Random tensor networks and holographic duality

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Outline

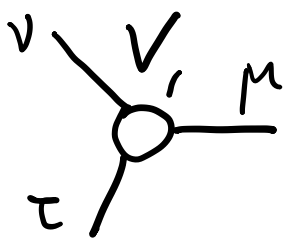
- Background: tensor networks and holographic duality
- Definition of random tensor networks
- Entanglement properties of random tensor networks
- Consequences in holography
  --Ryu-Takayanagi formula with quantum corrections
  --Quantum error correction properties of the holographic mapping
  --Correspondence in correlation functions
- Finite D corrections

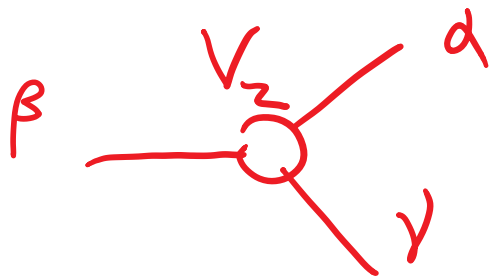
Reference

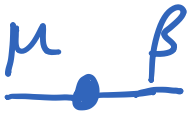
- Patrick Hayden, Sepehr Nezami, Xiao-Liang Qi, Nathaniel Thomas, Michael Walter, Zhao Yang, arxiv: 1601.01694

Tensor networks

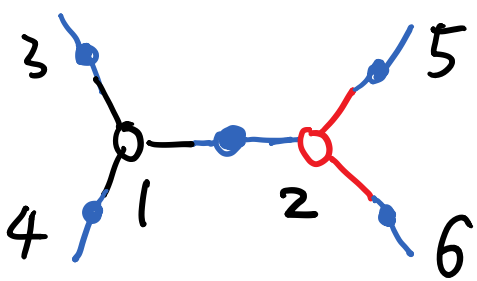
- Building many-body entangled states from few-qubit building blocks.
- Projected Entangled Pair States (PEPS) F. Verstraete, J.I. Cirac, 04'







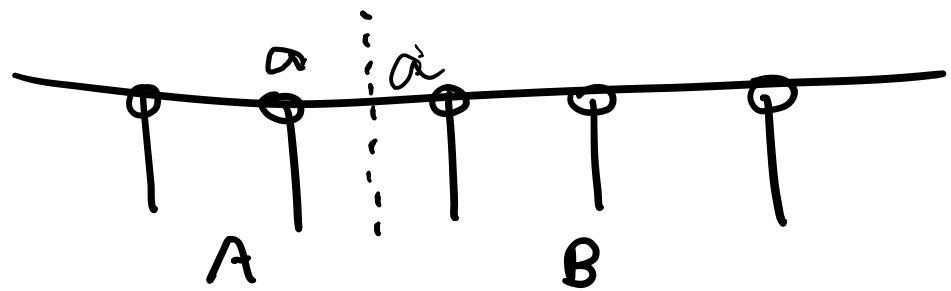
$T_{1\mu\nu\tau}|\mu\nu\tau\rangle = |V_1\rangle$ $T_{2\alpha\beta\gamma}|\alpha\beta\gamma\rangle = |V_2\rangle$ $|L\rangle = g^{\mu\beta}|\mu\beta\rangle$



$$|\Psi\rangle = \langle V_1|\langle V_2|L_{12}\rangle|L_{13}\rangle|L_{14}\rangle|L_{25}\rangle|L_{26}\rangle$$

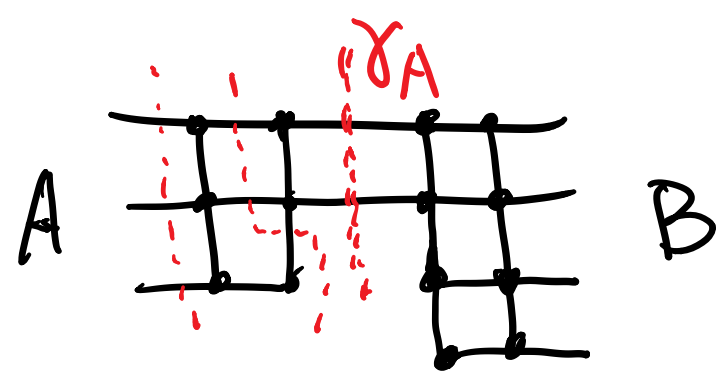
Tensor networks

- Advantage of tensor networks: geometrical manifestation of the entanglement structure
- $S_A \leq \log D_{min}(A)$
- or $S_A \leq |\gamma_A| \log D$. $|\gamma_A|$ is the minimal surface area bounding A region.

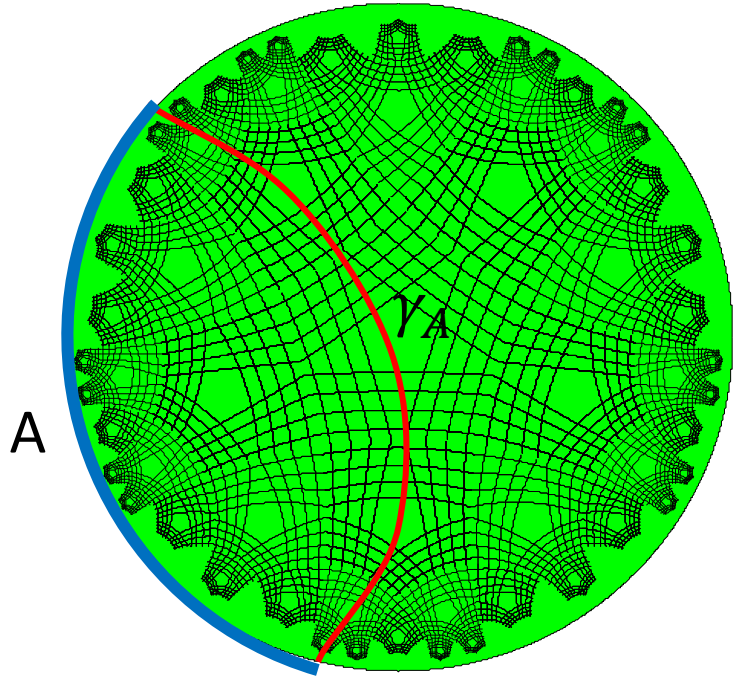


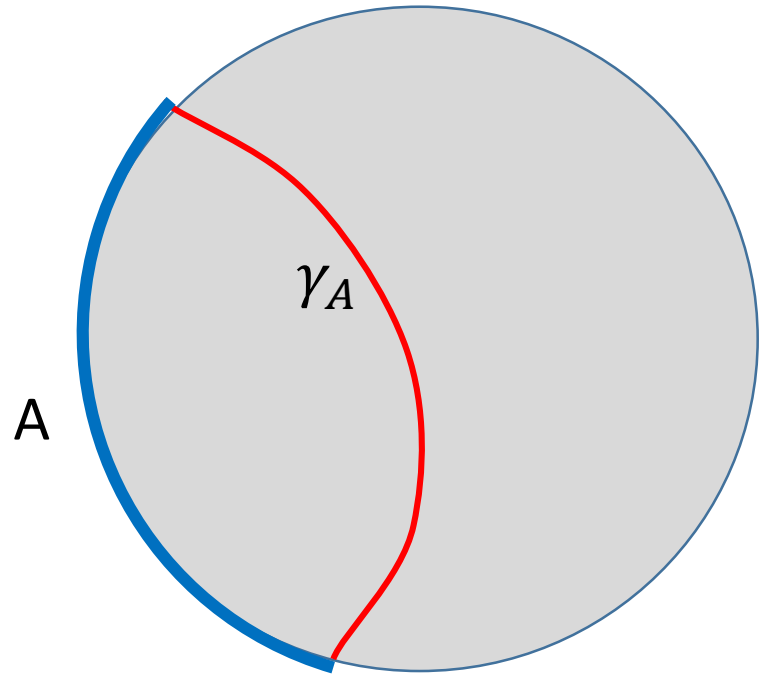
$$|\Psi\rangle = \sum_{a=1}^{D} |\Psi_a^A\rangle |\Psi_a^B\rangle$$

$$\Rightarrow rank(\rho_A) \leq D, S_A \leq \log D$$



Tensor networks and holographic duality





Tensor networks

$S \leq |\gamma_A| \log D$

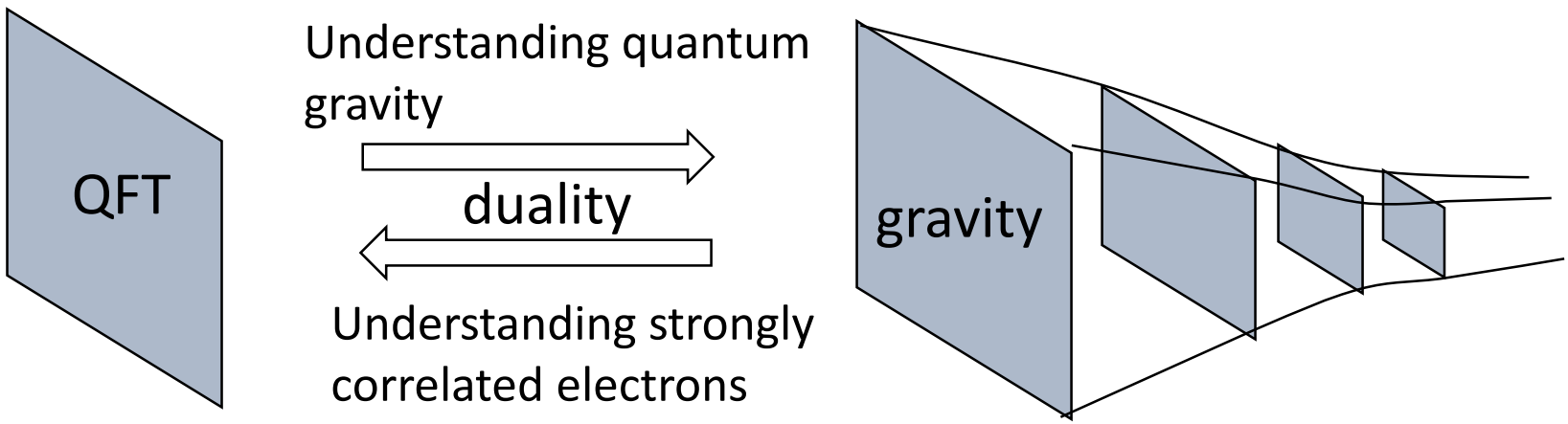
Holographic duality

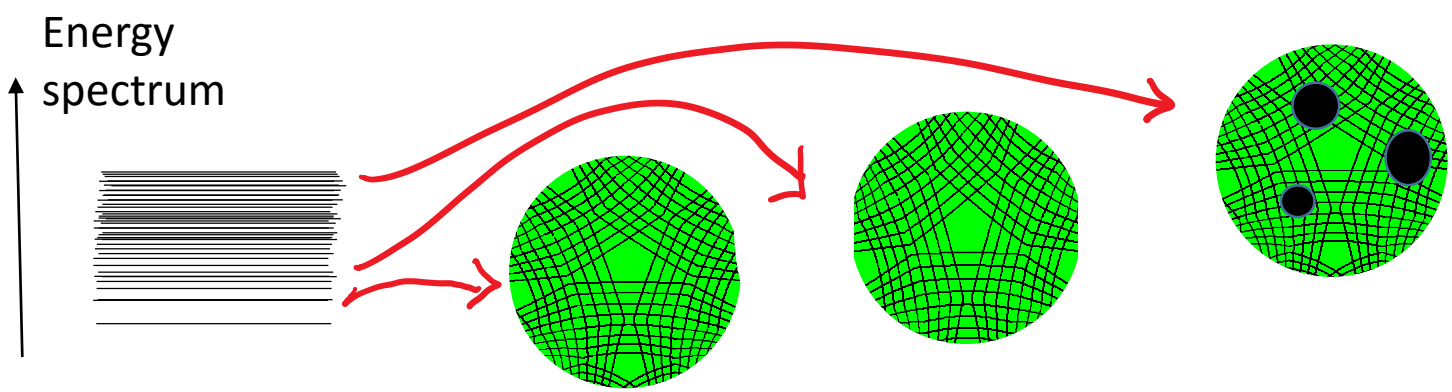
Ryu-Takayanagi formula

$$S = \frac{1}{4\pi G} |\gamma_A|$$

Ryu&Takayanagi '06, Swingle '09, Nozaki et al '12, XLQ '13, Pastawski et al '15, Yang et al '15

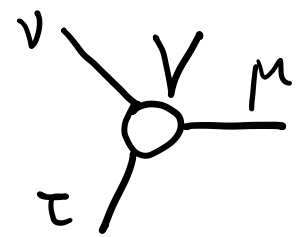
Tensor networks and holographic duality



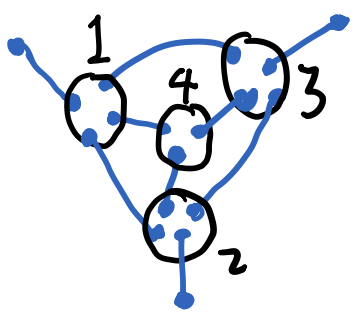


- The tensor network proposal: geometry emerges from the entanglement structure of quantum states
- How to proceed with this idea?

Random tensor networks



- Random tensor networks turns out to have "optimal" entanglement properties.
- A random tensor $V_{\mu\nu\tau}$ corresponds to a (Haar) random state in the Hilbert space $|V\rangle = V_{\mu\nu\tau}|\mu\rangle|\nu\rangle|\tau\rangle$.
- A random tensor network = EPR pairs with random projections on each site. (Similar to Gutzwiller projection)



$$|\Psi\rangle = \prod_x \langle V_x| \, |P\rangle$$

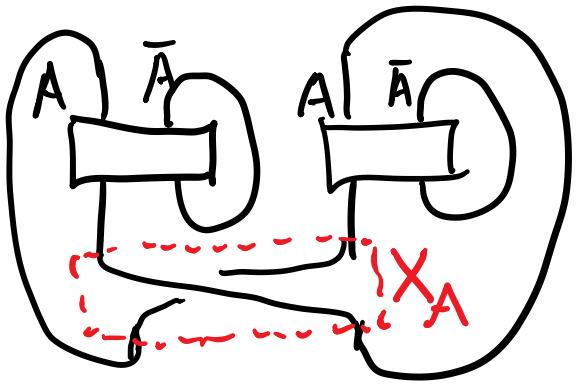
Random vertex states

Link states. EPR pairs or more general

- The link state can be EPR pairs $|\mathrm{P}\rangle = \prod_{xy} |L_{xy}\rangle$ but can also be more general

Entanglement properties of random tensor networks

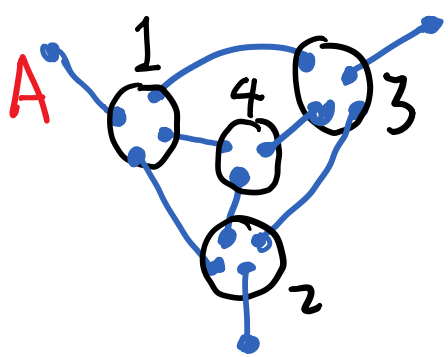
- $\rho = |\Psi\rangle\langle\Psi| = tr(\prod_x |V_x\rangle\langle V_x|\, \rho_P)$ is a linear function of $|V_x\rangle\langle V_x|$.
- Renyi entropies $S_A = \frac{1}{1-n}\log\frac{tr(\rho_A^n)}{tr(\rho)^n}$
- For any quantity that is polynomial in ρ, such as $tr(\rho_A^n)$, the random average can be easily obtained.

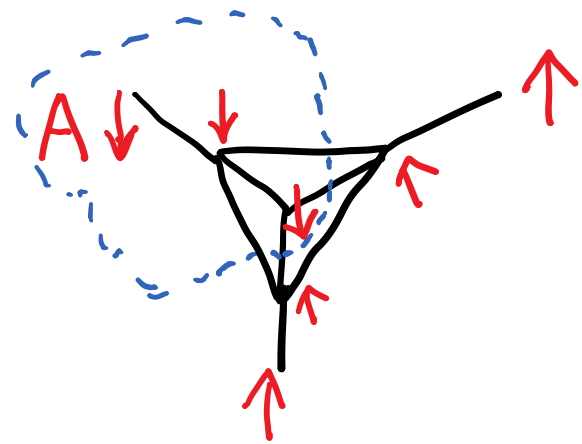


- For example,
  $tr(\rho_A^2) = tr(\rho \otimes \rho X_A)$
  $= tr(\rho_P \otimes \rho_P [X_A \otimes \prod_x |V_x\rangle\langle V_x| \otimes |V_x\rangle\langle V_x|])$
- Random average $\overline{|V_x\rangle\langle V_x| \otimes |V_x\rangle\langle V_x|} = \frac{1}{D_x^2+D_x}(I_x + X_x)$
- Random average ⇔ sum over an Ising variable at each x

Summary of the key results

- For a random tensor network
- $\overline{tr\left(\rho_A^2\right)} = Z_A = \sum_{\{\sigma_x=\pm 1\}} e^{-\mathcal{A}[\{\sigma_x\}]}$
- $\mathcal{A}[\{\sigma_x\}] = S(\{\sigma_x = -1\}; \rho_P)$ "the second Renyi entropy of $\sigma_x = -1$ domain for state $\rho_P = |P\rangle\langle P|$"
- Boundary condition: spin $\downarrow$ in A and $\uparrow$ elsewhere

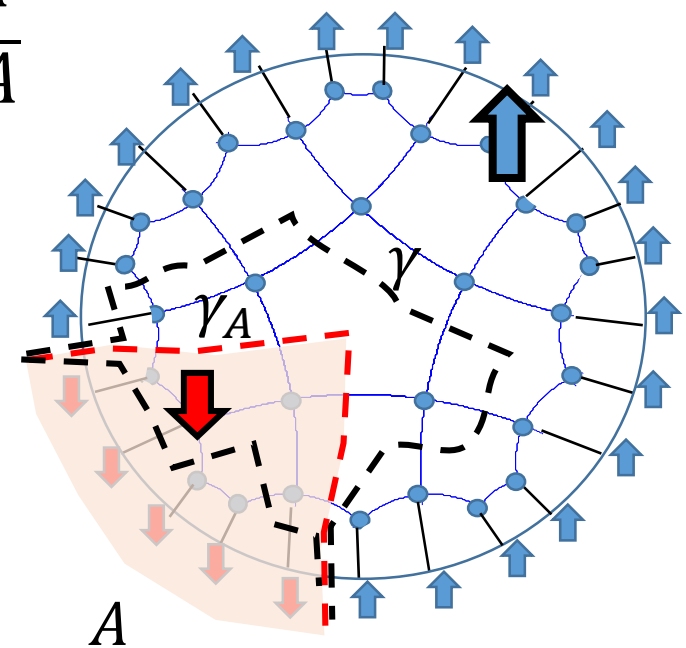




- The second Renyi entropy $S_A \simeq -\log \frac{Z_A}{Z_\emptyset}$ is the "cost of free energy" of flipping spins in A from $\uparrow$ to $\downarrow$.

RT formula

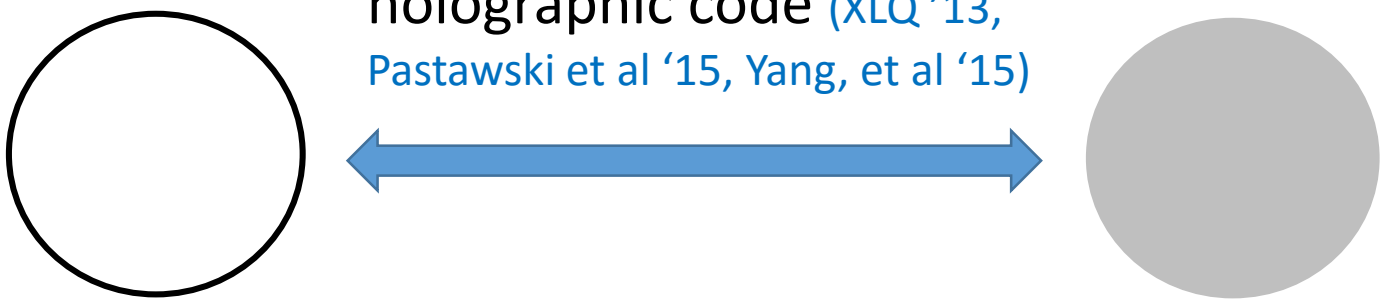
- If $|P\rangle = \prod_{xy} |L_{xy}\rangle$ consists of maximally entangled EPR pairs with rank D,
- $\mathcal{A}[\{\sigma_x\}] = -\frac{1}{2}\log D \sum_{xy} \sigma_x \sigma_y$
- Boundary cond. $\sigma_x = \begin{cases} -1, x \in A \\ +1, x \in \bar{A} \end{cases}$
- The action is proportional to the domain wall area.
- $\overline{tr(\rho_A^2)} = \sum_{\gamma \sim A} e^{-\log D|\gamma|}$
- $D \to \infty \Rightarrow$ low T limit of Ising model
- $\bar{S} \simeq -\log \overline{tr(\rho_A^2)} \simeq \log D\, |\gamma_A|$ (RT formula)



Quantum corrections to the RT formula

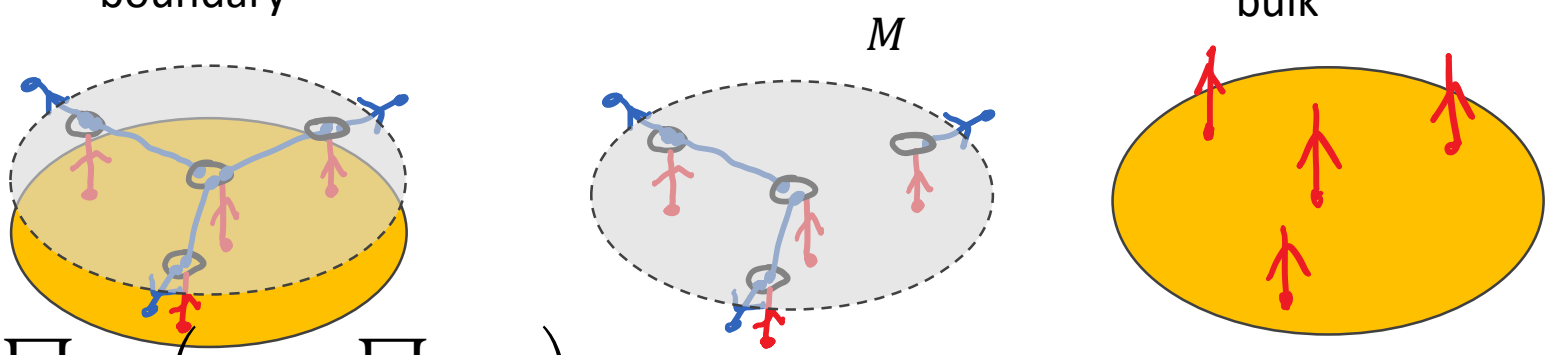
- To understand holographic duality, we need a tensor network description of the **mapping** between the two systems, rather than only a prescription of certain states.

Holographic mapping, or holographic code (XLQ '13, Pastawski et al '15, Yang, et al '15)



boundary

bulk



$$|\Psi_\partial\rangle = \prod_x \langle V_x| \left(|\Psi_b\rangle \otimes \prod_{xy} |L_{xy}\rangle \right) = M|\Psi_b\rangle$$

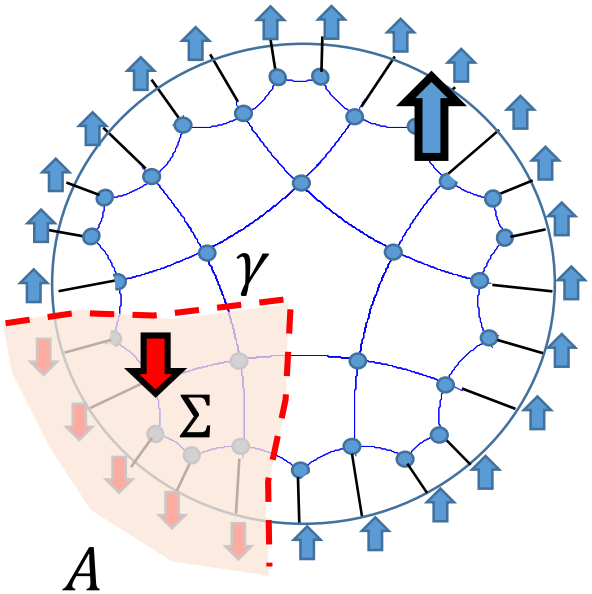
$|\Psi_b\rangle$

Quantum corrections to the RT formula

- Applying our general formula
- $\overline{tr(\rho_A^2)} = Z_{Ising} = \sum_{\{\sigma_x=\pm 1\}} e^{-A[\{\sigma_x\}]}$
- For state $|P\rangle = |\Psi_b\rangle \otimes \prod_{xy} |L_{xy}\rangle$
- $\mathcal{A}(\gamma) = \log D\,|\gamma| + S(\Sigma, |\Psi_b\rangle\langle\Psi_b|)$

