Math Trigonometry Review

CHSN Review Project

This review guide was written by Dara Adib. It was designed for a test on trigonometry administered by Jeanine Lennon to her Math 12H (4H/Precalculus) class on February 12, 2008, but also applies to trigonometry material in Math 11H (3H).

This is a development version of the text that should be considered a work-in-progress.

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Angles and Sectors of Circles

Radians

- To convert degrees to radians, multiply by $\frac{\pi}{180}$.
- To convert radians to degrees, multiply by $\frac{180}{\pi}$.
- One radian equals $\frac{180}{\pi}$ or approximately $57.296^\circ$.

Coterminal Angles

coterminal angles different angles that have the same initial and terminal ray

The differences between coterminal angles are multiples of $360^\circ$.

Examples: $60^\circ$, $-300^\circ$, $420^\circ$

Sectors

sector part of a circle formed by two radii and an arc

$s = r\theta$ (arc length = radius $\times$ angle)

The angle must be represented in radians.

apparent size the angle that an object subtends at one’s eyes; this explains why an object appears to be smaller when farther away

Trigonometric Functions

Reciprocal Functions

The following are reciprocal functions.

- $\csc x = \frac{1}{\sin x}$
- $\sec x = \frac{1}{\cos x}$
- $\cot x = \frac{1}{\tan x}$

As a result, the following are true as well.

- $\sin x = \frac{1}{\csc x}$
- $\cos x = \frac{1}{\sec x}$
- $\tan x = \frac{1}{\cot x}$
Cofunctions

The following are cofunctions.

- $\sin x = \cos(90 - x)$
- $\tan x = \cot(90 - x)$
- $\sec x = \csc(90 - x)$

As a result, the following are true as well.

- $\cos x = \sin(90 - x)$
- $\cot x = \tan(90 - x)$
- $\csc x = \sec(90 - x)$

Special Angles

Quadrantal Angles

**quadrantal angles** angles that have a terminal side coinciding with a coordinate axis

The value of the trigonometric function (i.e. sine, cosine, tangent) is determined by the coordinates of the points on a unit circle. By definition, the point $(x, y)$ on a unit circle corresponds to $(\cos \theta, \sin \theta)$.

As a result, quadrantal angles can be determined easily. Table 1 lists quadrantal angles and the values of trigonometric functions. Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, the tangent function of an angle can be found by dividing the sine function of the angle by the cosine function of the angle.
Quadrant | Reference Angle (β)  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>β = θ</td>
</tr>
<tr>
<td>II</td>
<td>β = 180° − θ</td>
</tr>
<tr>
<td>III</td>
<td>β = θ − 180°</td>
</tr>
<tr>
<td>IV</td>
<td>β = 360° − θ</td>
</tr>
</tbody>
</table>

Table 2: reference angles

**Angles greater than 90°**

Angles greater than 90° may be in different quadrants. The trigonometric functions vary over whether the trigonometric function of a certain angle is positive or negative. It is useful to remember the mnemonic device “All Students Take Calculus.”

- All trigonometric functions are positive in the first quadrant
- Sine and cosecant are positive in the second quadrant
- Tangent and cotangent are positive in the third quadrant
- Cosine and secant are positive in the fourth quadrant
- The remaining trigonometric functions in each quadrant are negative.

**Reference Angles**

The use of reference angles is a way to simplify the calculation of the values of trigonometric functions in different quadrants. Refer to Table 2, where θ is the angle (in degrees) and β is the reference angle.

**Sum and Difference Formulas**

**Sum Formulas**

**Sine**

\[ \sin(A + B) = \sin A \cos B + \cos A \sin B \]

**Cosine**

\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \]

**Tangent**

The derivation for the tangent of the sum of two angles follows.

\[ \tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} \]
\[
\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}
\]

\[
\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}
\]

\[
\tan A + \tan B
\]

\[
\frac{1}{1 - \tan A \tan B}
\]

**Difference Formulas**

**Sine**

\[
\sin(A - B) = \sin A \cos B - \cos A \sin B
\]

**Cosine**

\[
\cos(A - B) = \cos A \cos B + \sin A \sin B
\]

**Tangent**

The derivation for the tangent of the difference of two angles follows.

\[
\tan(A - B)
\]

\[
\frac{\sin(A - B)}{\cos(A - B)}
\]

\[
\frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}
\]

\[
\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}
\]

\[
\tan A - \tan B
\]

\[
\frac{1}{1 + \tan A \tan B}
\]

**Identities**

**Pythagorean Identities**

1. \(\sin^2 \theta + \cos^2 \theta = 1\)
2. \(1 + \tan^2 \theta = \sec^2 \theta\)
3. \(1 + \cot^2 \theta = \csc^2 \theta\)
**Quotient Identities**

1. \( \tan \theta = \frac{\sin \theta}{\cos \theta} \)
2. \( \cot \theta = \frac{\cos \theta}{\sin \theta} \)

**Graphing Trigonometric Functions**

**General Form**

The general form of a trigonometric function is one of the following (sine or cosine).

\[ y = a \sin b(x \pm c) \pm d \]

\[ y = a \cos b(x \pm c) \pm d \]

In other words, it can be understood to mean the following.

\[ y = \text{amplitude} \times \sin \left[ \text{frequency} \times (x - \text{horizontal translation}) \right] + \text{vertical translation} \]

\[ y = \text{amplitude} \times \cos \left[ \text{frequency} \times (x - \text{horizontal translation}) \right] + \text{vertical translation} \]

A negative amplitude means a reflection over the \( x \)-axis. The vertical translation may be kept in front only to prevent the ambiguity that the number may be part of the trigonometric function.

**Frequency and Period**

The following statements are true regarding the relationship between frequency and period.

\[ \text{frequency} = \frac{2\pi}{\text{period}} \]

\[ \text{period} = \frac{2\pi}{\text{frequency}} \]

**Sine and Cosine**

The sine function follows the form zero, maximum, zero, minimum, before any translation.
The cosine function follows the form maximum, zero, minimum, zero, before any translation.
As a result, sine and cosine are horizontal translations of each other.

**Double and Half Angle Formulas**

**Double Angle Formulas**

The following are determined by plugging in an angle twice into the sum formulas.
Sine
\[ \sin 2A = 2 \sin A \cos A \]

Cosine
\[ \cos 2A = \cos^2 A - \sin^2 A \]

More Cosine Formulas  However, more formulas for the double of an angle with cosine can be determined since \( \sin^2 A + \cos^2 A = 1 \) (first Pythagorean identity).

\[ \cos 2A = 1 - 2 \sin^2 A \]
\[ \cos 2A = 2 \cos^2 A - 1 \]

Tangent
\[ \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \]

Half Angle Formulas

Sine
\[ \sin \frac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{2}} \]

Cosine
\[ \cos \frac{1}{2}A = \pm \sqrt{\frac{1 + \cos A}{2}} \]

Tangent
\[ \tan \frac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \]