GLUING STABILITY CONDITIONS ON TOPOLOGICAL FUKAYA CATEGORIES

Some results of arXiv:1811.10592 and subsequent developments

OUTLINE

- Inspired by the concept of *Π*-stability in string theory, Bridgeland defined a notion of stability for a general triangulated category T
- The space of (Bridgeland) stability conditions on T denoted $Stab(T)$ naturally carries the structure of a complex manifold
- In principle, $Stab(T)$ could have many components, and interesting topology, but explicit calculations in several examples show that this is not the case
- In general, hard to prove general facts about the whole space $Stab(T)$, easier to construct components
- However in some cases the whole space is known, turns out to be connected and contractible
- Always rely on detailed, specific knowledge of the category T
- Here we will add a class of examples to this list, namely certain Fukaya categories of marked surfaces, related to gentle algebras
- Point of interest: rely on a local-to-global principle, don't need detailed information of T

$$T = \mathrm{colim}_i\, T_i$$

$$\{\text{ Compatible (relative) stability conditions on } T_i \} \leftrightarrow \{\text{ Stability conditions on } T \}$$

BRIDGELAND STABILITY CONDITIONS

Let T = triangulated k-linear category, fixed field k, assume *rank* $K_0(T) < \infty$

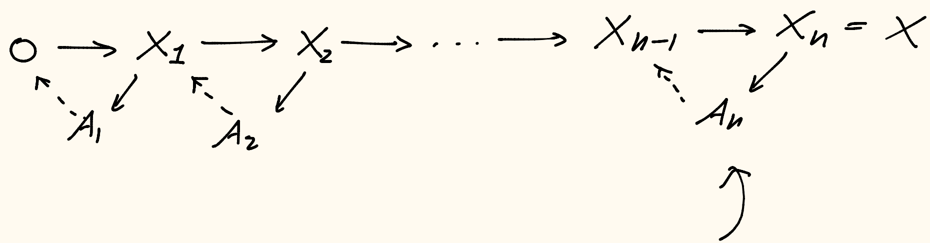
DEF (Bridgeland) A stability condition on T is a pair (Z,P)

Z = central charge $Z : K_0(T) \longrightarrow \mathbb{C}$

P = slicing $\mathcal{P} = \{\mathcal{P}_\phi\}_{\phi \in \mathbb{R}}$, each $\mathcal{P}_\phi$ full subcategory of T of semistable objects of phase ϕ

satisfying the conditions

- $\mathcal{P}_{\phi+1} = \mathcal{P}_\phi[1]$
- $\phi > \psi \Rightarrow \mathrm{Hom}(\mathcal{P}_\phi, \mathcal{P}_\psi) = 0$
- $\forall\, 0 \neq X \in T$,
  $\exists\, \phi_1 > \phi_2 > \cdots > \phi_n$
  $A_i \in \mathcal{P}_{\phi_i}$



(Harder-Narasimhan filtration)

- $0 \neq X \in \mathcal{P}_\phi$
  $\Rightarrow Z(X) = m(X)\, e^{i\pi\phi}, \quad m(X) \in \mathbb{R}_{>0}$

Equivalent to choice of bounded *t*-structure on *T*, together with compatible central charge

heart

$$(Z, \mathcal{P}) \longmapsto (\mathcal{H}, Z_{\mathcal{H}}) \quad \begin{cases} \mathcal{H} = \mathcal{P}(0,1] = \bigcup_{0<\phi\leq 1} \mathcal{P}_\phi \\ Z_{\mathcal{H}} = Z|_{\mathcal{H}} \end{cases}$$

DEF The set of stability conditions on *T* is denoted *Stab(T)*

THM (Bridgeland) The map Stab(*T*) → *Hom*(K_0(*T*), ℂ) defines the structure of a complex manifold on *Stab(T)* and is a local isomorphism of complex varieties

Stab(T) has a wall-and-chamber structure, with two types of walls

- Walls of the first type : $\phi(Z(X_1)) = \phi(Z(X_2))$, some $[X_1] \neq [X_2]$
- Walls of the second type : $Z(X) \in \mathbb{R}$, some X

Walls divide *Stab(T)* into chambers, inside of each chamber the heart *H* is constant

