rightarrow \Theta_M$ is smallest polyhedral cone containing $\bigcup_{M \in M_1 * M_2} \Theta_{M_1} \cap \Theta_{M_2}$

$M_1 * M_2 = \{0 \to M_1 \to M \to M_2 \to 0\}$

For $A = kQ$, need to understand $|\mathrm{supp}\, \vec{d}| = 1, 2$

1: easy; 2: Kronecker-type quiv.

Lem

a $\mathrm{supp}\, \vec{d} = \{v\} \Rightarrow \Theta_{\vec{d}} = \bigoplus_{i \neq v} \mathbb{R}[P_i]$

b $\mathrm{supp}\, \vec{d} = \{v, w\} \Rightarrow$ set $a = d_v / \gcd(d_v, d_w)$, $b = d_w / \gcd(d_v, d_w)$

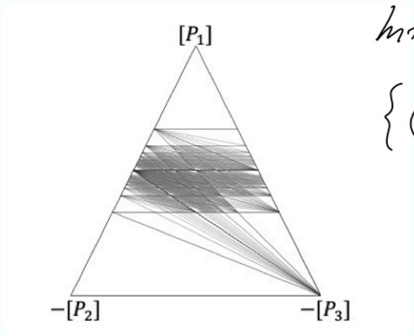
$m = |\{\text{edges } v \to w\}|$ $\quad v \overset{m}{\Rrightarrow} w$

$$\Theta_{\vec{d}} = \begin{cases} \left(\bigoplus_{i \neq v,w} \mathbb{R}[P_i]\right) \oplus \mathbb{R}_{\geq 0}\,(b[P_v] - a[P_w]) & a^2 + b^2 - mab \leq 1 \\ \bigoplus_{i \neq v,w} \mathbb{R}[P_i] & \text{otherwise} \end{cases}$$

c $|\mathrm{supp}\, \vec{d}| \geq 3 \Rightarrow \Theta_{\vec{d}}$ is smallest polyhedral cone $\supseteq \bigcup_{0 < \vec{c} < \vec{d}} (\Theta_{\vec{c}} \cap \Theta_{\vec{d} - \vec{c}})$

Example

$Q = 1 \rightrightarrows 2 \rightarrow 3$



Intersection of Wall with

$\left\{ (a_1, a_2, a_3) \,\middle|\, a_i > 0 \quad a_1 - a_2 - a_3 = 0 \right\}$

(Ex. 5.8 in Asai's paper)