Math Precalculus (12H/4H) Review

CHSN Review Project

Contents

This review guide was written by Dara Adib. Prateek Pratel checked the "Polar and Complex Numbers" chapter on page 9 for errors.

This is a development version of the text that should be considered a work-in-progress.

This review guide and other review material are developed by the CHSN Review Project.

Functions

This chapter was designed for a test on functions administered by Jeanine Lennon to her Math 12H (4H/Precalculus) class on January 4, 2008.

Definitions

function relation in which each first coordinate (usually x value) cooresponds to only one last coordinate (usually y value); passes vertical line test

vertical line test test on a graph to determine if a relation is a function

domain set of all first coordinates (usually x values)

range set of all last coordinates (usually y values)

restricted domain values for which the function is undefined

i.e. $x \mid x \in R, x \neq 0, x \neq \pm 5$

inverse of a relation reversed domain and range of a relation

$R^{-1} = \{(y, x)\}$

one-to-one function function whose inverse is also a function; passes both vertical and horizontal line tests

periodic function function in a cycle such that $f(x + p) = f(x)$ where p represents period

amplitude $\frac{1}{2}(max - min)$ of a periodic function

frequency number of cycles per 360° (degrees) or 2π (radians)

period duration of one cycle

composite function (composition of functions) application of one function on the result of other function

asymptote line that a graph approaches, but never intersects

Operations on Functions

Explanation

$(f + g)(x)$ is equivalent (and equal) to $f(x) + g(x)$.

$\left(\frac{f}{g}\right)(x)$ is equivalent (and equal) to $\frac{f(x)}{g(x)}$, $g(x) \neq 0$ (to keep the denominator from being zero).

x	$f(x)$	$g(x)$
-2	7	3
-1	3	1
0	8	0
1	4	1

Table 0.1: common x values

x	$f(x)$	$g(x)$	$\frac{f}{g}$
-2	7	3	$\frac{7}{3}$
-1	3	1	3
0	8	0	undefined
1	4	1	4

Table 0.2: y values for the operation $\frac{f}{g}$

For All Common Values

On the other hand, $f + g$ means finding $f(x) + g(x)$ for all common x values. The answer would be displayed as a set of ordered pairs. For example: $\{(a, b), (c, d), (e, g)...\}$.

It should be understood that operations on functions can only occur when x is in the domain of both functions.

Example

Table 0.1 consists of common x values between $f(x)$ and $g(x)$. Other values may be provided, but only the common x values are important.

Table 0.2 adds a fourth column, which contains the y values for the operation $\frac{f}{g}$. Since $\frac{8}{0}$ is undefined, it will not be present in the answer.

Answer $\{(-2, \frac{7}{3}), (-1, 3), (1, 4)\}$

Composite Functions

A composition of functions (composite function) consists of the application of one function on the result of other function. In other words, a function is applied to another function. After a value (usually y value) is determined by one function, it is substituted into the other function (usually x

Reflection	Coordinates	Function
$r_{x-axis}(x, y)$	$(x, -y)$	$y = -f(x)$
$r_{y-axis}(x, y)$	$(-x, y)$	$y = f(-x)$
$r_{y=x}(x, y)$	(y, x)	$x = f(y)$
$R_O(x, y)$	$(-x, -y)$	$y = -f(-x)$

Table 0.3: reflection

value). The functions may or not be commutative, so the order of the functions in the composition should be taken into account.

$f(g(x))$ and $(f \circ g)(x)$ are equivalent. In both cases, $g(x)$ is determined first and the result is plugged into the x value of $f(x)$.

> If you find $f(g(x))$ to be easier, thank the mid-20th century mathematicians that came up with this notation after determining that $(f \circ g)(x)$ was too confusing.

Reflections and Symmetry

Reflection

When functions are reflected over a certain line or point (i.e. x-axis or origin), coordinates ((x, y)) of points in the function and the function itself ($y = f(x)$) are affected as seen in Table 0.3.

Symmetry

Functions symmetric over a certain line or point (i.e. x-axis or origin) contain both (x, y) and the coordinates shown in Table 0.3.

