The ribbon quiver complex and operations on Hochschild invariants

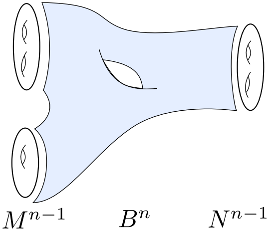
Alex Takeda
joint work with M. Kontsevich and Y. Vlassopoulos

Institut des Hautes Études Scientifiques

9 December 2021

A *topological quantum field theory* (TQFT) is an object of physics that resembles familiar quantum field theories (such as quantum electrodynamics) but whose observables are topological invariants $\implies$ only depend on topology of space-time.

Mathematical formalism = symmetric monoidal functor out of some category of cobordisms into linear category, e.g. vector spaces



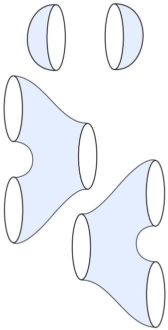
$\xrightarrow{Z} (Z(B) : Z(M) \to Z(M))$
a morphism in the target category (vector spaces etc.)

Add adjectives (oriented, framed TQFT etc.) by requiring corresponding structure on cobordisms.

2d TQFTs

The philosophy of the "cobordism hypothesis" (Baez, Dolan, Lurie et al.) is that each such type of TQFT is characterized by a type of structure on the target object.

Our working example: 2d oriented TQFTs with target in vector spaces. By classification of surfaces, cobordisms generated by



So the vector space A associated to the circle must have maps given by the following diagrams, that is:

- unit $\mathbb{k} \to A$
- trace map $\mathrm{tr} : A \to \mathbb{k}$
- commutative product $A \otimes A \to A$
- cocommutative coproduct $A \to A \otimes A$

satisfying some equations coming from composing surfaces.

So A must have the structure of a finite-dimensional commutative Frobenius algebra (Abrams).

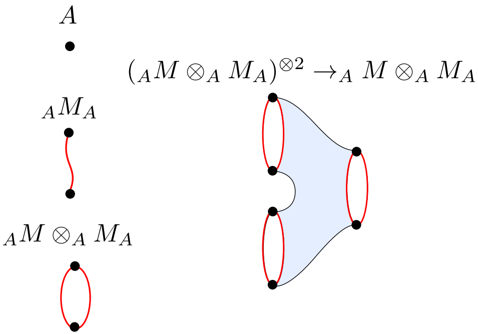
Fully extended oriented 2d TQFTs

More precisely, the cobordism hypothesis is about *fully extended* TQFTs, i.e. functors from higher cobordism categories.

A $(2,1,0)$-TQFT is a symmetric monoidal 2-functor

$$\mathrm{Cob}_2^2 \to \mathrm{Alg}_{\Bbbk}^2$$

to the 2-category of $\Bbbk$-algebras and bimodules.



Fully extended oriented 2d TQFTs

Theorem (Moore,Segal,Lazaroiu et al.)

The data of a $(2,1,0)$-TQFT is equivalent to the structure of a finite-dimensional semisimple Frobenius algebra on the vector space $A = Z(\bullet)$

1D equations imply that $M = A$ is the diagonal bimodule, so associated to the circle we get

$$Z(S^1) = A \otimes_{A\otimes A} A = A/[A, A]$$

with dual space $Z(S^1)^\vee = z(A)$ given by the center of A, which is a commutative Frobenius algebra.

Fully extended oriented 2d TQFTs

Not so interesting; in particular it only sees the set of cobordisms (up to homeomorphism), and nothing about the topology of the *space* of cobordisms.

One could ask about a theory that sees $H^{\neq 0}$ information. For example, Gromov-Witten invariants are about (virtual) fundamental classes of $\overline{\mathcal{M}_{g,n}}$ which are not in H^0.

Two new features: passing from $H^0(\mathcal{M}_{g,n}) \to H^*(\mathcal{M}_{g,n})$ and passing from $\mathcal{M}_{g,n} \to \overline{\mathcal{M}_{g,n}}$.

Cohomological Field Theories

There are some different mathematical formalisms for dealing with all this data; let us first discuss one formalism for non-fully extended case.

Let V be a fin.dim. $\mathbb{Z}$ (or $\mathbb{Z}/2$)-graded vector space, with a distinguished element $1 \in V$ and a symmetric nondegenerate pairing η.

Definition

A cohomological field theory (CohFT) is a collection of classes

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}_{g,n}}) \otimes (V^\vee)^{\otimes n},$$

invariant under the action of the symmetric group and satisfying some relations.

Relations encode the combinatorics of the boundary strata in $\overline{\mathcal{M}_{g,n}}$.

Cohomological Field Theories

Interpretation: V is the vector space associated to the circle, elements $\Omega_{g,n}$ give operations

$$\Omega_{g,n}(\alpha) : V^{\otimes p} \to V^{\otimes n-p}$$

for any homology class $\alpha \in H_*(\overline{\mathcal{M}_{g,n}})$.

Can have many different variants; using $\mathcal{M}_{g,n}$ instead of compactification, or real blowup $\widetilde{\mathcal{M}}_{g,n}$ of $\overline{\mathcal{M}_{g,n}}$, etc.

Basically, this formalism promotes the *non-fully extended* TQFT to a CohFT.

Chain level fully extended CohFTs

There is another approach: construct explicit chain models of $\mathcal{M}_{g,n}$ and explicitly describe the operations each chain represents.

For that, need more structure on V; turns out this structure exists once we require the theory to come from a fully extended theory.

Recall that in the TQFT case, we had $Z(\bullet) = A$ a f.d. semisimple Frobenius algebra and $Z(S^1) = V = A/[A, A]$.

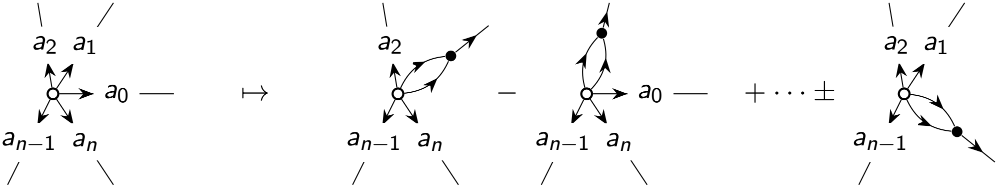
In a CohFT we replace the two-sided tensor of bimodules $A \otimes_{A \otimes A} A = A/[A, A]$ by its derived version, giving an appropriate sequence of groups, of which it is the degree zero part.

Hochschild homology and cohomology

Given an associative algebra A, the derived tensor product $A \otimes^{\mathbb{L}}_{A \otimes A} A$ calculates the *Hochschild homology* $HH_*(A)$. Degree zero part: $HH_0(A) = A/[A, A]$.

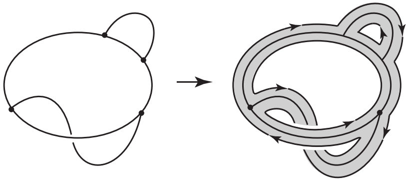
For our purposes, will use explicit definition by the bar complex $C_n(A) = A^{\otimes n+1}$, differential $C_{n+1}(A) \to C_n(A)$ given by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto a_0 a_1 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n + \cdots \pm a_n a_0 \otimes \cdots \otimes a_{n-1}$$



We will assign this complex $C_*(A)$ to the circle.

Since the seminal works of Strebel, Penner, Kontsevich and others, it has been known that there are cell complexes for $\mathcal{M}_{g,n}$ whose cells are labeled by ribbon graphs (or fatgraphs). A ribbon graph should be thought of as a retract of a punctured surface.



Alternatively, it is a graph with a fixed cyclic ordering at each vertex. We can put a metric on these graphs, and there is a natural topology on all metric ribbon graphs

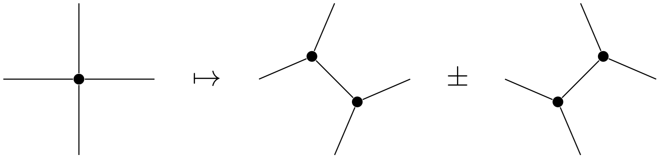
Theorem (Kontsevich, Penner et al.)

There is a homeomorphism (of orbifolds) $\mathcal{M}_{g,n} \times (S^1)^n \to \mathrm{Met}_{g,n}$ to the space of metric ribbon graphs.

Graph complexes

The space $\mathrm{Met}_{g,n}$ is stratified by the topological type of each graph, so we get a cell complex of ribbon graphs $(RG_{g,n}, \partial)$ calculating the homology of $\mathcal{M}_{g,n}$.

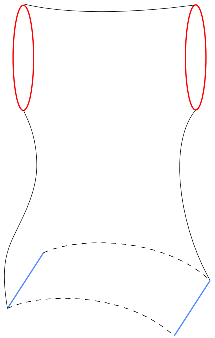
The differential is obtained by summing (with a sign) all the possible *separations* over vertices of valence ≥ 3



Open-closed theories

In a fully extended theory, it is also natural to consider not only *closed* inputs and outputs (here, just the circle) but also *open* ones (here, intervals).

Therefore our theory needs to assign operations for open-closed cobordisms, i.e. surfaces whose boundary is partitioned into five subsets:

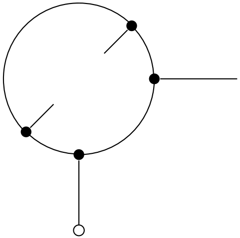


- closed inputs (red circles on left)
- open inputs (blue interval on left)
- closed outputs (red circles on right)
- open outputs (blue interval on right)
- free boundary (dashed)

The complex of “Black-and-white graphs”

There is a cell complex for such moduli spaces, known in the literature as the complex BW of “black-and-white graphs” (Costello, Tradler-Zeinalian, Egas, Wahl-Westerland et al.)

Each such graph is a ribbon graph with two marked subsets of its vertices, black (valence ≥ 3) and white,



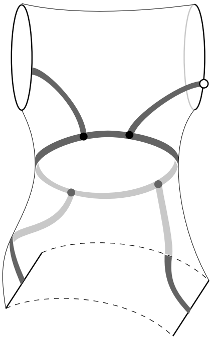
such that (1) every vertex of valence ≥ 2 is marked either black or white, and (2) every white vertex has a distinguished incident half-edge.

The complex of "black-and-white graphs"

Similarly to the Kontsevich graph complex $RG_{g,n}$, this complex has a natural grading and a differential ∂, and $H_*(BW, \partial)$ computes (rationally) the homology of the spaces of open-closed cobordisms.

For each arrangement of open/closed in/out boundaries, we list which graphs can appear:

- Unlabeled leaf, on boundary with more than one leaf: open in or out
- Unlabeled leaf, on boundary with one leaf: open in or out, or closed in
- White vertex = closed out



Subcomplex: A-infinity operad

To each such graph Γ, we would like to associate an operation

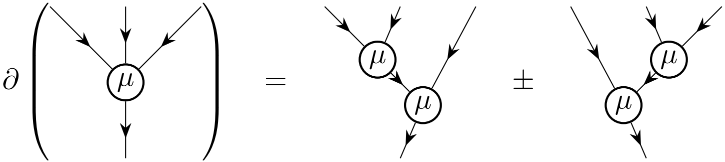
$$\Gamma : A^{\otimes n_1} \otimes C_*(A)^{\otimes m_1} \to A^{\otimes n_2} \otimes C_*(A)^{\otimes m_2}$$

We need some structure on the graded vector space A. For example, let us consider only genus zero graphs with open inputs and one output.

To produce an operation, we orient the edges according to in/out. For vertex with n incoming and one outgoing, we pick a map $\mu^n : A^{\otimes n} \to A$.

Subcomplex: A-infinity operad

The relations that must be satisfied by the μ^n maps, and their degrees, can be deduced from requirement that the operation given by Γ is compatible with the differential



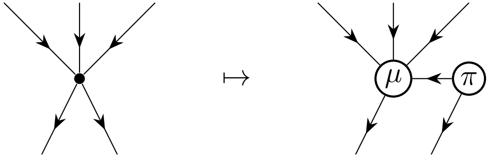
Degree of $\mu^n = 2 - n$, and for each n we have the relation

$$\sum_{r+s-1=n} \pm\mu^r(\dots, \mu^s(\dots), \dots) = 0$$

This defines the structure of an A_∞-algebra on A. Note that $(\mu^1)^2 = 0$ gives a differential on A.

Cyclic Calabi-Yau structures

For more general graphs, pick appropriate orientation of the edges, and end up with vertices with $k > 1$ outgoing edges.



In order to evaluate these, pick bilinear form $\langle , \rangle : A \otimes A \to \Bbbk$, and use its inverse π $k-1$ times. In order for the result not to depend on choices, must have identity

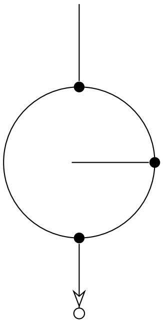
$$\langle a_0, \mu^r(a_1, \ldots, a_r)\rangle = \pm\langle a_1, \mu^r(a_2, \ldots, a_r, a_0)\rangle$$

Definition

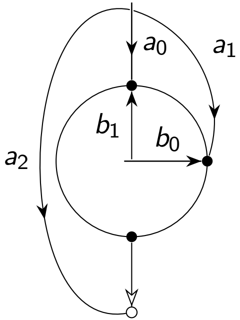
A cyclic Calabi-Yau algebra of dimension d is a finite-dimensional A_∞-algebra with a nondegenerate pairing ω of degree d satisfying the cyclic invariance property above.

Action of black-and-white graphs

To input Hochschild chains, recall that $C_n(A) = A^{\otimes n+1}$. We regard $a_0 \otimes \cdots \otimes a_n$ as n arrows, sending a_0 along the edge, and adding extra arrows for $a_1, \ldots, a_n$, connecting them to vertices *in all possible ways*.



$$(a_0 \otimes a_1 \otimes a_2, b_0 \otimes b_1) \quad = $$



$$+ \ldots$$

We then sum over all these choices. To read the output at a white vertex, we start from the its marked incident edge (white arrow). Note that by definition, we must always have *at least one input*.

Action of black-and-white graphs

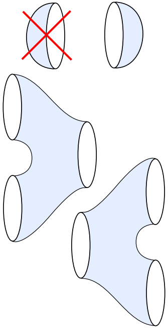
Compatibility of this action with the differential then gives:

Theorem (Tradler-Zeinalian, Wahl-Westeland, Costello et al.)

A cyclic CY algebra A determines a fully-extended open-closed TQFT, which evaluates* $Z(\bullet) = A$ *and* $Z(S^1) = C_*(A)$.

For different d, this corresponds to taking coefficients in $\mathcal{L}^d$ for a certain line bundle $\mathcal{L}$ on the moduli spaces.

This is an incomplete TQFT; it only assigns values to cobordisms with non-empty incoming end. Alternatively, cobordisms generated by index 1 and index 2 handles (but not index zero)



(Proper) Calabi-Yau structures

The notion of cyclic CY is not homotopically invariant; given A_∞-quasi-isomorphism $A' \to A$, $\langle , \rangle$ doesn't pull back to a cyclic CY structure. Instead need the following definition, which uses the notion of cyclic homology

Definition

A (proper) CY structure of dimension d on A is a map

$$\omega : CC_*(A) \to \mathbb{k}[-d]$$

from the cyclic complex, with a nondegeneracy condition.

Cyclic homology is another invariant; should be thought as the homotopy fixed points of an S^1-action on $C_*(A)$, with a canonical map $C_*(A) \to CC_*(A)$.

In characteristic zero, given A with a proper CY structure, can always find a quasi-isomorphic A' with a cyclic CY structure. In particular, $H^*(A)$ needs to be finite-dimensional.

Examples

In algebraic geometry: let X be proper Calabi-Yau variety of dimension d, and take some algebra A such that Mod_A is derived equivalent to $D^b(X)$ (pick $\mathrm{End}(E)$ for some generating object). Then $HH_d(A) = H^0(X, \omega_X) = H^0(X, \mathcal{O}_X) = \mathbb{k}$, and any nonzero map out of this gives a Calabi-Yau structure.

In Floer theory: for a suitable Lagrangian L in a compact symplectic manifold M, its Floer cohomology algebra $CF^*(L, L)$ admits a cyclic CY structure (Fukaya). This can be extended as a proper CY structure to the Fukaya category of M (Abouzaid-Fukaya-Oh-Ohta-Ono, Ganatra et al.)

De Rham complex: let M be an orientable d-manifold and $A = \Omega^*(M)$ its dg algebra of de Rham forms. Seen as an A_∞-algebra with $\mu^{\geq 3} = 0$, the fundamental class of M defines a proper CY structure on A.

Finiteness and dualizability

We ended up with the properness condition on A ultimately because we required a value for the 'cap' (index 2 handle).

There is another finiteness condition one can impose; in our setting it translates to A being *homologically smooth*, i.e. the bimodule ${}_A A_A$ has a finite-length resolution in terms of free bimodules.

If A is smooth, there is a different notion of Calabi-Yau structure that applies.

Definition (Ginzburg, Kontsevich, Soibelman)

A (smooth) CY structure of dimension d on A is a map

$$\omega : \mathbb{k}[d] \to CC_*^-(A)$$

to the negative cyclic complex, with a nondegeneracy condition.

The negative cyclic complex should be thought as the homotopy fixed points of an S^1-action on $C_*(A)$, with a canonical map $CC_*^-(A) \to C_*(A)$.

Finiteness and dualizability

Names come from relations to coherent sheaves: let $D(X)$ be (a dg enhancement) of the bounded derived category of coherent sheaves on X. Then $D(X)$ is proper/smooth if X is proper/smooth.

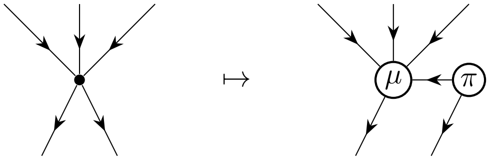
A smooth CY structure is 'dual' to a proper CY structure, and should give rise to a fully extended open-closed TQFT for surfaces generated by index 0 and index 1 handles. This has been long known in some abstract sense, but without an explicit description of operations.

Two questions:

1. Given a smooth CY structure, how to construct TQFT operations?
2. If A is neither smooth nor proper, can we at least construct the operations given by surfaces generated by index 1 handles? That is, not requiring a cup or a cap?

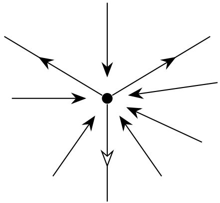
Finiteness and dualizability

To answer second question: recall how we evaluated vertices in black and white graphs:



If we don't have an invertible pairing $\langle , \rangle$, will have to evaluate vertices like the one above without dualizing. So we need a collection of maps

$$A^{\otimes r_1} \otimes \cdots \otimes A^{\otimes r_k} \to A^{\otimes k}$$



for every $k \geq 1$ and tuple $r_1, \ldots, r_k$. We collect all these maps for a fixed k into an element $m_{(k)}$.

Pre-Calabi-Yau structures

Definition

A *pre-Calabi-Yau* structure of dimension d on A is a collection of maps $m_{(k)}$, of a certain degree, satisfying a cyclic symmetry condition and such that $m = m_{(1)} + m_{(2)} + \dots$ satisfies an equation

$$m \circ m = 0$$

for a certain 'necklace product'.

Our main results (roughly phrased) are then:

Theorem

There is a cell complex for the moduli of open-closed surfaces with at least one input and at least one output (the ribbon quiver complex), which describes explicitly the corresponding TQFT operations on A and $C_*(A)$ when A is a pre-CY algebra.

Theorem

A smooth CY structure on A determines, by an inductive construction, maps $m_{(k)}$ forming a pre-CY structure on A.

Break

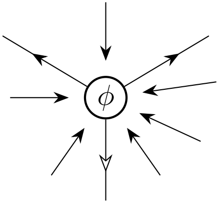
Definitions

Disclaimer: definitions first made by Kontsevich and Vlassopoulos in 2013, since then appeared in many places with different grading and sign conventions (Iyudu, Yeung)

Definition

The space of k-higher Hochschild cochains on A is the graded vector space

$$C^*_{(k)}(A) = \prod_{n_1,\dots,n_k} \mathrm{Hom}_{\Bbbk}\left(A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_k}, A^{\otimes k}\right).$$



$\phi : A[1]^{\otimes 2} \otimes A[1] \otimes A[1]^{\otimes 3} \to A^{\otimes 3}$ is an element of $C^*_{(3)}(A)$. White arrow marks the first output.

The cyclic group $\mathbb{Z}/k$ acts on this space by rotating the inputs and outputs. For some d, we define the action $(\mathbb{Z}_k, d)$ of dimension d as having a sign given by the Koszul sign (as elements of $A[1]$) together with an extra $(-1)^{(k-1)(d-1)}$.

Definition

The space of *cyclic k-cochains* of dimension d on $\mathcal{A}$ is defined as

$$C^*_{(k,d)}(A) := (C^*_{(k)}(A))^{(\mathbb{Z}_k,d)}[(d-2)(k-1)]$$

We assemble all these complexes into the *tangent complex* of dimension d.

$$C^*_{[d]}(A) := \prod_{k\geq 1} C^*_{(k,d)}(A)$$

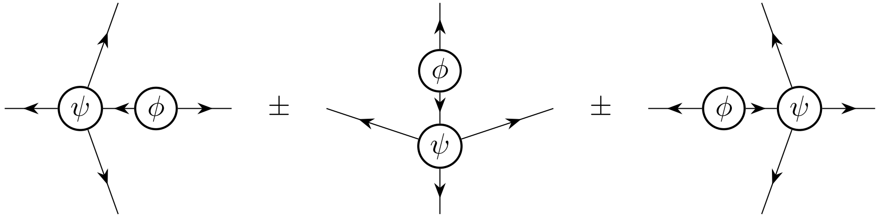
Definitions

There is a binary operation $\underset{\text{nec}}{\circ}$ called the *necklace product*

$$\underset{\text{nec}}{\circ} : C^*_{(k)}(A) \otimes C^*_{(\ell)}(A) \to C^*_{(k+\ell-1)}(A)$$

given by summing over diagrams such as

$$\psi \underset{\text{nec}}{\circ} \phi \quad =$$



This restricts to cyclic cochains of dimension d, and the corresponding bracket

$$[\phi, \psi]_{\text{nec}} = \phi \underset{\text{nec}}{\circ} \psi - (-1)^{(|\phi|-1)(|\psi|-1)} \psi \underset{\text{nec}}{\circ} \phi$$

defines a graded Lie algebra structure on $C^*_{(k,d)}(A)[1]$.

Definition

A pre-CY structure of dimension d on A is a solution $m = m_{(1)} + m_{(2)} + \dots$ to the Maurer-Cartan equation $m \underset{\text{nec}}{\circ} m = 0$ of degree 1 in the graded Lie algebra $C^*_{(k,d)}(A)[1]$.

Expanding the first couple terms, we get

1. $m_{(1)} \underset{\text{nec}}{\circ} m_{(1)} = 0 \implies \mu = m_{(1)}$ is A_∞-structure,
2. $[\mu, m_{(2)}]_{\text{nec}} = 0 \implies m_{(2)}$ defines a class in $H^d_{(2)}(A) = H^2(C^*_{(2,d)}(A), [\mu, -]_{\text{nec}})$
3. $m_{(2)} \underset{\text{nec}}{\circ} m_{(2)} = [\mu, m_{(3)}]_{\text{nec}} \dots$

Obstruction theory

We can interpret the higher Hochschild, cyclic, and tangent cohomology in terms of the obstruction theory of pre-CY structures.

Let $\mathcal{M}_{d-\mathrm{pre-CY}}$ denote the derived moduli stack of pre-CY structures on A; its tangent space at a given point m is calculated by the tangent complex $(C^*_{[d]}(A), [m,-]_{\mathrm{nec}})$.

This admits a natural filtration whose graded pieces are given by the higher cyclic cohomology $(C^*_{(k,d)}(A), [\mu,-]_{\mathrm{nec}})$, and there is a spectral sequence starting from the higher cyclic cohomology

$$E_1^{p,q} = HC^{p+q}_{(p,d)}(A)$$

and converging to the tangent cohomology

$$E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}_{[d]}(A).$$

Obstruction theory

Since higher cyclic cohomology is computed by cyclically anti/invariant higher Hochschild cochains, we have:

Theorem

If $HH^{d\ell-d-2\ell+4}_{(\ell)}(A) = 0$ *for every* $\ell \geq 3$, *then any cocycle* $m_{(2)} \in C^2_{(2,d)}(\mathcal{A})$ *can be extended to a pre-CY structure of dimension* d *on* $\mathcal{A}$.

$HH^*_{(k)}$ on smooth/proper algebras

In general, higher Hochschild cohomology must be calculated directly from definitions. But in homologically unital, smooth or proper case we have:

Theorem

If A is homologically unital and proper, there is a quasi-isomorphism

$$C^*_{(k)}(A) \xrightarrow{\sim} \mathrm{Hom}_{A-A}((A^\vee)^{\otimes_A (k-1)}, A)$$

to morphisms of (A,A)-bimodules; $A^\vee$ is the linear dual.
If A is homologically unital and smooth, there is a quasi-isomorphism

$$C^*_{(k)}(A) \xrightarrow{\sim} \mathrm{Hom}_{A-A}(A_\Delta, (A^!)^{\otimes_A (k-1)})$$

where $A^!$ is the bimodule dual $A^! = \mathrm{Hom}_{A-A}(A, A\otimes A)$.

Example: varieties with anticanonical section

Let X be a smooth quasi-projective variety of dimension d. Its bounded derived category $D(X)$ admits a generator E (Bondal-van der Bergh) so we take $A = \mathrm{End}(E)$.

Under equivalence ${}_A\mathrm{Mod}_A \simeq D(X \times X)$, the bimodule dual $A^!$ corresponds to $\Delta_*(\omega_X^{-1})[-d]$, that is, the pushforward of the anticanonical sheaf along the inclusion of the diagonal $\Delta : X \to X \times X$.

For any $k \geq 2$ we have $HH^*_{(k)}(A) = \mathrm{Ext}^{*-d(k-1)}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^{1-k})$, so any section s of ω_X^{-1} gives a suitable $m_{(2)}$.

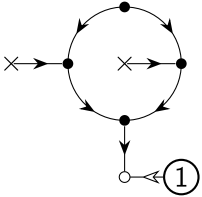
Because for every $k \geq 3$ we have $dk - d - 2k + 4 - d(k-1) = 4 - 2k < 0$, we have

Theorem

Any anticanonical section defines a pre-CY structure of dimension d on A.

Ribbon quivers

For simplicity, let us describe the ribbon quiver complex with only closed in/outputs. A ribbon quiver is an *acyclic oriented* ribbon graph, with markings similar to the black-and-white graphs.



Its valence one sources are labeled $\times$; they correspond to inputs. All of its sinks are labeled $\circ$; they correspond to outputs. Outputs have a choice of distinguished incoming edge; they can also have a $\mathbf{1}$ attached to them, in which case that edge is distinguished.

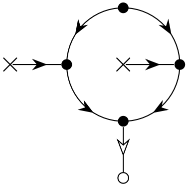
There are some limitations: for instance, there can only be one $\times$ vertex on each boundary circle, no vertices with one input and one output...

Ribbon quivers

Each ribbon quiver $\vec{\Gamma}$ has a genus (genus of the surface it represents) and a d-degree (for each d), given by

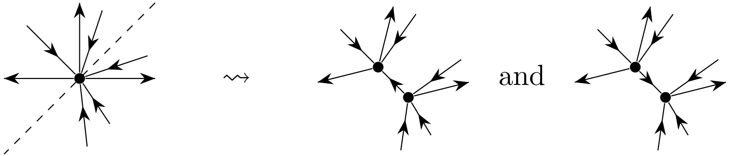
$$\deg_d(\vec{\Gamma}) = \sum_{v=\circ}(in(v) - 1) + \sum_{others}((2-d)out(v) + d + in(v) - 4)$$

When $d = 0$, the minimum possible degree is zero, and is attained by quivers whose sinks are all valence 1, flow vertices have two inputs and one output, unmarked sources are valence 2, and no 1 vertices.



Differential and composition

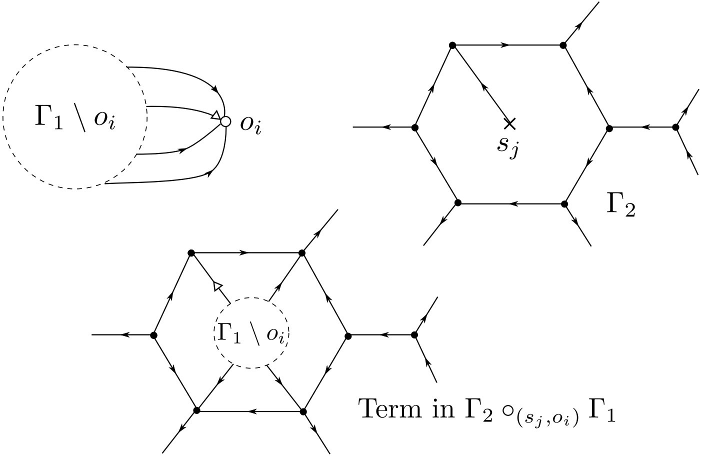
The differential on the ribbon quiver complex is given by summing over separations of its vertices. For any dashed line partitioning the incident edges:



Satisfying some rules: for instance, a separation cannot create more sinks, or lead to vertices with one input and one output.

Differential and composition

Any two ribbon quivers $\vec{\Gamma}_1, \vec{\Gamma}_2$ can be composed along a $(\circ, \times)$ pair: connecting the distinguished edge incident to $\circ$ to the edge incident to $\times$; if there are more edges we sum over ways of distributing them to vertices of Γ_2.



Orientations on ribbon quivers

Strictly speaking, the complex Q^d of ribbon quivers is not spanned by ribbon quivers, but rather by ribbon quivers *with d-orientation*. The set of orientations is given by

$$(\{\text{total ordering of edges and vertices}\} \times \{\pm 1\})/\text{symmetric group}$$

where each permutation acts on edges with weight $(-1)^{d-1}$ and vertices with weight $(-1)^d$. For example, we have

$$(e_1 e_2 v_1 e_3 v_2 v_3 \dots v_n)_d = (-1)^{(d-1)}(e_2 e_1 v_1 e_3 v_2 v_3 \dots v_n)_d = (-1)^{d-1}(e_2 e_1 e_3 v_1 v_2 v_3 \dots v_n)_d$$

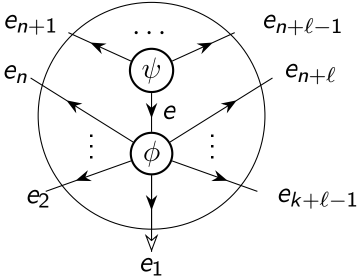
Effectively, if d is even, it is an orientation on the vector space spanned by edges, and if d is odd, spanned by vertices. The signs in differential admit an easy description in these terms: if a vertex v is separated to an edge $e : a \to b$, we have

$$\partial((\vec{\Gamma},((o_1 o_2 \dots o_n \dots 1_i \dots v_N \dots \mathbf{v} \dots v_1 \dots x_m \dots x_1))) = \sum_{\substack{v \in V(\Gamma) \\ (e:a\to b)\text{separation of } v}} \left(\vec{\Gamma}_{(e:a\to b)}, (\mathbf{e}\ \mathbf{a}\ o_1 o_2 \dots o_n \dots 1_i \dots v_N \dots \mathbf{b} \dots v_1 \dots x_m \dots x_1)\right)$$

Orientations on ribbon quivers

The signs in the necklace product also admit an easy description in terms of orientations:

$$\phi \underset{\text{nec}}{\circ} \psi = \left(\vphantom{\int} \right.$$



$$\left. , (e_1 e_2 \dots e_{k+\ell-1} \phi e \psi) \right) + (\text{cyc})$$

Given a pre-CY structure m on A, we can insert Hochschild chains into each $\times$ vertex as before with the black-and-white graphs, apply $m_{(k)}$ on each vertex with k outputs, and read out Hochschild chains at each $\circ$ vertex.

Orientations on ribbon quivers

In order to define the correct sign, we first put the orientation in normal form:

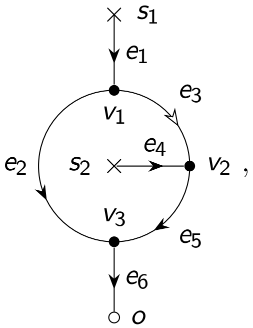
$$(e_{11}, \dots, e_{1k_1}, v_1, \dots, v_{n-1}, e_{n1}, \dots, e_{nk_n}, v_n)$$

where $v_1, v_2, \dots, v_n$ is non-decreasing in the partial order, $(e_{i1}, \dots, e_{ik_i})$ are the edges *going out* of v_i in some order compatible with the clockwise cyclic order.

and iteratively apply each vertex to the inputs, using the Koszul sign (for elements of $A[1]$). The 1-vertices output the unit Hochschild cochain $1 \in A$ (assume strict unit).

Orientations on ribbon quivers

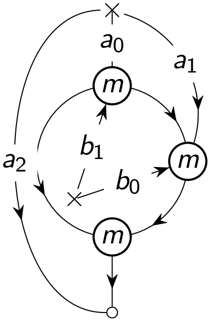
We consider the graph below with the normal form ordering



$$(o\ e_6\ v_3\ e_5\ v_2\ e_3\ e_2\ v_1\ e_4\ s_2\ e_1\ s_1)$$

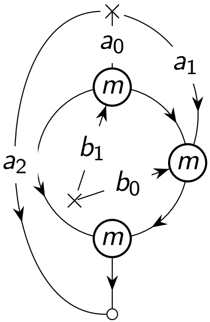
and input a pair of Hochschild chains $a_0 \otimes a_1 \otimes a_2$ and $b_0 \otimes b_1$.

Orientations on ribbon quivers



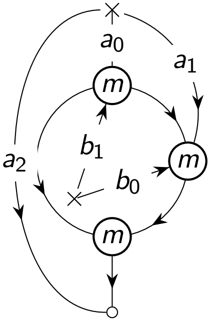
, $(o\ e_6\ v_3\ e_5\ v_2\ e_3\ e_2\ v_1\ e_4\ s_2\ e_1\ s_1)(a_0\ a_1\ a_2\ b_0\ b_1)$

We evaluate the source vertices and edges, and then permute to evaluate v_1 (at the top of the circle)



$$, \quad (o\ e_6\ v_3\ e_5\ v_2\ e_3\ e_2\ v_1)(b_1\ a_0\ a_1\ a_2\ b_0) \times (-1)^{\bar{b}_1(\bar{a}_0+\bar{a}_1+\bar{a}_2+\bar{b}_0)}$$

evaluate ϕ, suppose for instance that $\phi(b_1; a_0) = c_1 \otimes c_2$



$$, \quad (o\ e_6\ v_3\ e_5\ v_2)(c_1\ c_2\ a_1\ a_2\ b_0) \times (-1)^{\bar{b}_1(\bar{a}_0+\bar{a}_1+\bar{a}_2+\bar{b}_0)}$$

and continue this process until we reach the output o.

Topological interpretation

Theorem

This action above commutes with the Hochschild differential and gives a map of complexes

$$Q^d(m,n) \otimes (C_*(A))^{\otimes m} \to (C_*(A))^{\otimes n}$$

Let us discuss the case $d = 0$. Using a variation on the theory of Strebel differentials, we prove:

Theorem

The complex $Q_g^{d=0}(m,n)$ computes the (rational) homology of the space $\mathcal{M}_{g,\vec{m}+\vec{n}}$ of surfaces with $m+n$ punctures with a distinguished direction at each puncture, for $m \geq 1$ and $n \geq 1$

This complex is a refinement of the analogous complex of black-and-white graphs, and when the algebra A is finite-dimensional and cyclic CY, the action is exactly the one described by Tradler-Zeinalian, Wahl-Westerland et al.

Topological interpretation

When $d \neq 0$, instead we prove that $Q_g^d(m, n)$ calculates the cohomology of $\mathcal{M}_{g,\vec{m}+\vec{n}}$ with coefficients in $\mathcal{L}^d$, where $\mathcal{L}$ is a line bundle concentrated in cohomological degree $-\chi(\Sigma)$. This is a certain tautological tangent bundle.

Theorem

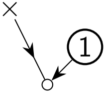
In the absence of 'free boundary circles', that is, boundary components of Σ without any inputs or outputs, the line bundle $\mathcal{L}$ is trivial (up to shift). So in this case all the complexes Q^d are isomorphic up to shift.

Some genus zero graph

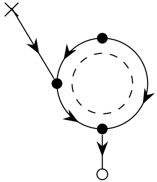
Let us discuss some examples in genus zero

$m = 1, n = 1$

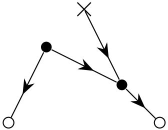
(1)  $\deg = 0$

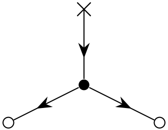
(2) 

$\deg = 1$

(3)  $\deg = -d$

$m = 1, n = 2$

(4)  $\deg = -d$

(5)  $\deg = -d + 1$

We give the ribbon structure from the embedding into the page. Example (1) is the identity, (2) gives the Connes differential, (5) gives a term in the coproduct.

Smooth CY and pre-CY

A pre-CY structure defines a partially defined TQFT, with cobordisms generated by handles of index 1. Recall that a smooth/proper CY structure does too, but with handles of index (0 and 1)/(1 and 2).

Abstractly, one should be able to start with a proper/smooth CY structure and obtain a pre-CY structure. Proper case is easy and less interesting, since we have the formalism of black-and-white graphs. We now

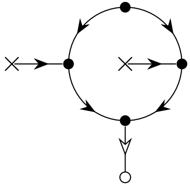
Smooth CY structure of dimension d is given by a class $\lambda \in HC_d^-(A)$ in negative cyclic homology. Use model $CC_*^-(A) = (C_*(A)[[u]], b + uB)$, so $\lambda = \lambda_0 + u\lambda_1 + u^2\lambda_2 + \dots$, with

$$b\lambda_0 = 0, \quad B\lambda_0 = b\lambda_1, \quad B\lambda_1 = b\lambda_2, \dots$$

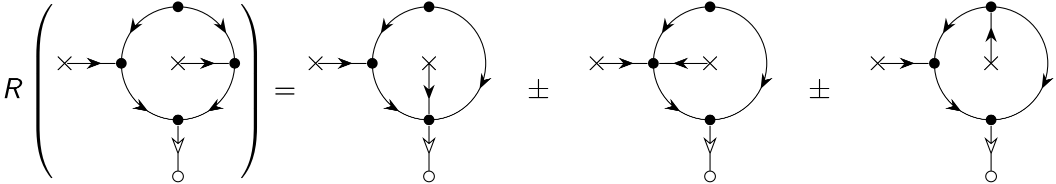
where u is of (cohomological) degree $+2$.

The R-differential

Each $\times$-vertex is surrounded by its boundary cycle, which is homotopic to one of the boundary components of the corresponding surface.



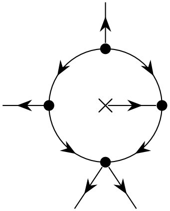
The edge coming from a given $\times$ can either land on an edge of this boundary cycle or an already-existing vertex. If edge, we define an operation R of (homological) degree $+1$



Recall that $B(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n + \ldots$; for some Hochschild chain α we then have by unitality $\Gamma(B\alpha) = R\Gamma(\alpha)$.