Contribution of EPR pairs. RT formula

Contribution of the bulk matter field, or residual bulk entanglement



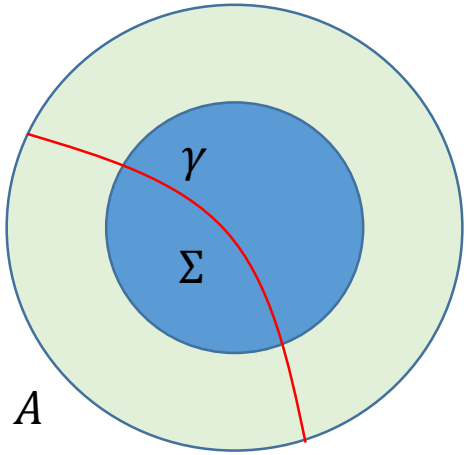
- Large D limit, $S_A \simeq \log D\,|\gamma_A| + S(\mathrm{E}_A, |\Psi_b\rangle\langle\Psi_b|)$
- Consistent with AdS/CFT results (Faulkner, Lewkowycz & Maldacena'13)

Example: Hawking-Page transition

- Entanglement of bulk matter field modifies the minimal surface. This provides a way to see "back reaction" of matter to geometry.
- Example: consider $|\Psi_b\rangle$ to be a random state in a bulk region. The tensor network is a discretization of hyperbolic space.
- Minimize $S = \log D\,|\gamma| + S_{bulk}(\Sigma)$
- Random state

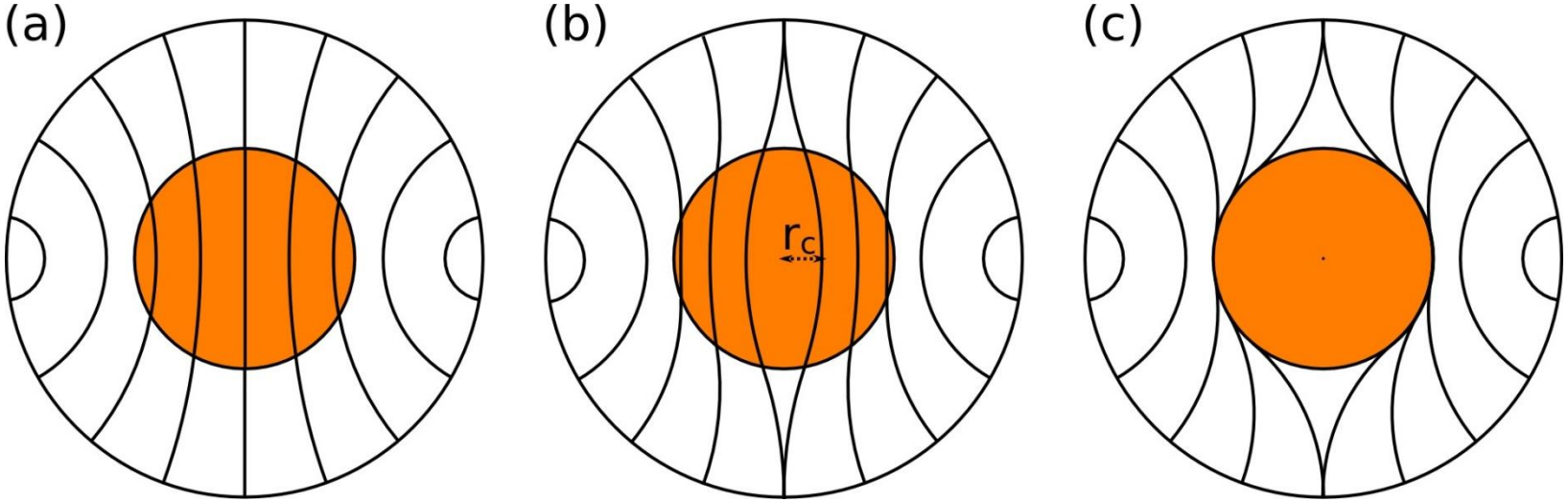
$$S_{bulk}(\Sigma) = \log \frac{D_b^{V_T}+1}{D_b^{|\Sigma|}+D_b^{V_T-|\Sigma|}}.$$

(E Lubkin & T Lubkin Int J of Theo Phys 32 933 (1993))

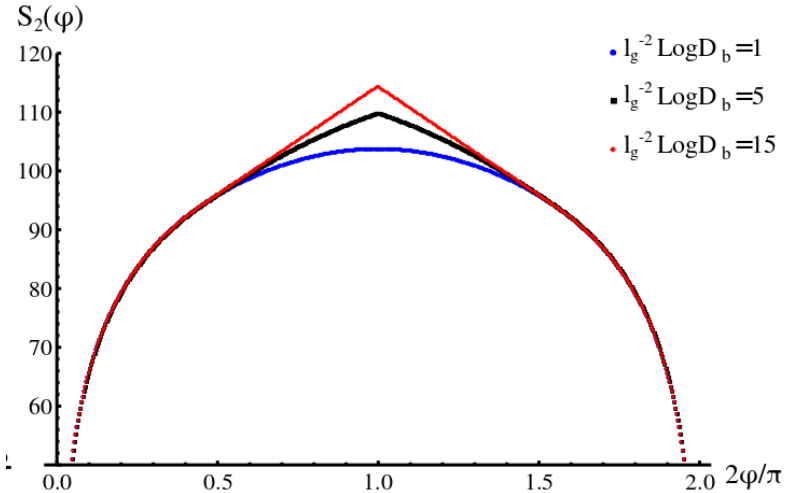


Example: Hawking-Page transition

- Topology of the minimal surfaces change upon increase of bulk state entanglement



- Black-hole formation driven by entanglement
- Black-hole phase ↔ cusp in the entropy curve $S(l)$ of the boundary



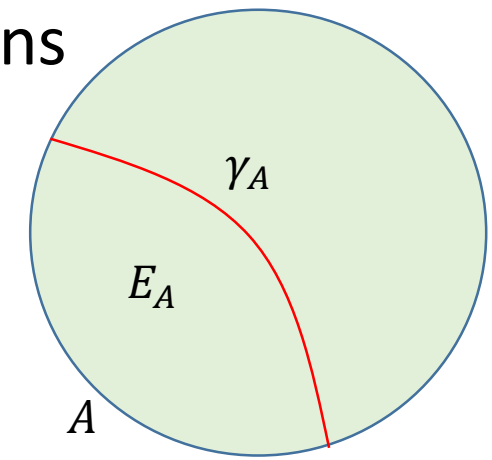
Some general comments

- In the RT formula with quantum corrections

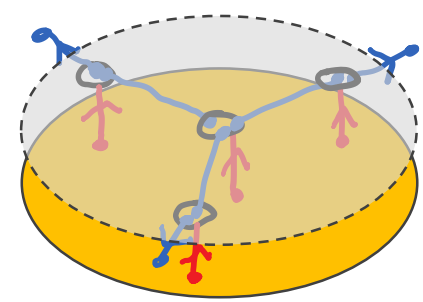
$$S_A \simeq \frac{1}{4G_N}|\gamma_A| + S(E_A, \rho_{bulk})$$

geometry

matter



- In random tensor networks, both terms come from quantum entanglement in the "parent state" $S_A \simeq S(E_A, \rho_P)$ which reduces to the two terms when $\rho_P = \rho_b \otimes \prod_{xy} |L_{xy}\rangle\langle L_{xy}|$.

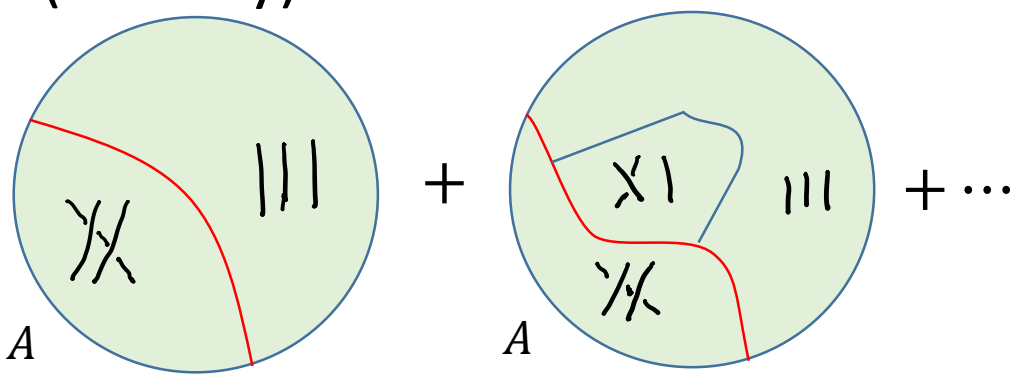


- The separation of the two terms is artificial
- Short-range entanglement ⇒ geometry
  Longer-range entanglement ⇒ matter

Generalization to higher Renyi entropies

- n-th Renyi entropy $S_n(A) = \frac{1}{1-n} \log tr(\rho_A^n)$
- For random tensor network state $\rho = tr_{bulk}(\prod_x |V_x\rangle\langle V_x| \, \rho_P)$. Using $\overline{|V_x\rangle\langle V_x|^{\otimes n}} = C_n^{-1} \sum_{g_x \in S^n} g_x$
- $\overline{tr(\rho_A^n)} = C^{-1} \sum_{\{g_x \in S^n\}} tr\left(\rho_P^{\otimes n} \prod_x g_x\right)$
  with the boundary condition $g_x = \mathrm{C_n}$ (cyclic permutation) in A and $g_x = I$ (identity) elsewhere.

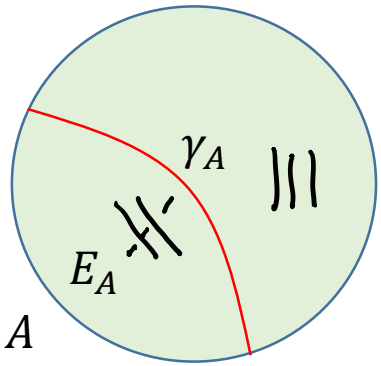
$\overline{tr(\rho_A^n)} =$



- These terms are generally multipartite entanglement measures. (LU invariants Leifer, Linden, Winter PRA 69 052304 (2004))

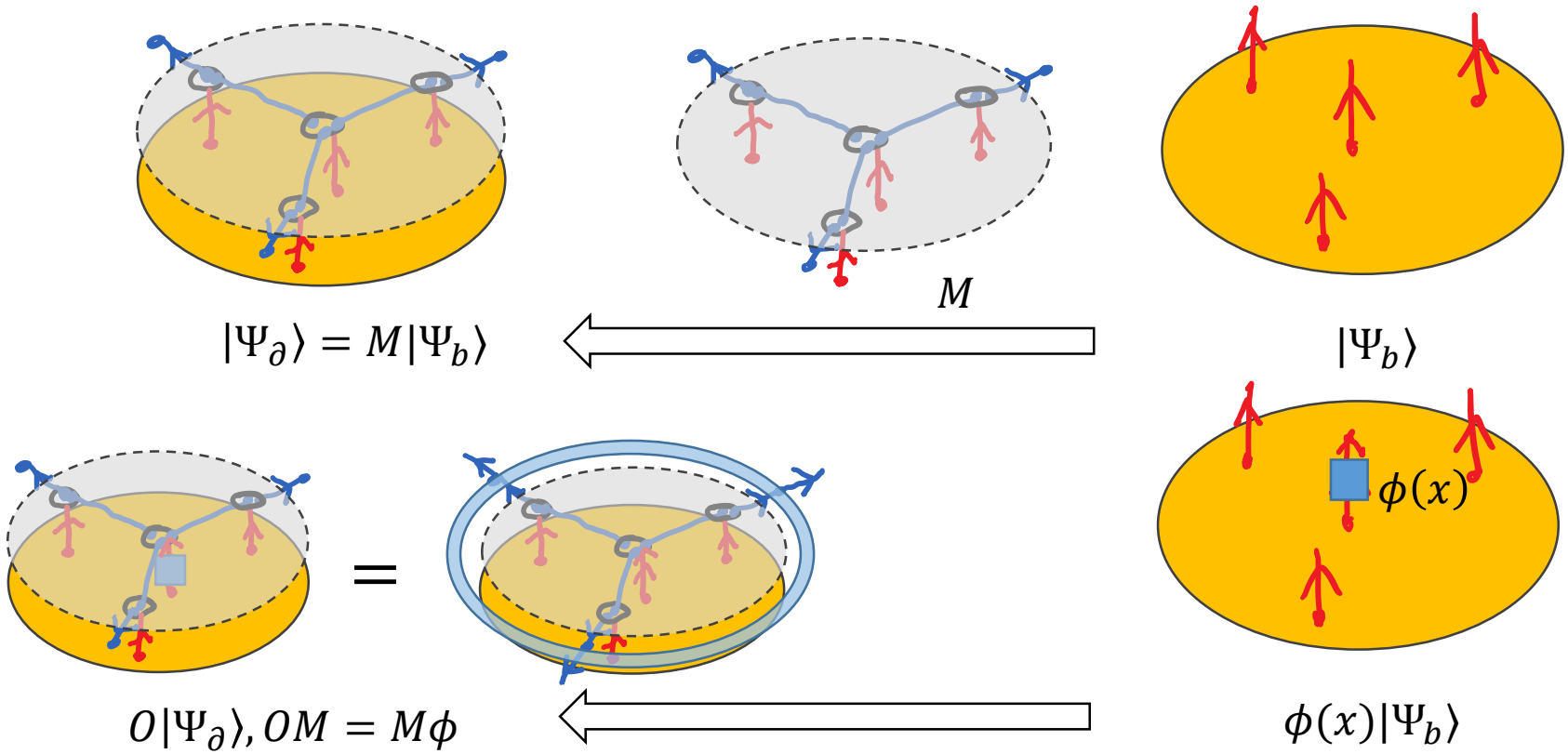
Generalization to higher Renyi entropies

- If $\rho_P = |P\rangle\langle P| = \prod_{xy} |L_{xy}\rangle\langle L_{xy}|$ is made by EPR pairs,
- $\overline{tr\left(\rho_A^n\right)} = \sum_{\{g_x \in S^n\}} e^{-A[\{g_x\}]}$
- $\mathcal{A}[\{g_x\}] = -\log D \sum_{xy} \chi\left(g_x^{-1} g_y\right)$
- $\chi\left(g_x^{-1} g_y\right) =$ # of loops in permutation $g_x^{-1} g_y$.
- In large D limit, $\overline{tr\left(\rho_A^n\right)} \simeq e^{-A_{min}} = e^{(1-n)\log D|\gamma_A|}$
- $\overline{S_n} \simeq \log D\, |\gamma_A|$
- In large D limit, $\overline{S}_n$ is independent from n
- Thus we expect $S_{vN} = \log D\, |\gamma_A|$

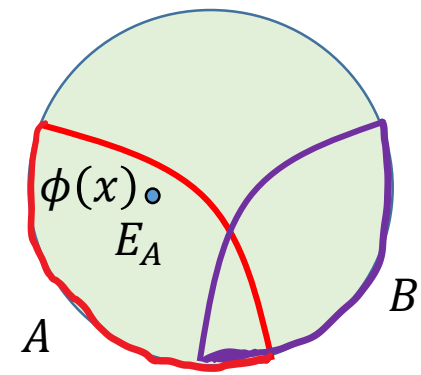


Error correction properties of the holographic mapping

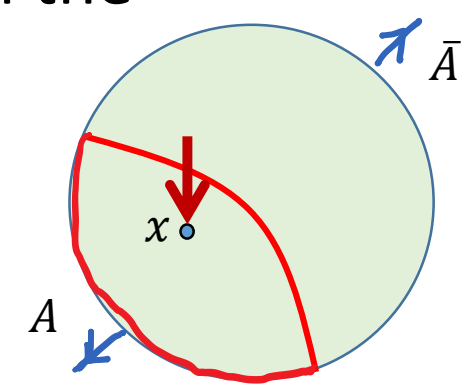
- The holographic mapping defined by a random tensor network also defines a mapping between bulk and boundary operators



Error correction properties of the holographic mapping

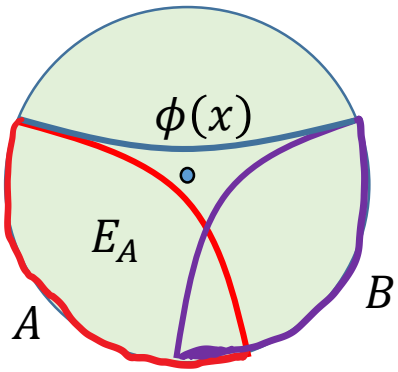


- If we have a small bulk matter field Hilbert space D_b, the correspondence between bulk operator $\phi(x)$ and boundary operator is not one-to-one.
- For region A, there exists O_A supported on A such that $O_A M = M\phi(x)$ if and only if $x \in E_A$ is in the entanglement wedge.



- Proof: the equivalent statement is that the mutual information $I_{x\bar{A}} = S_x + S_{\bar{A}} - S_{x\bar{A}} = 0$
- In the Ising model, this means the boundary fields at x and $\bar{A}$ are uncorrelated.

Error correction properties of the holographic mapping



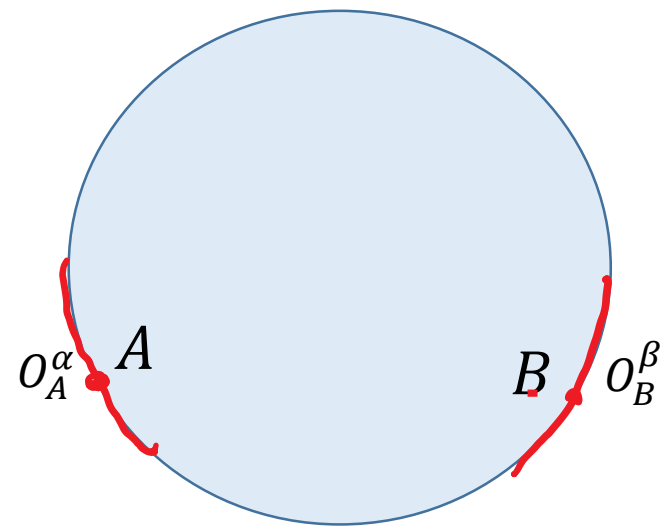
- Example: at the center point x, $\phi(x)$ can be represented on any boundary region with size > ½ system size
- The further away x is from the boundary, the more nonlocal are the operators O corresponding to $\phi(x)$
- This property means the holographic mapping is a *quantum error correction code*. (Almheiri, Dong, Harlow '14) When we measure the state at the center point x, errors made in any region < ½ system size can be corrected.
- This error correction property is important for the holographic duality to be both local (in entanglement wedge) and isotropic.