Even and Odd Functions

even functions functions symmetric over the y-axis

odd functions functions symmetric over the origin

Determining Symmetry

To algebraically determine symmetry over a certain line or point, replace the values listed below. Then, simplify the equation and determine if the two equations are equivalent. If the equations are equivalent, the graph is symmetric over the specified line or point.

x-axis negate the y values

y-axis negate the x values

y=x reverse x and y values (substitute the x and y values with each other)

origin negate both the x and y values

Example

Equation of graph: $x^2 + xy = 4$

x-axis

1. Negate y values: $x^2 - xy = 4$
2. Simplify, if necessary (already simplified): $x^2 - xy = 4$
3. Compare with graph equation: not equivalent
4. Not symmetric over the x-axis

y-axis

1. Negate x values: $(-x)^2 - xy = 4$
2. Simplify, if necessary: $x^2 - xy = 4$
3. Compare with graph equation: not equivalent
4. Not symmetric over the y-axis

y=x

1. Reverse x and y values: $y^2 + yx = 4$
2. Simplify, if necessary: $y^2 + xy = 4$
3. Compare with graph equation: not equivalent
4. Not symmetric over the line $y = x$

origin

1. Negate x and y values: $(-x)^2 + (-x)(-y) = 4$
2. Simplify, if necessary: $x^2 + xy = 4$
3. Compare with graph equation: equivalent
4. Symmetric over the origin

Periodic Functions

Definitions

See definitions on page 3.

Effects of Different Equations

The bulleted examples below are compared with $y = f(x)$

- $y = 2 \times f(x)$ doubles the amplitude
- $y = f(2x)$ doubles the frequency; halves the period (periodic shrink)
- $y = f(\frac{1}{2}x)$ halves the frequency; doubles the period (periodic stretch)

Other Effects

The following text in this section may be incorrect.[1] The "ceiling function" is not a periodic function. The bulleted examples below are compared with $y = \lceil x \rceil$.

- $y = 2\lceil x \rceil$ doubles y values
- $y = \lceil 2x \rceil$ doubles x values

Inverses of Relations

A function is a relation, but a relation does not necessarily have to pass the vertical line test.

Definitions

See definitions on page 3.

Determining the Inverse of a Relation

Reverse the x and y values of an equation and if necessary, solve for y. In this way, the inverse of $y = 2x + 1$ will be $y = \frac{x-1}{2}$.

Other Situations

You may not need to necessarily determine the equation of a relation's inverse. For example, one can use a y value on a chart of values of a relation instead of the x value of its inverse.

Translations of Functions

To translate a function, values are either added or subtracted to part of a function. The examples below are compared with $y = f(x)$.

- $y = f(x) + a$ moves the graph a units up; add a to y values
- $y = f(x + b)$ moves the graph b units to the left; subtract b from x values

The examples below are compared with $y = |x|$

- $y = |x| + a$ moves the graph a units up; add a to y values
- $y = |x + b|$ moves the graph b units to the left; subtract b from x values

Keep in mind that absolute value will cause the graph of an example like $y = ||x| - c|$ to be shaped like a W (assuming c is positive).

[1] Corrections and other feedback sent to the author or to the CHSN Review Project are greatly appreciated.

Asymptotes

Definition

See definitions on page 3. For more information on asymptotes, see the *Math Calculus Review*.

Determining Asymptotes

To determine asymptotes algebraically, you must solve for either x or y—isolate the variable.

Vertical Asymptotes

Solve the equation for y and determine the x value where the denominator of the fraction would equal zero.

Horizontal Asymptotes

Solve the equation for x and determine the y value where the denominator of the fraction would equal zero.

Important Information

Some equations may not have any asymptotes. These include, but are not limited to, linear equations. Beware, a linear equation may be obfuscated to appear to be another type of equation. Therefore, remember to factor numerators and denominators and attempt to simplify fractions as much as possible before determining asymptotes. Also understand that $\frac{0}{0}$ is not a valid equation or asymptote.