The complex of one-loop ribbon quivers

Consider the complex $\mathcal{T}_{(k)}$ of ribbon quivers with one $\times$-input, one loop around it, and some number k of outgoing legs



We now define the complexes of one-loop ribbon quivers

$$\mathcal{T}^{-}_{(k)} = (\mathcal{T}_{(k)}[[u]], \partial + uR)$$

Given a pre-CY structure m on A, we then have a map

$$CC^{-}_{*}(A) \otimes \mathcal{T}^{-}_{(k)} \to C^{*}_{(k)}(A)$$

commuting with the differentials induced by $b + uB, \partial + uR$.

Chain-level nondegeneracy

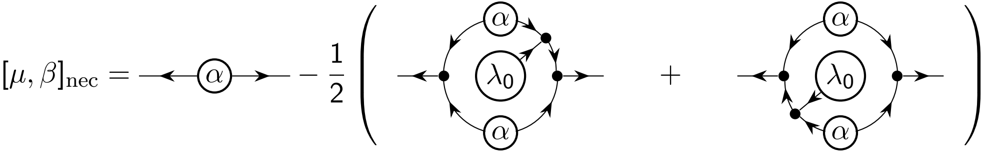
However, this complex also allows one to construct the pre-CY structure, inductively, starting from a smooth CY structure.

$\lambda \in CC_d^-(A)$ defines a smooth CY structure if

$$\lambda_0 \in C_d(A) \simeq \mathrm{Hom}_{A-A}(A^!, A[-d])$$

is invertible. Let $\alpha \in \mathrm{Hom}_{A-A}(A, A^![d]) \simeq C^d_{(2)}(A)$ be a symmetrized chain representing its inverse.

This implies that there is $\beta \in C^{d-1}_{(2)}(A)$ satisfying the equation



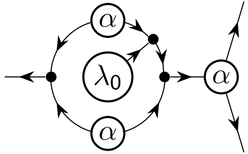
giving a chain-level picture of the nondegeneracy condition.

Inductive construction of pre-CY structure

We first observe that the complex $\mathcal{T}^-_{(k)}$ computes the homology of $\mathcal{M}_{0,\vec{2}}/S^1 \underset{\mathbb{Q}}{\simeq} \bullet \implies H_{\neq 0}(\mathcal{T}^-_{(k)}) = 0.$

We now construct a pre-CY structure with $m_{(2)} = \alpha$. Equation for $m_{(3)}$ is

$$[\mu, m_{(3)}]_{\mathrm{nec}} = m_{(2)} \underset{\mathrm{nec}}{\circ} m_{(2)} = \frac{1}{2}[\alpha, [\mu, \beta]_{\mathrm{nec}}]_{\mathrm{nec}} + \frac{1}{2}\Bigg($$



$$+ \dots \Bigg)$$

This last expression can be interpreted as $\Delta^0_{(2,2)}(\lambda)|_{u=0}$; $\Delta^0_{(2,2)}$ is of degree 1. We can show that it lifts to a closed class $\Delta_{(2,2)} = \Delta^0_{(2,2)} + u\Delta^{-1}_{(2,2)}$

Inductive construction of pre-CY structure

Therefore by our calculation there must be some element $\Gamma_{(3)} = \Gamma^0_{(3)} + u^{-1}\Gamma^1_{(3)}$ such that $(\partial + uR)\Gamma_{(3)} = \Delta_{(2,2)}$.

So we can find $m_{(3)}$ solving our equation by

$$\begin{aligned} m_{(3)} &= [\alpha, \beta]_{\mathrm{nec}} + \Gamma_{(3)}(\lambda)|_{u=0} \\ &= [\alpha, \beta]_{\mathrm{nec}} + \Gamma^0_{(3)}(\lambda_0) + \Gamma^1_{(3)}(\lambda_1) \end{aligned}$$

and so on for $m_{(4)}, \ldots$. We then find that:

Theorem

The element $m_{(k)}$ depends only on $\alpha, \beta, \lambda_0, \ldots, \lambda_{k-2}$. Similar bound exists for all the TQFT operations of fixed genus and number of inputs/outputs.

Examples: loop spaces

Let M be a closed, connected, orientble d-manifold, and consider chains on its based loop space $A = C_*(\Omega M)$. We have the 'Goodwillie equivalence'

$$C_*(A) \simeq C_*(LM)$$

to chains on the free loop space, sending the circle action on the Hochschild complex to loop rotation, so we get a canonical map

$$i_* : C_*(M) \to CC_*^-(A)$$

to the negative cyclic complex.

Theorem (Cohen, Ganatra)

If $\lambda \in CC_*^-(A)$ is cohomologous to the pushforward of the fundamental class of M, then it defines a smooth CY structure of dimension d on M.

Therefore by our construction one should recover many operations on loop space; we conjecture that these agree with the explicit constructions of Sullivan, Goresky, Hingston etc (loop product, coproduct...)

Examples: Fukaya-Seidel categories

Our solution generalizes to a construction of a pre-CY structure on B from a smooth relative CY structure on a map $f : A \to B$ (Brav-Dyckerhoff).

Recently Seidel ('Fukaya A_∞-structures... V') constructed this structure at the $m_{(2)}$ level. Consider a Lefschetz fibration $p : E \to \mathbb{R}^2$, with symplectic fiber M. This gives a restriction functor

$$Q : FS(E) \to Fuk(M)$$

from the Fukaya-Seidel category of the fibration to the (infinitesimal) Fukaya category of the fiber M.

Seidel constructs a class $[\delta] \in C^*_{(2)}(FS(E))$, and conjectures that it extends to a pre-CY structure ('log CY structure'). We conjecture that there is a dual picture (involving smooth 'wrapped' categories) for which a pre-CY structure and its operations can be constructed from a relative CY structure on $Q^\vee$.

The end, thank you!