Math Trigonometry Review

CHSN Review Project

This review guide was written by Dara Adib. It was designed for a test on trigonometry administered by Jeanine Lennon to her Math 12H (4H/Precalculus) class on February 12, 2008, but also applies to trigonometry material in Math 11H (3H).

This is a development version of the text that should be considered a work-in-progress.

This review guide and other review material are developed by the CHSN Review Project.

Angles and Sectors of Circles

Radians

- To convert degrees to radians, multiply by $\frac{\pi}{180}$.
- To convert radians to degrees, multiply by $\frac{180}{\pi}$.
- One radian equals $\frac{180}{\pi}$ or approximately 57.296°.

Coterminal Angles

coterminal angles different angles that have the same initial and terminal ray

The differences between coterminal angles are multiples of 360°.

Examples: $60^\circ, -300^\circ, 420^\circ$

Sectors

sector part of a circle formed by two radii and an arc

$s = r\theta$ (arc length = radius × angle)

The angle must be represented in radians.

apparent size the angle that an object subtends at one's eyes; this explains why an object appears to be smaller when farther away

Trigonometric Functions

Reciprocal Functions

The following are reciprocal functions.

- $\csc x = \dfrac{1}{\sin x}$
- $\sec x = \dfrac{1}{\cos x}$
- $\cot x = \dfrac{1}{\tan x}$

As a result, the following are true as well.

- $\sin x = \dfrac{1}{\csc x}$
- $\cos x = \dfrac{1}{\sec x}$
- $\tan x = \dfrac{1}{\cot x}$

θ (degrees)	θ (radians)	Point	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	$(1, 0)$	0	1	0
90°	$\frac{\pi}{2}$	$(0, 1)$	1	0	undefined
180°	π	$(-1, 0)$	0	-1	0
270°	$\frac{3\pi}{2}$	$(0, -1)$	-1	0	undefined

θ (degrees)	θ (radians)	Point	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0	$(1, 0)$	undefined	1	undefined
90°	$\frac{\pi}{2}$	$(0, 1)$	0	undefined	1
180°	π	$(-1, 0)$	undefined	-1	undefined
270°	$\frac{3\pi}{2}$	$(0, -1)$	0	undefined	-1

Table 1: quadrantal angles

Cofunctions

The following are cofunctions.

- $\sin x = \cos(90 - x)$
- $\tan x = \cot(90 - x)$
- $\sec x = \csc(90 - x)$

As a result, the following are true as well.

- $\cos x = \sin(90 - x)$
- $\cot x = \tan(90 - x)$
- $\csc x = \sec(90 - x)$

Special Angles

Quadrantal Angles

quadrantal angles angles that have a terminal side coinciding with a coordinate axis

The value of the trigonometric function (i.e. sine, cosine, tangent) is determined by the coordinates of the points on a unit circle. By definition, the point (x, y) on a unit circle corresponds to $(\cos\theta, \sin\theta)$.

As a result, quadrantal angles can be determined easily. Table 1 lists quadrantal angles and the values of trigonometric functions. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, the tangent function of an angle can be found by dividing the sine function of the angle by the cosine function of the angle.

Quadrant	Reference Angle (β)
I	$\beta = \theta$
II	$\beta = 180^\circ - \theta$
III	$\beta = \theta - 180^\circ$
IV	$\beta = 360^\circ - \theta$

Table 2: reference angles

Angles greater than 90°

Angles greater than 90° may be in different quadrants. The trigonometric functions vary over whether the trigonometric function of a certain angle is positive or negative. It is useful to remember the mnemonic device "All Students Take Calculus."

- All trigonometric functions are positive in the first quadrant
- Sine and cosecant are positive in the second quadrant
- Tangent and cotangent are positive in the third quadrant
- Cosine and secant are positive in the fourth quadrant
- The remaining trigonometric functions in each quadrant are negative.

Reference Angles

The use of reference angles is a way to simplify the calculation of the values of trigonometric functions in different quadrants. Refer to Table 2, where θ is the angle (in degrees) and β is the reference angle.

Sum and Difference Formulas

Sum Formulas

Sine

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

Cosine

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

Tangent

The derivation for the tangent of the sum of two angles follows.

$\tan(A + B)$

$\frac{\sin(A + B)}{\cos(A + B)}$

$$\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$\frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Difference Formulas

Sine

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Cosine

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Tangent

The derivation for the tangent of the difference of two angles follows.

$$\tan(A - B)$$

$$\frac{\sin(A - B)}{\cos(A - B)}$$

$$\frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

$$\frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Identities

Pythagorean Identities

1. $\sin^2 \theta + \cos^2 \theta = 1$
2. $1 + \tan^2 \theta = \sec^2 \theta$
3. $1 + \cot^2 \theta = \csc^2 \theta$

Quotient Identities

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

2. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Graphing Trigonometric Functions

General Form

The general form of a trigonometric function is one of the following (sine or cosine).

$y = a \sin b(x \pm c) \pm d$

$y = a \cos b(x \pm c) \pm d$

In other words, it can be understood to mean the following.

$y = \text{amplitude} \times \sin\left[\text{frequency}\,(x - \text{horizontal translation})\right] + \text{vertical translation}$

$y = \text{amplitude} \times \cos\left[\text{frequency}\,(x - \text{horizontal translation})\right] + \text{vertical translation}$

A negative amplitude means a reflection over the x-axis. The vertical translation may be kept in front only to prevent the ambiguity that the number may be part of the trigonometric function.

Frequency and Period

The following statements are true regarding the relationship between frequency and period.

$\text{frequency} = \dfrac{2\pi}{\text{period}}$

$\text{period} = \dfrac{2\pi}{\text{frequency}}$

Sine and Cosine

The sine function follows the form zero, maximum, zero, minimum, before any translation.

The cosine function follows the form maximum, zero, minimum, zero, before any translation.

As a result, sine and cosine are horizontal translations of each other.

Double and Half Angle Formulas

Double Angle Formulas

The following are determined by plugging in an angle twice into the sum formulas.

Sine

$\sin 2A = 2 \sin A \cos A$

Cosine

$\cos 2A = \cos^2 A - \sin^2 A$

More Cosine Formulas However, more formulas for the double of an angle with cosine can be determined since $\sin^2 A + \cos^2 A = 1$ (first Pythagorean identity).

$\cos 2A = 1 - 2 \sin^2 A$

$\cos 2A = 2 \cos^2 A - 1$

Tangent

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Half Angle Formulas

Sine

$$\sin \frac{1}{2}A = \pm\sqrt{\frac{1 - \cos A}{2}}$$

Cosine

$$\cos \frac{1}{2}A = \pm\sqrt{\frac{1 + \cos A}{2}}$$

Tangent

$$\tan \frac{1}{2}A = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$$