Polar and Complex Numbers

This chapter was designed for a test on polar and complex numbers administered by Jeanine Lennon to her Math 12H (4H/Precalculus) class on March 5, 2008. Prateek Pratel checked this chapter for errors.

Polar Coordinates

Cartesian (rectangular) coordinates a point, P, on a plane is described in terms of x and y, where x and y are the respective horizontal and vertical distances from the origin

Form: (x, y)

polar coordinates a point, P, on a plane is described by specifying the distance, r, from the origin and the angle, θ, measured counter-clockwise from the positive x-axis to the line joining P to the origin

Form: (r, θ)

coterminal angles angles that coincide (when placed in standard position); added or subtracted multiples of 360° (including 360°)

reference angles way to simplify the calculation of the values of trigonometric functions in different quadrants

Refer to Table 0.4, where θ is the angle and β is the reference angle.

Converting Coordinates

Polar-to-Cartesian

Polar coordinates are expressed in the form (r, θ), while Cartesian coordinates are expressed in the form (x, y). The two following equations may be used to find the appropriate values for x and y based on r and θ, which may be substituted into (x, y). There is no requirement for r to be positive.

Quadrant	Reference Angle (β)
I	$\beta = \theta$
II	$\beta = 180° - \theta$
III	$\beta = \theta - 180°$
IV	$\beta = 360° - \theta$

Table 0.4: reference angles

$x = r\cos\theta$

$y = r\sin\theta$

Therefore, the form for conversion from polar coordinates to Cartesian coordinates is expressed as $(r\cos\theta, r\sin\theta)$.

It should be noted that the polar and Cartesian coordinates must be in the same quadrant (hint: reference angles).

Explanation

These formulas are derived by inscribing a right triangle with hypotenuse r in a circle on a coordinate plane, such that θ is an angle from the positive x-axis to the line joining a point to the origin. If the point assumes the coordinates (x, y), the following two equations are valid.

$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

By multiplying both sides of the equations by r, the method for conversion is derived.

$x = r\cos\theta$

$y = r\sin\theta$

Cartesian-to-Polar

Cartesian coordinates are expressed in the form (x, y) , while polar coordinates are expressed in the form (r, θ). The two following equations may be used to find the appropriate values for r and θ based on x and y, which may be substituted into (r, θ).

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\frac{y}{x}, x \neq 0$

Therefore, the form for conversion from Cartesian coordinates to polar coordinates is expressed as $(\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}), x \neq 0$.

It should be noted that the Cartesian and polar coordinates must be in the same quadrant (hint: reference angles). If $x = 0$, the second equation cannot be used to find θ since it will be undefined. Instead, one must determine if θ is 90° or 270°, depending on whether y is positive or negative, respectively.

Explanation

These formulas are derived by inscribing a right triangle with hypotenuse r in a circle on a coordinate plane, such that θ is an angle from the positive x-axis to the line joining a point to the origin. If the point assumes the coordinates (x, y), the following two equations are valid.

$r^2 = x^2 + y^2$ (Pythagorean Theorem)

$\tan^{-1}\frac{y}{x}, x \neq 0$

The square root of both sides is taken from the first equation based on the Pythagorean Theorem.

$r = \pm\sqrt{x^2 + y^2}$

The negative value of r can be rejected for practical purposes.

$r = \sqrt{x^2 + y^2}$

The second equation can be rewritten to isolate θ on one side of the equation.

$\theta = \tan^{-1}\frac{y}{x}, x \neq 0$

Polar Inequalities

Polar inequalities are fairly simple. They are expressed in the following form.

$a \leq r \leq b$

$c \leq \theta \leq d$

a represents the smallest value of r in the range, while b represents the largest value of r in the range

c represents the smallest angle of θ in the range, while d represents the largest angle of θ in the range

Etiquette

For math etiquette, one should follow the following guidelines when writing polar inequalities.

$a \geq 0$

$b > 0$

$0^\circ \leq c < 360^\circ$

$0^\circ < d \leq 360^\circ$

In other words, the radii should be positive, while the angles should be between 0° and 360°.