Warning walls are dense in general

BRIDGELAND STABILITY CONDITIONS : EXAMPLE

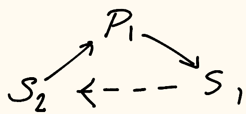
EX.1 $T = D^b(\mathrm{Rep}\, A_2)$

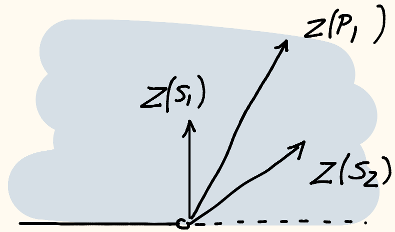
$A_2 = (1 \to 2)$

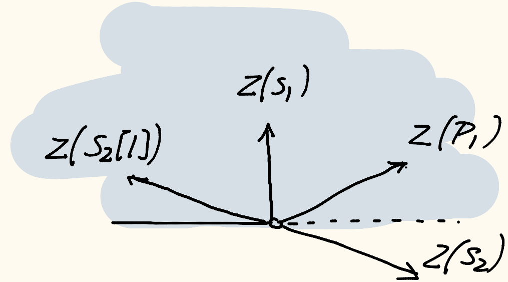
$S_1 = (k \to 0)$

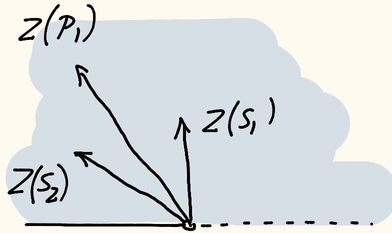
$S_2 = (0 \to k)$

$P_1 = (k \to k)$

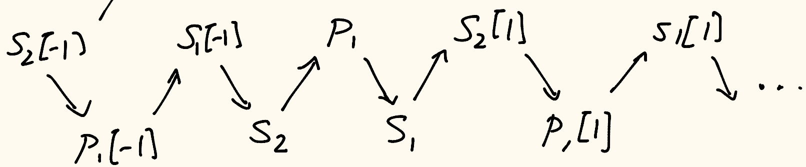


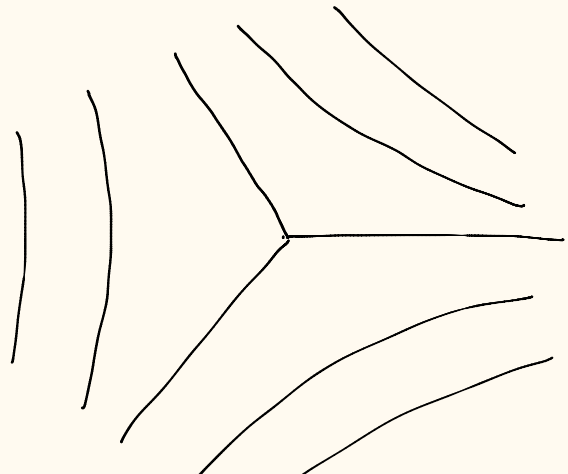






AR quiver





EX.2 $T = D^b(\mathrm{Rep}\, K) \cong D^b(\mathrm{Coh}\, \mathbb{P}^1)$

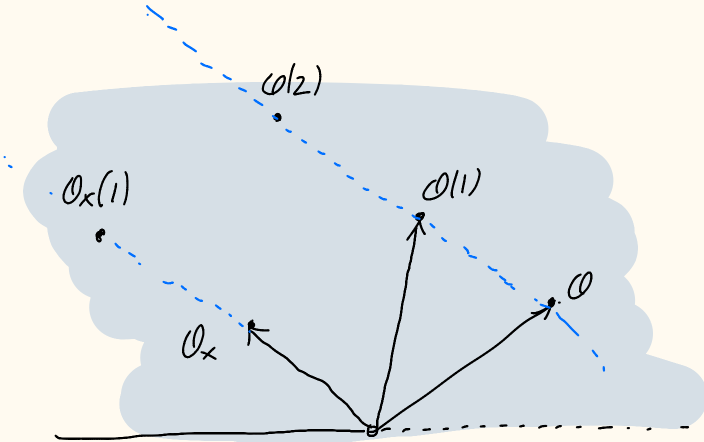
$$K = \left(1 \overset{x}{\underset{y}{\rightrightarrows}} 2\right)$$

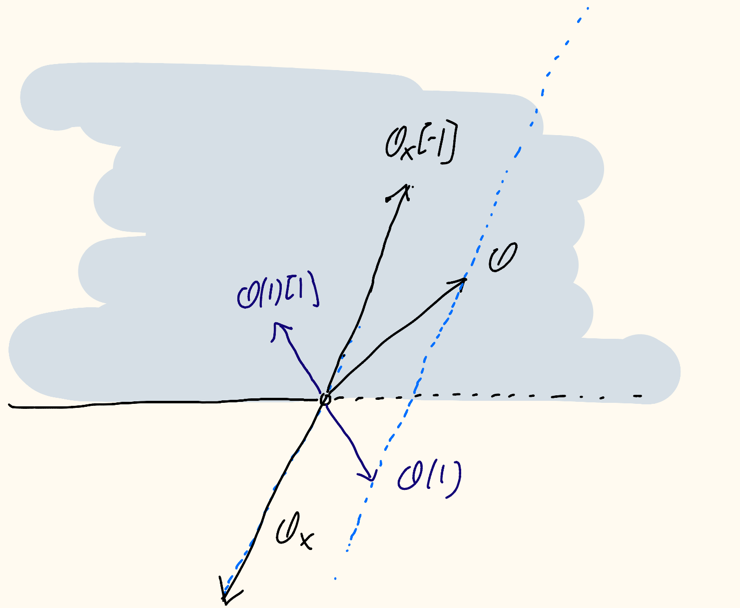
$$(k \rightrightarrows 0) \iff \mathcal{O}$$

$$(0 \rightrightarrows k) \iff \mathcal{O}(1)[-1]$$

$$(k \overset{a}{\underset{b}{\rightrightarrows}} k) \iff \mathcal{O}_x \,,\; x = [a : b]$$

$$\mathcal{O} \xrightarrow{x} \mathcal{O}(1) \longrightarrow \mathcal{O}_x$$



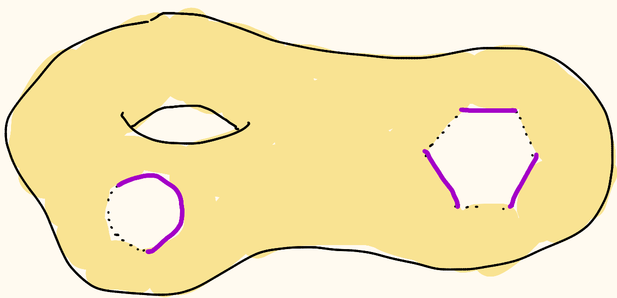


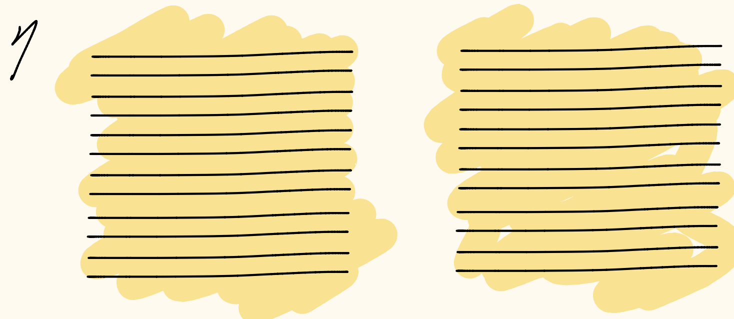
MARKED SURFACES

DEF A graded marked surface is a triple (Σ, M, η) where

- $(\Sigma, \partial\Sigma)$ is a compact surface with boundary
- $M \subseteq \partial\Sigma$ is the marked part of the boundary NOTE unmarked = stops
- $\eta \in \Gamma(\Sigma, \mathbb{P}T\Sigma)$

In this talk: every boundary component has at least one marked and at least one unmarked interval





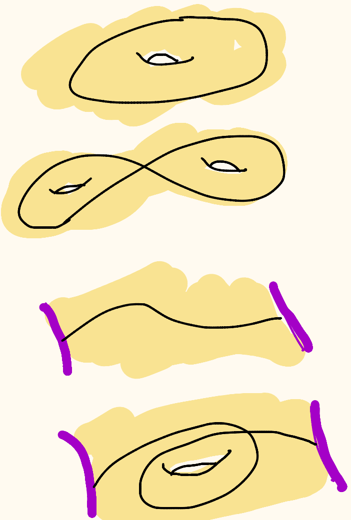
THE FUKAYA CATEGORY OF A MARKED SURFACE

To a (graded marked) surface Σ we associate its (derived) topological Fukaya category $\mathscr{F}(\Sigma)$ (aka partially wrapped Fukaya category)

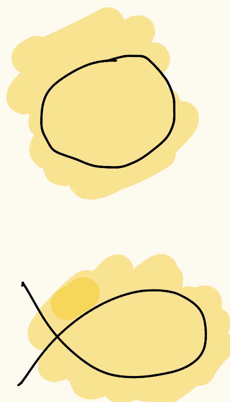
Indecomposable objects = admissible curves + irreducible local system

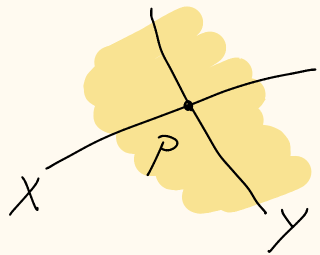
Morphisms = Floer complex of intersections + shared marked boundaries with differential given by counting bigons

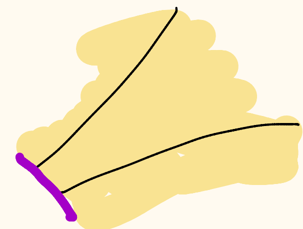
Adm. curves



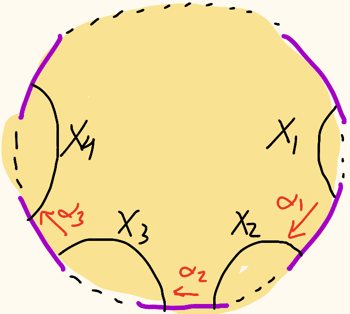
Non-adm.

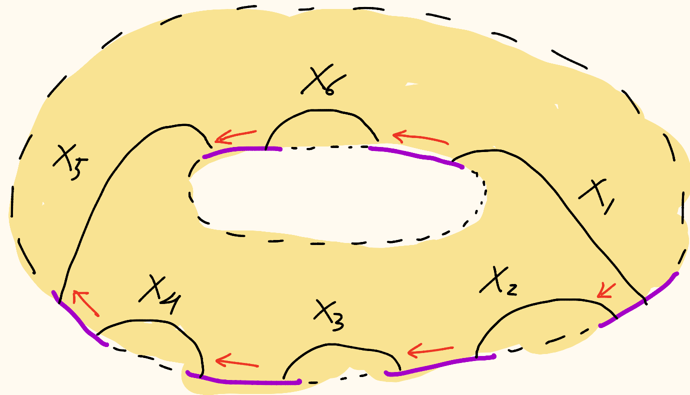






EXAMPLES





$p = 4$

$q = 2$

Choose grading on X_i s.t.

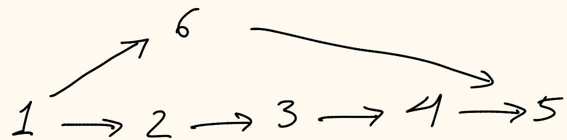
$\deg \alpha_i = 1$

$\mathcal{F}(\Sigma) \xleftrightarrow{\sim} D^b(\operatorname{Rep} A_{n-1})$

$X_i \longleftrightarrow (0 \to \cdots \to \overset{i}{k} \to \cdots \to 0) = S_i$

Annulus grading $= 0 \Rightarrow$ can choose $\deg \alpha_i = 1$

$\mathcal{F}(\Sigma) \longleftrightarrow D^b(\operatorname{Rep} \hat{A}_{p+q-1})$



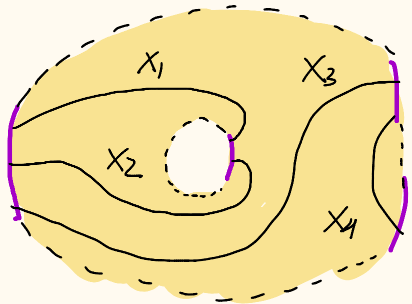
RELATION TO GENTLE ALGEBRAS

DEF (Assem-Skowroński) A gentle algebra $A = kQ/I$ given by a connected quiver Q with relations I such that

