Gluing stability conditions

Or: charts on spaces of holomorphic differentials

Alex A. Takeda

UC Berkeley/MSRI

July 12, 2019

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- In "fully stopped" case, this can show every stability condition is given by quadratic differential

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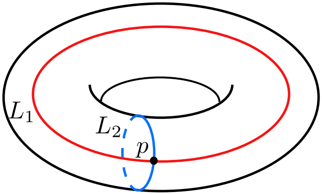

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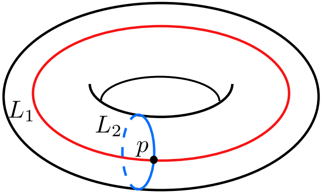

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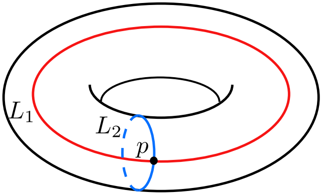

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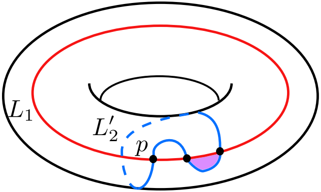

Comes in many flavors: as an A_∞-category, as a triangulated category $\mathcal{D}^\pi(\mathcal{F}(M))$ etc.

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Use counts of holomorphic disks to correct for variation of intersections under Hamiltonian isotopy.

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Theorem (Bridgeland)

The set of stability conditions $\mathrm{Stab}(\mathcal{D})$ *is naturally a complex manifold, and the map* $\mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C})$ *is a local homeomorphism.*

The Thomas-Yau conjecture

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Let (M, J, g, Ω) be a Calabi-Yau m-fold with Calabi-Yau m-form Ω. A Lagrangian L is a *special Lagrangian of phase* ϕ if the phase of Ω is constant on L, ie. $Im(e^{i\pi\phi}\Omega|_L) = 0$.

Consider its triangulated Fukaya category $\mathcal{D}^\pi(\mathcal{F}(M))$.

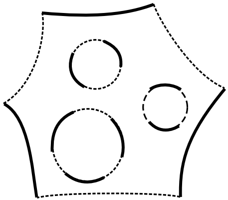
Conjecture (Thomas-Yau, Bridgeland, Smith etc.)

There is a stability condition $(Z, \mathcal{P})$ on $\mathcal{D}^\pi(\mathcal{F}(M))$ such that

- *Central charge is given by $Z : [L] \mapsto \int_L \Omega$*
- *Object represented by L is semistable of phase ϕ if there is a special Lagrangian $L' \sim L$ of phase ϕ.*

"Toy model": Fukaya categories of marked surfaces

Instead of Fukaya category of Calabi-Yau, look at marked surface (Σ, M). **Solid** $=$ marked, dashed $=$ unmarked



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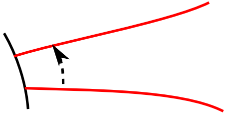

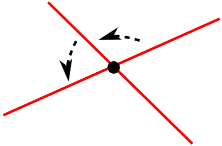

Morphisms have a degree given by relative grading of curves. At intersection $p \in \gamma_1 \cap \gamma_2$, degrees add up to one $i_p(\gamma_1, \gamma_2) + i_p(\gamma_2, \gamma_1) = 1$. Something like a 1D Calabi-Yau category.

Quadratic differentials

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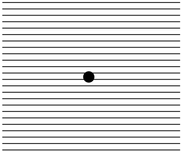

Figure: Regular point $\varphi \sim dz^2$

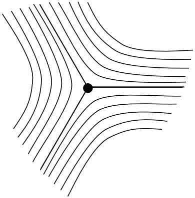

Figure: Simple pole $\varphi \sim zdz^2$

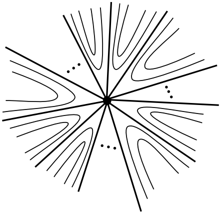

Figure: Exponential-type singularity $\varphi \sim \exp(1/z^3)dz^2$

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If Σ has infinite area, the critical leaves cut Σ into horizontal strips.

Finitely many have finite height, with unique geodesics connecting singularities = saddle connections

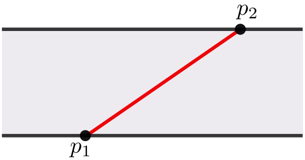
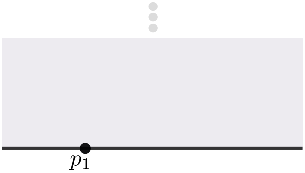


Figure: Horizontal strip of finite height



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Horizontal strip decomposition: examples

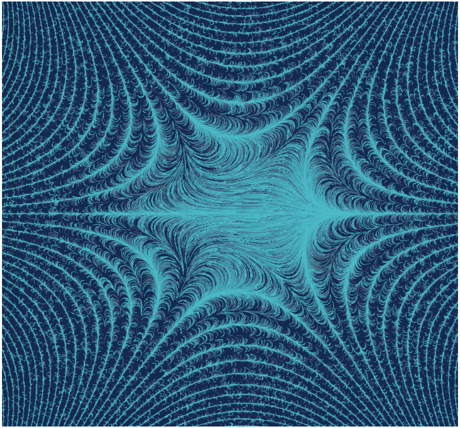

Figure: Horizontal foliation of exponential-type quadratic differential $\exp(z^3)dz^2$ on the complex plane

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- Let $\mathcal{M}(\Sigma)$ be the moduli space of such differentials