Graphing Polar Equations

Graphing

Manually

A polar equation can be graphed by hand. This work is tedious, and requires the creation of a table of values. Instructions for graphing manually are not included, as the test this review report is designed for does not require manually calculating values and graphing.

Graphing Calculator

A graphing calculator must be placed in polar mode. Instructions for modern Texas Instruments (TI) graphing calculators follow, though they may be altered for use with other graphing calculators as well.

1. Change “Mode” to radians or degrees as necessary
2. Change “Mode” from “FUNC” (function) to “POL” (polar)
3. Adjust “Window” as necessary

“Steps” (of θ) correspond to the number of calculations done to graph the equation. A lower step means more calculations, while a higher step means less calculations. More calculations mean more accuracy in graphing (less distortion), but longer periods to graph. Less calculations mean less accuracy in graphing (more distortion), but shorter periods to graph.

Polar-to-Cartesian

Generally, a polar equation is written in the form $r = a\sin\theta$ or $r = a\cos\theta$, while an equation in the Cartesian coordinate plane is written in the form $x^2 + y^2 = ay$ or $x^2 + y^2 = ax$. The two following equations may be used to find the appropriate coefficients of x and y based on the coefficients of $\cos\theta$ and $\sin\theta$.

$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

The appropriate coefficient of $x^2 + y^2$ may be determined based on the coefficient of r^2.

$r^2 = x^2 + y^2$

Substitution is key in these cases.

Example

An example of a conversion from a polar equation to a Cartesian equation follows. $r = 2\cos\theta$ is the polar function, while $x^2 + y^2 = 2x$ is the determined equation (in the Cartesian coordinate plane).

$r = 2\cos\theta$

$r = 2\left(\frac{x}{r}\right)$

$r^2 = 2x$

$x^2 + y^2 = 2x$

No further work is required on this determined equation.

Complex Numbers

The rectangular form of a complex number is $a + bi$. The polar form of a complex number is $r\operatorname{cis}\theta$. $r\operatorname{cis}\theta$ is an abbreviation for $r(\cos\theta + i\sin\theta)$. The same rules and methods to convert from polar form to Cartesian form and vice-versa apply to complex numbers in polar form as they apply to real numbers in polar form.

Conversions

If a complex number is represented as an ordered pair, represent the complex number in the other form as an ordered pair. If a complex number is not represented as an ordered pair, do not represent the complex number in the other form as an ordered pair.

Cartesian-to-Polar

The Cartesian complex number $a + bi$ can be expressed as the following polar number:

$\sqrt{a^2 + b^2}[\cos(\tan^{-1}\dfrac{b}{a}) + i\sin(\tan^{-1}\dfrac{b}{a})]$, or abbreviated as $\sqrt{a^2 + b^2}\,\text{cis}\,(\tan^{-1}\dfrac{b}{a})$.

It should be noted that the Cartesian and polar coordinates must be in the same quadrant (hint: reference angles).

Explanation The absolute value of a complex number, expressed as $|z|$, represents the length of the vector of a graphed complex number.

$|z| = \sqrt{a^2 + b^2}$ (Pythagorean Theorem)

Since the length of the vector is equivalent to r (the radius), $r = \sqrt{a^2 + b^2}$.

Polar-to-Cartesian

The polar complex number $r\,\text{cis}\,\theta$ can be expressed as the Cartesian number $r\cos\theta + (r\sin\theta)i$. It should be noted that the polar and Cartesian coordinates must be in the same quadrant (hint: reference angles).

Operations

Answer in exact form (i.e. with square roots) when specified or when using special angles (chart available from the CHSN Review Project). Otherwise, decimal rounded form may be used.

Cartesian

FOIL may be used to multiply two complex numbers in Cartesian form.

In general, if $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 \times z_2 = a_1 a_2 + a_1 b_2 i + a_2 b_1 i - b_1 b_2$. The last term is negative and does not contain any imaginary numbers since $i^2 = -1$.

Polar

The majority of operations can only be done in polar form. If a Cartesian complex number is provided and a Cartesian (rectangular) answer is requested, the provided Cartesian complex number must be first converted to polar form, the operation done, and finally the resulting polar complex number converted back into Cartesian form.