- each vertex has at most two outgoing and at most two incoming vertices
- I is generated by paths of length 2
- for each arrow x there is at most one arrow y such that $xy \in I$ and at most one arrow z such that $zx \in I$
- for each arrow x there is at most one arrow y such that $xy \notin I$ and at most one arrow z such that $zx \notin I$

EX

$$1 \underset{d}{\overset{a}{\rightrightarrows}} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \qquad bd = 0 \qquad \Longleftrightarrow$$



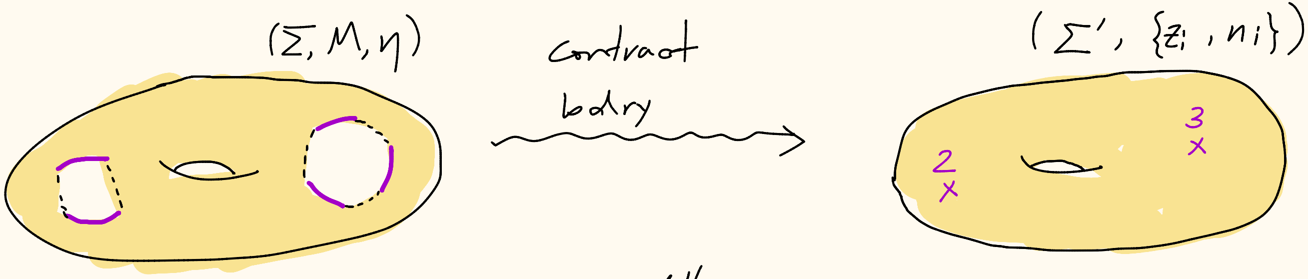
with winding $|d| - |a|$

THM (Opper-Plamondon-Schroll, Lekili-Polishchuk) For any homologically smooth $\mathbb{Z}$-graded gentle algebra A there is a marked surface Σ such that $D(A) \cong \mathcal{F}(\Sigma)$ where $D(A)$ is the perfect derived category of dg A-modules.

Every boundary component has at least one stop/marked part $\Leftrightarrow$ A is homologically smooth and proper

STABILITY CONDITIONS

Haiden, Katzarkov and Kontsevich constructed stability conditions on $\mathscr{F}(\Sigma)$ using quadratic differentials with exponential-type singularities.



$$\mathcal{M}(\Sigma) = \left\{ (X, \varphi) \;\middle|\; \begin{array}{l} \text{Riemann surf } X \overset{\text{diff}}{\cong} \Sigma' \\ \varphi \text{ quadratic differential on } X \end{array} \begin{array}{l} \text{holom. on } X \setminus \{z_i\} \\ \sim \exp\left(\frac{1}{p_i(z)}\right) \cdot p_i \text{ polyn. deg } n_i \\ \quad \text{around } z_i \end{array} \right\}$$

THM (Haiden, Katzarkov, Kontsevich) There is a map of complex manifolds $\mathscr{M}(\Sigma) \to \mathit{Stab}(\mathscr{F}(\Sigma))$ which is moreover an isomorphism of complex manifolds to a union of connected components.

Concretely, given a quadratic differential φ we get a flat metric g with conical singularities at each marked point, and the stability condition is given by

- Central charge $$Z(X) = \int_{supp(X)} \sqrt{\varphi}$$
- Semistable objects = objects represented by geodesics (stable objects: simple geodesics with indecomposable local system)

Semistable objects can be supported on either immersed intervals or embedded circles

Flat metric and exponential-type singularities

$$\varphi \rightsquigarrow \begin{cases} \text{metric } g = |\varphi| \\ \text{horizontal foliation} \end{cases}$$

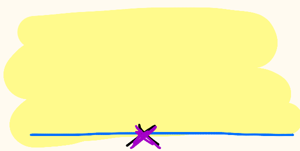
φ holom $\longrightarrow$ g flat

singularities of φ $\longrightarrow$ conical singularities of g

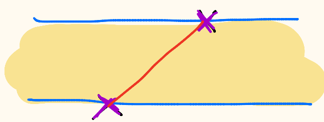
Exp.-type singularities

Decompose Σ into

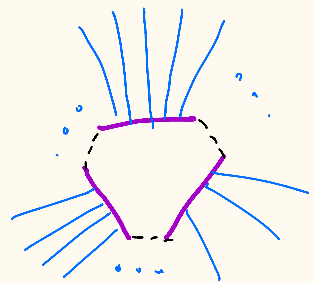
half-planes



fin-height strip



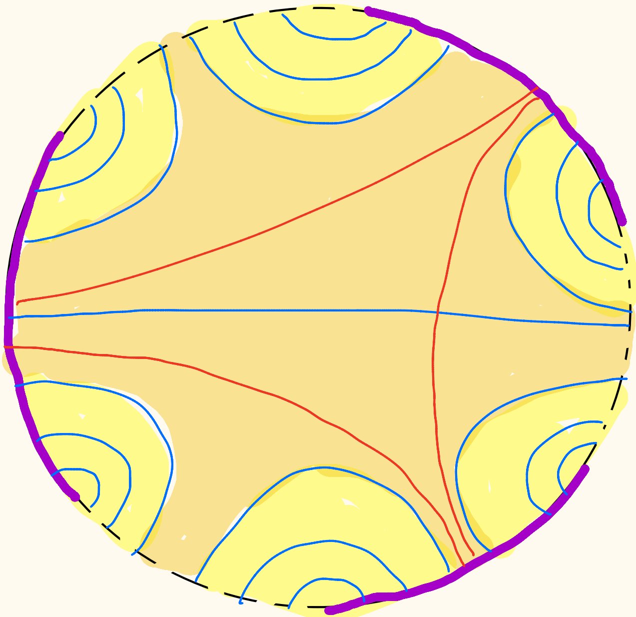
$\exp\left(\frac{1}{z^3}\right)$



"$3\mathbb{Z}$" many half-planes

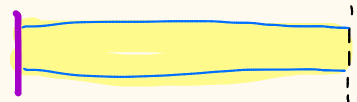
COMBINATORIAL DESCRIPTION OF THE STRIP DECOMPOSITION

EX $\Sigma = \Delta_3$, $\mathscr{F}(\Sigma) = D^b(\mathrm{Rep}\,A_2)$, $\varphi = \exp(z^3 + az + b)dz^2$

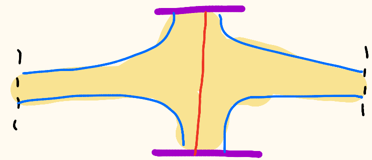


2 types of pieces

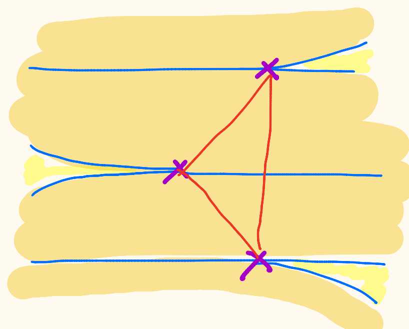
1/2-plane

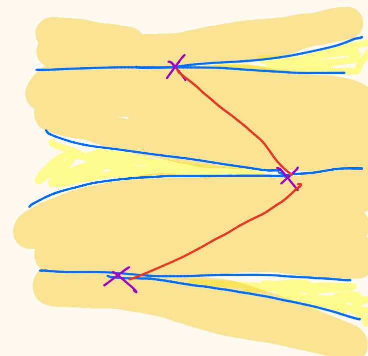


hor. strip



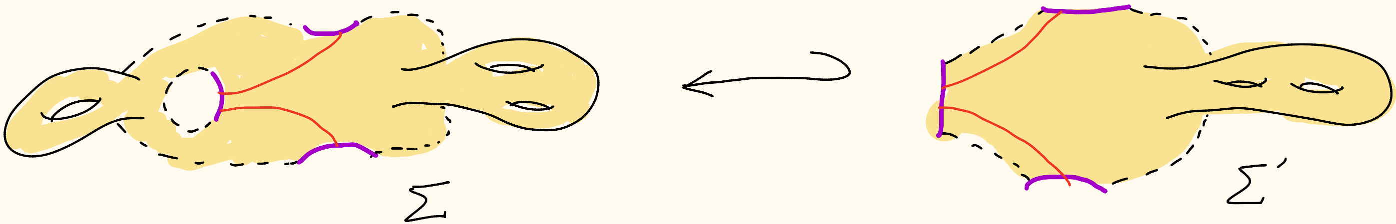
$$\left\{ \begin{matrix} \text{quad. diff} \\ \varphi \end{matrix} \right\} \overset{\sim}{\Longleftrightarrow} \left\{ \begin{matrix} \text{strip decomp.} \\ + \ (Z(X),\ \text{phase}(X)) \end{matrix} \right\}$$





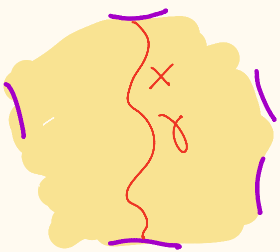
LOCAL-TO-GLOBAL BEHAVIOR

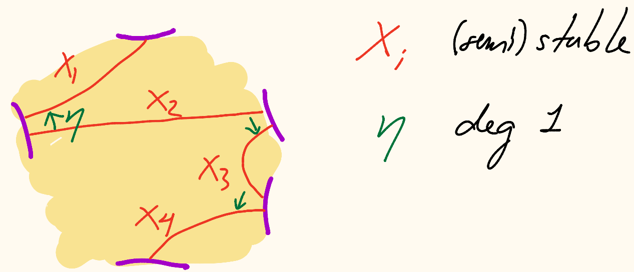
Observation Bridgeland stability conditions in general do not have functoriality properties, but quadratic differentials do.



Question Can we understand this functoriality just in terms of the stability conditions themselves?

Observation Fix an HKK stability condition, and pick and object X supported on embedded interval γ. The HN decomposition of X takes the following form:



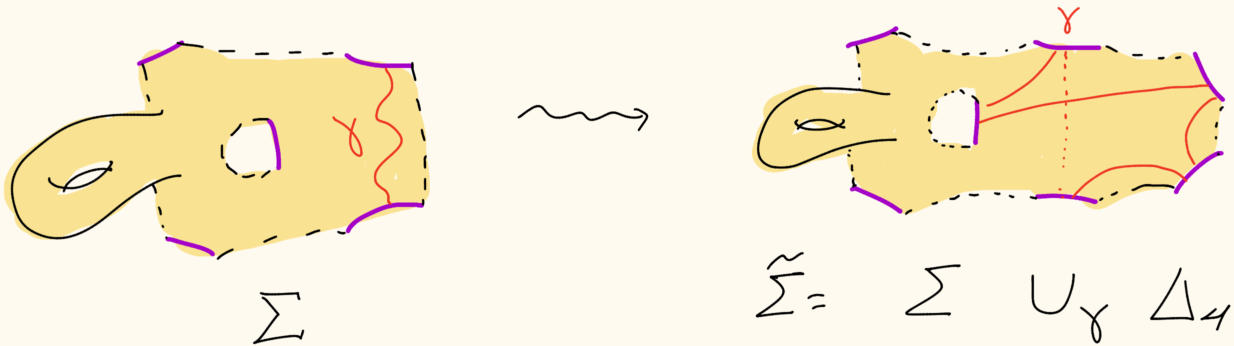


RELATIVE STABILITY CONDITIONS

Let us try to replicate this just in terms of stability conditions

DEF A relative stability condition on (Σ, γ) is the data of:

- A natural number $n \geq 3$
- A stability condition $\sigma \in Stab(\mathscr{F}(\tilde{\Sigma}))$ where $\tilde{\Sigma} = \Sigma \cup_\gamma \Delta_n$

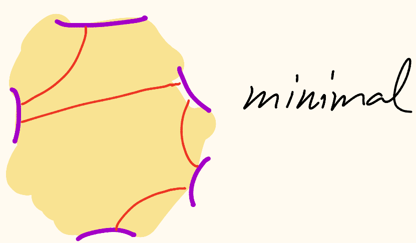


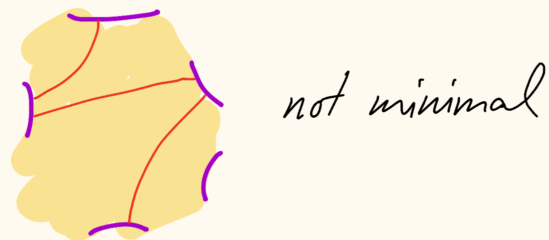
(note that we do not make any assumption on σ being HKK)

PROPERTIES

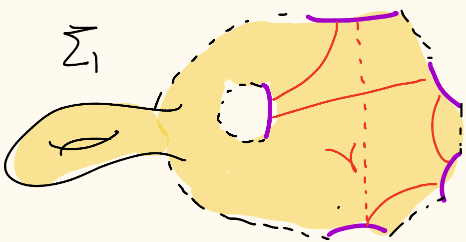
A relative stability condition $\sigma \in Stab(\mathscr{F}(\tilde{\Sigma}))$ restricts to a stability condition on $\mathscr{F}(\Delta_r) = D^b(Rep\, A_{r+1})$, for some $r \geq n$

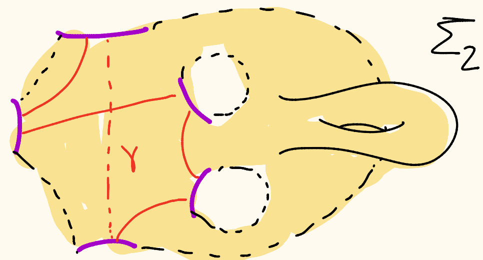
We say such a condition is minimal if the corresponding decomposition of X (object supported on γ) hits all the marked parts of Δ_r





Two minimal relative stability conditions σ_1, σ_2 on $(\Sigma_1, \gamma), (\Sigma_2, \gamma)$ are compatible if they restrict to the same stability condition on $\mathscr{F}(\Delta_r)$





THE SPACE OF RELATIVE STABILITY CONDITIONS

Let us denote $RelStab(\Sigma, \gamma)$ the set of minimal relative stability conditions. There is an identification

$$RelStab(\Sigma, \gamma) = \bigcup_n \ Stab(\mathscr{F}(\Sigma \cup_\gamma \Delta_n))/\sim$$

LEM The space $RelStab(\Sigma, \gamma)$ with the quotient topology is an (infinite-dimensional) Hausdorff space.

For a decomposition $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ are also cutting and gluing maps

$$\begin{aligned} &cut\colon \ Stab(\mathscr{F}(\Sigma)) \to \Gamma \\ &glue\colon \ \Gamma \to Stab(\mathscr{F}(\Sigma)) \qquad \text{where } \Gamma \subseteq RelStab(\Sigma_1, \gamma) \times RelStab(\Sigma_2, \gamma) \end{aligned}$$

which are continuous for that topology.

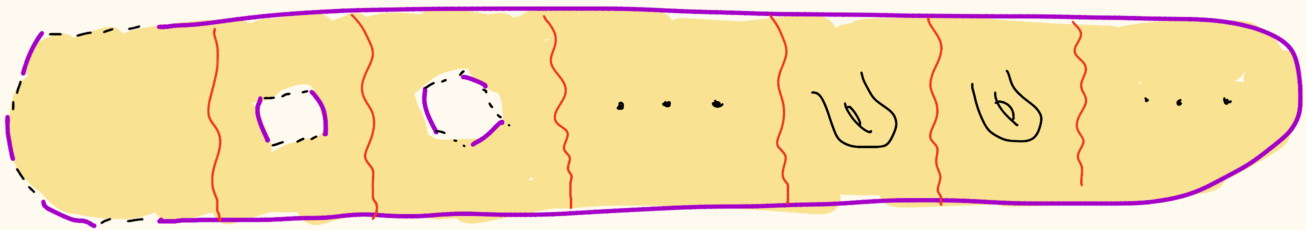
THE LOCAL-TO-GLOBAL PRINCIPLE

The following lemma holds for the entire space *Stab*:

LEM The maps *cut* and *glue* are homeomorphisms.

So if we can cut Σ into smaller pieces and understand the relative stability conditions on those, we understand all of *Stab*($\mathscr{F}$(Σ)).

Now given any surface Σ we consider the following decomposition



There are three base cases to consider: the disk, the annulus and the punctured torus

FINITE-HEART STABILITY CONDITIONS

DEF A stability condition is finite-heart if the corresponding heart is a finite abelian category, ie. every object has finite length and there are finitely many isomorphism classes of simples.