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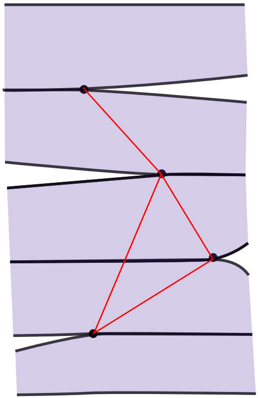

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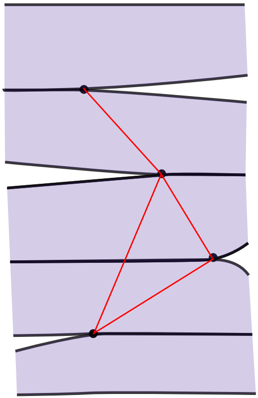

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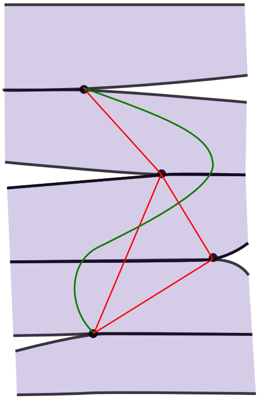

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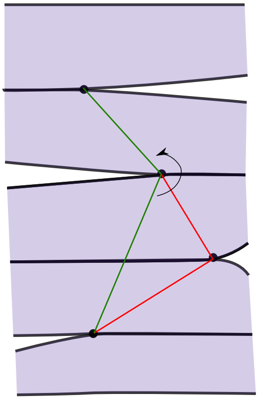

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Theorem (Haiden-Katzarkov-Kontsevich)

Consider $\mathcal{M}(\Sigma)$ the moduli space of such quadratic differentials. Then the map

$$\mathcal{M}(\Sigma) \to \mathsf{Stab}(\mathcal{F}(\Sigma))$$

is a homeomorphism to a union of connected components.

Question: does this cover all the stability conditions on $\mathcal{F}(\Sigma)$? Or are there 'exotic', non-geometric components?

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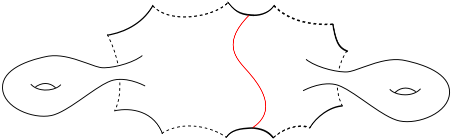

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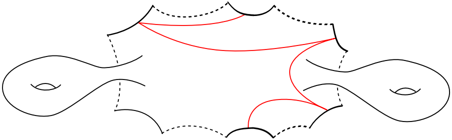

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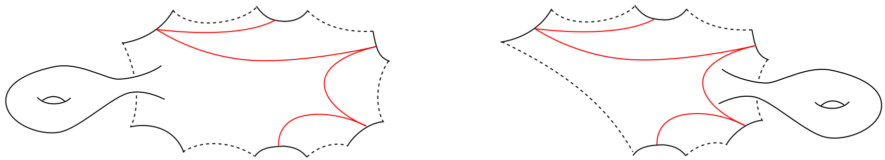

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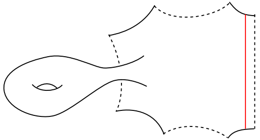

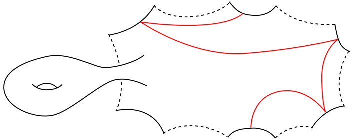

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Definition

A relative quadratic differential on (Σ_L, γ) is a quadratic differential on $\tilde{\Sigma}_L = \Sigma \cup_\gamma \Delta_{n+2}$

Relative spaces

We define the space of relative quadratic differentials

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Filtered by finite parts

$$\mathcal{M}^N(\Sigma_L, \gamma) = \bigsqcup_{n=0}^{n=N} \mathcal{M}(\Sigma_L \cup_\gamma \Delta_{n+2}) \Big/ \sim$$

Cutting and gluing relative quadratic differentials

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and cutting and gluing maps

$$\mathcal{M}(\Sigma) \xrightarrow{cut} \Gamma \xrightarrow{glue} \mathcal{M}(\Sigma)$$

Theorem (T.)

1. *These maps are both homeomorphisms and compose to the identity.*
2. Γ *sits in some finite part of the relative stability spaces, ie.*

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As a corollary: iterating this cutting procedure, for a decomposition of Σ *into simple pieces* Σ_i*, this gives charts on* $\mathcal{M}(\Sigma)$ *in terms of moduli spaces* Σ_i*.*

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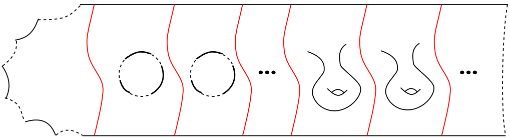

into disks, annuli and punctured tori.

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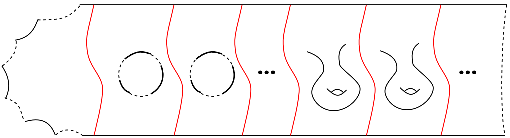
This defines spaces of relative stability conditions $\mathrm{RelStab}(\Sigma, \gamma)$, and cutting/gluing maps.

Take any ('fully stopped') marked surface Σ. One can cut it the following way



into disks, annuli and punctured tori.

Cutting/gluing combined with local computations for these cases gives:

Theorem (T.)

The map $\mathcal{M}(\Sigma) \to \mathrm{Stab}(\mathcal{F}(\Sigma))$ is a homeomorphism, ie. there are no other 'non-geometric' components of $\mathrm{Stab}(\mathcal{F}(\Sigma))$.

Questions

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Thanks!