Multiplication and Division In general, if $z_1 = r_1\,\text{cis}\,\theta_1$ and $z_2 = r_2\,\text{cis}\,\theta_2$, then $z_1 \times z_2 = r_1 \times r_2\,\text{cis}\,(\theta_1 + \theta_2)$. Likewise, under the same conditions, $\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}\,(\theta_1 - \theta_2)$.

De Moivre's Theorem Let n be any integer, then $(r \operatorname{cis} \theta)^n = r^n \operatorname{cis} n\theta$.

To determine the roots of a complex number, n can be substituted with $\frac{1}{x}$, where x represents the x^{th} root . A complex number raised to the power of $\frac{1}{x}$ will have x number of roots. This is due to the fact that equivalent complex numbers in polar form (with coterminal angles) will have reference angles $360°$ apart. When $360°$is divided by x, x unique angles will be created (between $0°$ and $360°$), with $^{360°}/_{x}$ as the difference in the angles of the roots. As a result, a non-zero complex number will have two square roots, three cube roots, four fourth roots, etc.

In general, $(r \operatorname{cis} \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \operatorname{cis} \dfrac{\theta + 360K}{n}$, where $K = 0, 1, 2, 3 ... n - 1$.

Sequences and Series

This chapter was designed for a test on sequences and series administered by Jeanine Lennon to her Math 12H (4H/Precalculus) class on March 14, 2008. This chapter refers to the position of a term as n, and refers to the total number of terms in a sequence or series as k.

Arithmetic Sequences

Arithmetic sequences (also known as lists and progressions) have a constant difference. Terms increase or decrease by adding or subtracting a fix quantity. The constant difference is known as d, which is the difference between any two consecutive terms.

Arithmetic sequences use the following equation to determine the n^{th} term.

$t_n = t_1 + (n-1)d$ (provided on test unlabeled)

t_1 represents the first term, while n represents the position of a term (i.e. first, second, third).

Rewritten in the following equation, one may determine the number of terms in an arithmetic sequence.

$n = \frac{t_n - t_1}{d} + 1$

Note Multiples of a number are an arithmetic sequence.

Example

Sequence $2, 5, 8, 11, 14, 17$

The above sequence is an example of an arithmetic sequence, and $d = 3$.

Arithmetic Mean

Sequence $\ldots a, x, b \ldots$

If the above sequence is given, x is considered the arithmetic mean of a and b. Therefore, $x = \frac{a+b}{2}$.

Explanation

If one attempts to solve determine the value of x based on a and b, it would be determined that $d = x - a$ and $d = b - x$. Through substitution, $x - a = b - x$. This can in turn be rewritten as $x = \frac{a+b}{2}$, the same as the arithmetic mean of a and b.

Recursive Arithmetic Sequences

recursive equation an equation for sequences in which the value of a term is defined based on the preceding term(s).

Recursive formulas must have the following:

1. an initial condition, t_1
2. a recursive equation

A recursive arithmetic formula is a different method of expressing an arithmetic sequence, instead of using standard form.

Standard form $t_n = t_1 + (n-1)d$

Recursive form $t_n = d + t_{n-1}$

Geometric Sequences

Geometric sequences have a constant ratio. All terms are connected by a common ratio. The constant ratio is known as r, which is the quotient determined by the division of any two consecutive terms.

Geometric sequences use the following equation to determine the n^{th} term.

$t_n = t_1 \times r^{n-1}$ (provided on test unlabeled)

t_1 represents the first term, while t_n represents the n^{th} term.

Note Geometric sequences with alternating signs of terms have negative r $(-r)$ values. Consequently, when a term and its position in a sequence (n) are provided and n is even, multiple values for r may be determined. In this case, r and $-r$ $(\pm r)$ may both be correct values for r.

Example

Sequence $4, 8, 16, 32, 64, 128$

The above sequence is an example of a geometric sequence, and $r = 2$.

Geometric Mean

Sequence $\ldots a, x, b \ldots$

If the above sequence is given, x is considered the geometric mean of a and b. Therefore, $x = \sqrt{a \times b}$.

Explanation

If one attempts to solve determine the value of x based on a and b, it would be determined that $r = \frac{x}{a}$ and $r = \frac{b}{x}$.

Through substitution, $\frac{x}{a} = \frac{b}{x}$. This can in turn be rewritten as $x = \sqrt{a \times b}$, the same as the geometric mean of a and b.

Series

series the sum of a sequence, list, or progression

S_k represents the summation of k number of terms in a sequence.

The formulas follow, and are different for arithmetic and geometric series.

Arithmetic Series

The following formula can be used to determine the value of an arithmetic series, S_k.

$S_k = \frac{k(t_1 + t_k)}{2}$ (provided unlabeled on test as $S_n = \frac{n(t_1 + t_n)}{2}$)

k represents the total number of terms in a sequence or series.

Geometric Series

The following formula can be used to determine the value of a geometric series, S_k.

$S_k = \frac{t_1(1 - r^k)}{1 - r}$ (provided unlabeled on test as $S_n = \frac{t_1(1 - r^n)}{1 - r}$)

k represents the total number of terms in a sequence or series.

Infinite Geometric Series

The sum of an infinite geometric series would be expressed by the following statement.

$S = t_1 + t_1 r + t_1 r^2 + t_1 r^3 \ldots$

Infinite geometric series either converge or diverge.

Converging Infinite Geometric Series

Converging infinite geometric series have a sum, and the value of r in the series fits in $-1 < r < 1$. In other words, r must be between -1 and 1, non-inclusive.

The following formula would be used to determine the sum of the infinite geometric series.

$S = \frac{t_1}{1 - r}, -1 < r < 1$ (provided on test, possibly without domain)

Interval of Convergence In some cases r, may be provided or can be determined in terms of another variable. For example, r may equal $x + 1$. In this case, one simply substitutes r with $x + 1$ into the domain provided for converging infinite geometric series.

$-1 < r < 1$

$-1 < x + 1 < 1$

To isolate the variable, x, one may subtract 1 from all sides of the relation.

$-2 < x < 0$

It has now been determined that if $r = x = 1$, the interval that convergence will occur (resulting in a non-infinite sum) is between -2 and 0, non-inclusive.

Diverging Infinite Geometric Series

If the value of r in the infinite geometric series does not fit in $-1 < r < 1$, then the sum of the series is infinity and is said to diverge.

Sigma Notation

Summations may also be expressed through sigma notation.

$$\sum_{i=n}^{m} x_i$$

In the above expression of sigma notation, i is the index of summation, n is the lower limit of summation, and m is the upper limit of summation. It is possible than m may be infinity (∞), consequently resulting in no limit on summation.

The sigma notation specified above would be equivalent to the following expression.

$$x_m + x_{m+1} + x_{m+2} + \cdots + x_{n-1} + x_n$$

Sigma notation can be used for both arithmetic and geometric series

Arithmetic Series

Sigma notation has the following form for arithmetic series.

$$\sum_{n=t_1}^{t_k} dn$$

The constant difference is known as d, which is the difference between any two consecutive terms. k represents the total number of terms in a sequence or series. n represents the position of a term (i.e. first, second, third). It is possible than t_k may be infinity (∞), consequently resulting in no limit on summation.

Geometric Series

Sigma notation has the following form for geometric series.

$$\sum_{n=t_1}^{t_k} (t_1)(r)^{n-1}$$

The constant ratio is known as r, which is the quotient determined by the division of any two consecutive terms. k represents the total number of terms in a sequence or series. n represents the position of a term (i.e. first, second, third). It is possible than t_k may be infinity (∞), consequently resulting in no limit on summation.

Alternating Signs

Arithmetic series with alternating signs of terms are in fact both arithmetic and geometric series, with d defined and $r = -1$. As a result, the format used for arithmetic sequences is represented by the following expression.

$$\sum_{n=t_1}^{t_k} (dn)(-1)^{k-1}$$