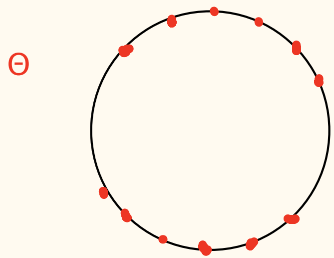
For a given stability condition, set Θ = {phases of semistable objects} $\subseteq S^1$

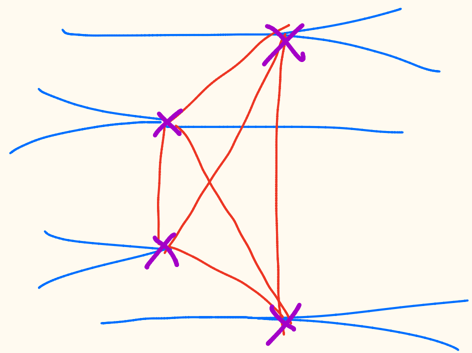
LEM For a category T such that $rank(K_0(T)) < \infty$, and a given stability condition, if Θ has a gap containing phase zero, then the stability condition is finite-heart.

THM (HKK) Any finite-heart stability condition on $\mathscr{F}(\Sigma)$ is HKK. $\Rightarrow$ also any deformation of a finite-heart stab. condition

THE DISK

The category for the disk is $\mathscr{F}(\Delta_i) = D^b(Rep\, A_{i+1})$



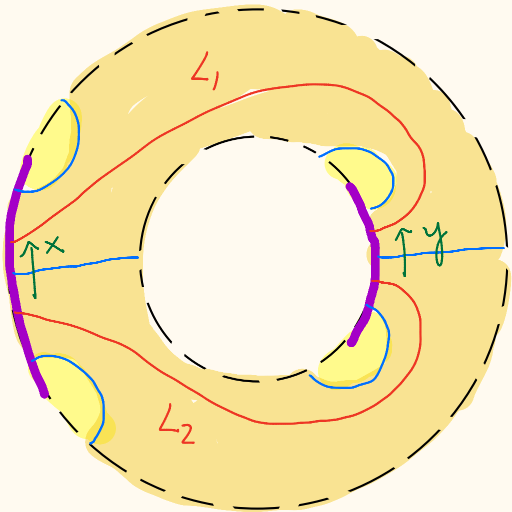


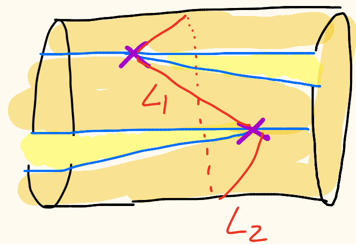
All stability conditions are finite-heart, and Φ is discrete. $\Rightarrow$ is HKK

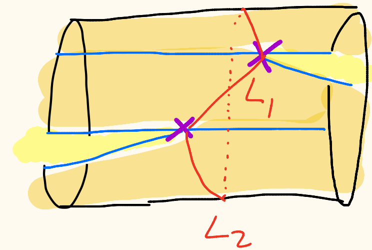
THE ANNULUS

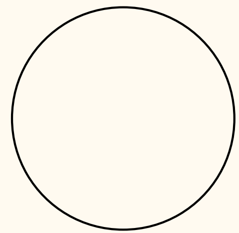
The category for the annulus is $\mathscr{F}(\Delta^*_i) = D^b(Rep\ \hat{A}_{i+1})$.

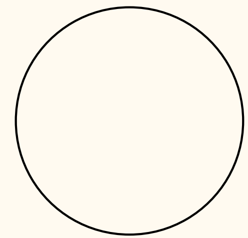
THM (HKK) Any stability condition, possibly after infinitesimal phase rotation, is finite-heart. Θ either is discrete or has two accumulation points. ⇒ is HKK







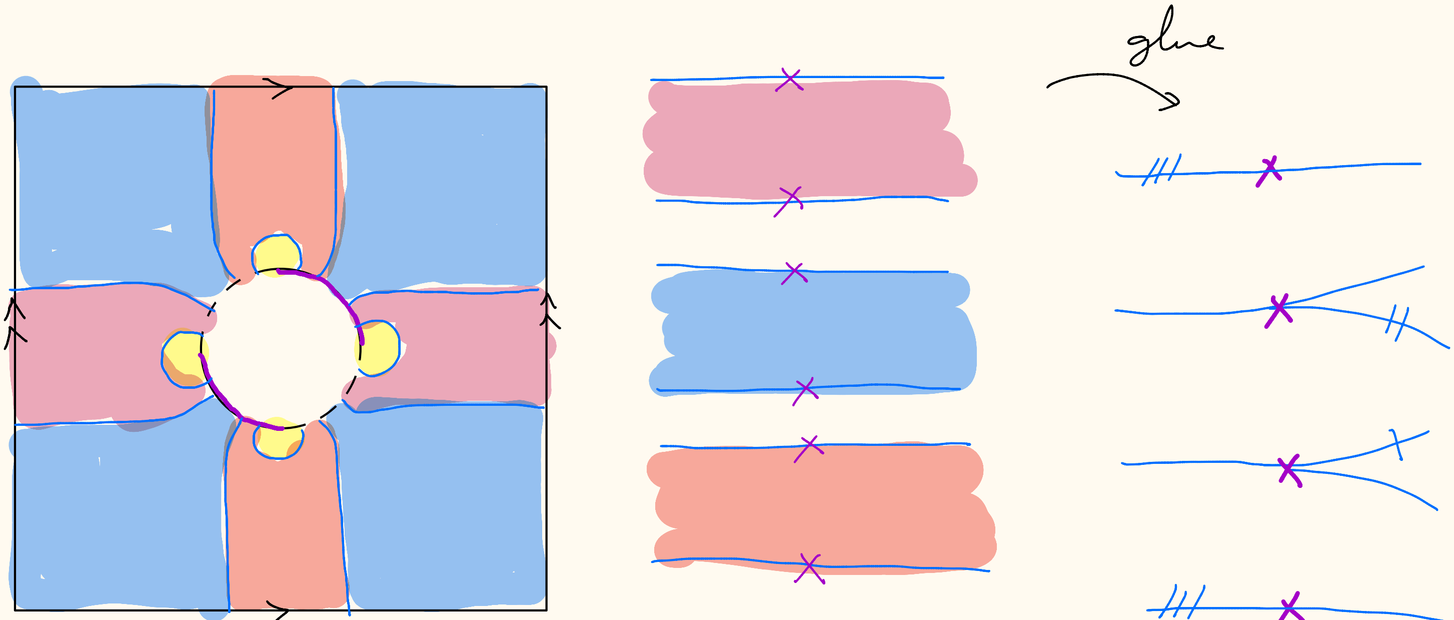


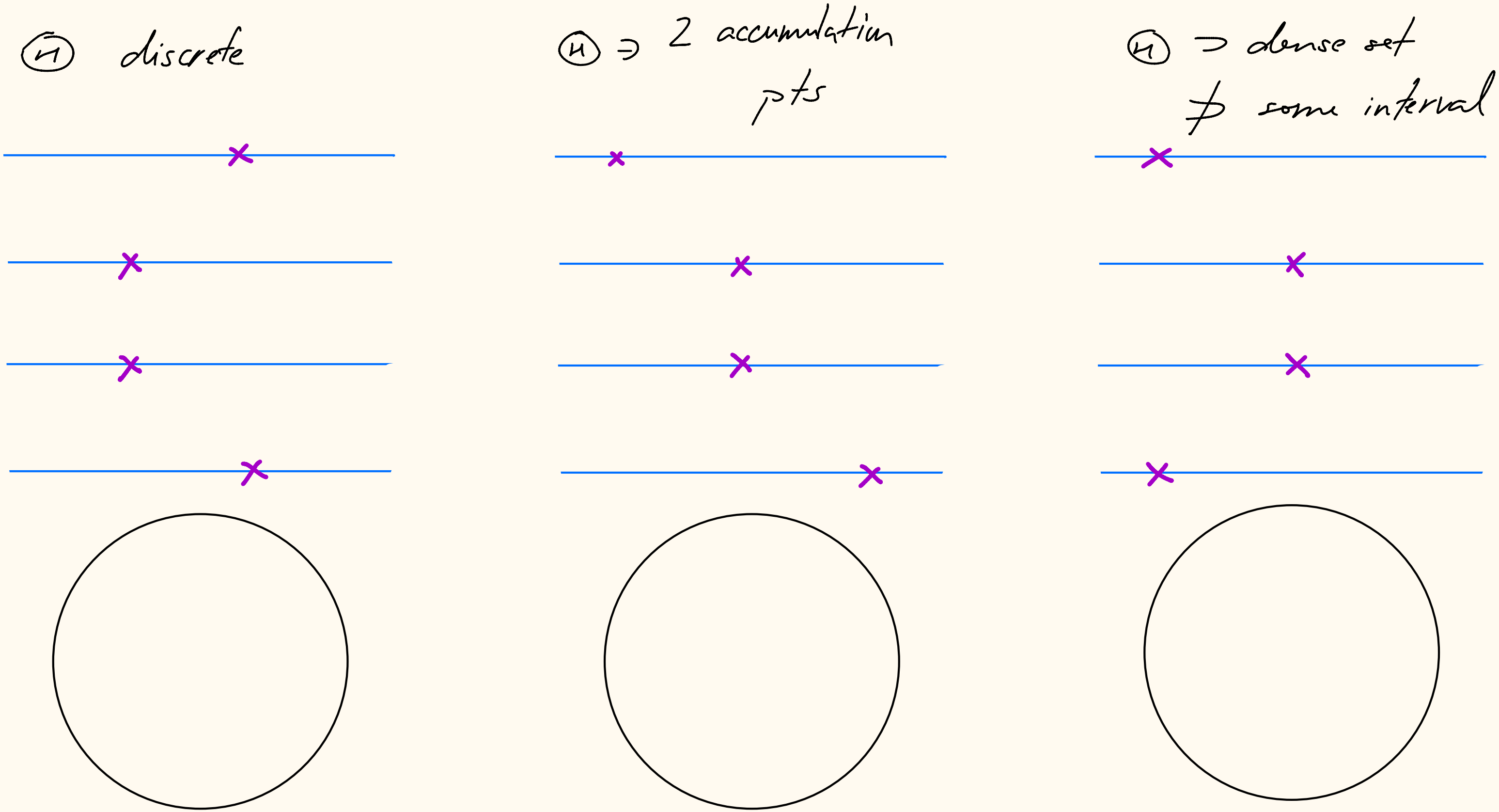


THE PUNCTURED TORUS

LEM Any stability condition σ on $\mathscr{F}(T^*_r)$, possibly (after infinitesimal deformation) and finite phase rotation, is finite-heart.

Θ may be dense, but only on a proper subset of the circle (ie. has a gap) ⇒ σ is HKK





THM For any (fully stopped) surface Σ, every stability condition is HKK.

(STILL) UNWRITTEN WORK

THM* For any fully stopped surface Σ the stability space *Stab*($\mathscr{F}(\Sigma)$) is contractible

$\Rightarrow$ *Stab(D(A))* is contractible for any homologically smooth and proper $\mathbb{Z}$-graded gentle algebra *A*.

Idea of proof:

- every (relative) stability space has a maximally degenerate locus *MaxDeg* (maximal alignment of phases of stable objects), which sits inside the closure of every chamber
- *MaxDeg* itself is generally disconnected, but it has a neighborhood which is connected
- Cutting/gluing/compatibility preserves the maximally degenerate loci