Bulk-boundary correspondence of correlation functions

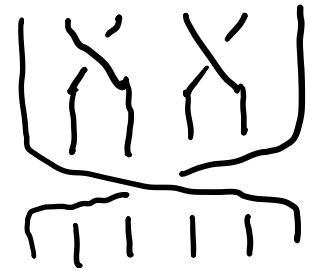
- Our approach applies to more generic entanglement properties (LU invariants) beyond Renyi entropies.
- Boundary two-point functions $\langle \Psi_\partial | O_1 O_2 | \Psi_\partial \rangle$
- We want a LU invariant that characterizes two-point functions
- Define the correlation matrix

$$M_{AB}^{\alpha\beta} = \left\langle \Psi_\partial \left| O_A^\alpha O_B^\beta \right| \Psi_\partial \right\rangle$$

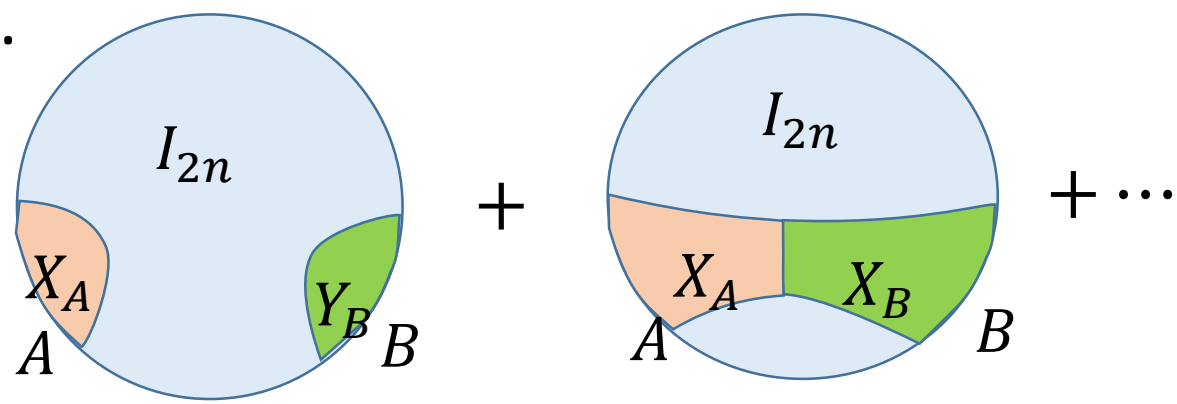
- O_A^α is an orthonormal basis of operators in A region.
- $tr\left((M_{AB}^{+} M_{AB})^n\right)$ are LU invariants



Bulk-boundary correspondence of correlation functions

- $C_{2n}(AB) = tr\left((M_{AB}^{+} M_{AB})^{n}\right) = \mathrm{tr}(|\Psi_{\partial}\rangle\langle\Psi_{\partial}|^{\otimes 2n} \mathrm{X_A Y_B})$
- $X_A =$  $Y_B =$ 

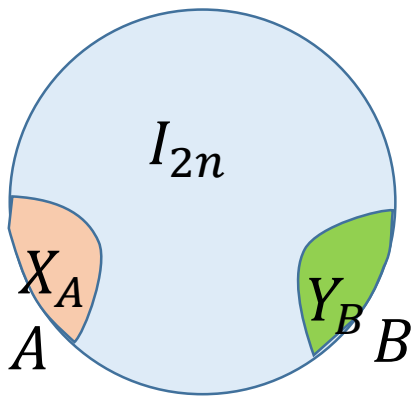
- For random tensor networks, $C_{2n}(AB)$ is a partition function of S_{2n} spin model with boundary conditions X_A and Y_B.

$C_{2n}(AB) =$ 

Bulk-boundary correspondence of correlation functions

- Large D limit

$$C_{2n}(AB) =$$



$$= D^{-n|\gamma_A|-n|\gamma_B|} C_{2n}(E_A E_B)$$

- Singular values of correlation matrices M_{AB} are identical to those of $M_{E_A E_B}$ of the bulk state. i.e. in suitable basis,

$$\langle O_A^n O_B^n \rangle_{boundary} = D^{-\frac{1}{2}(|\gamma_A|+|\gamma_B|)} \times \langle \phi_{E_A}^n \phi_{E_B}^n \rangle_{bulk}$$

- "Scaling dimension gap" of the boundary theory (in the limit of $D \to \infty, D_b$ finite.

Finite D corrections

- We can bound the deviation of $S_n(A)$ away from the RT value

- $\forall \delta, \exists D_c(\delta, V)$ such that for $D \gg D_c$ $\left|S_n(A) - S_n^{RT}(A)\right| < \delta$ holds with probability $1 - \frac{D_c}{D}$

- This conclusion can be proved rigorously for $D_c = \alpha \delta^{-2} e^{c_{2n} V}$
- Under reasonable physical conditions (behavior of Free energy in local spin models), D_c can be improved to $D_c = \alpha' \delta^{-2} V^{\frac{2}{\Delta_{2n}}}$

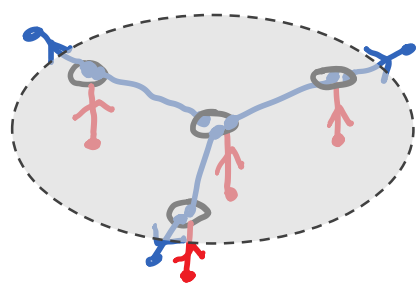
Conclusion

- Random tensor networks are a large class of interesting highly entangled states.

Entanglement measures of RTN → (random average) → Partition functions of classical spin models

- Random tensor networks are (best so far) candidates of tensor network holographic mapping

QFT in d-dimension ↔ (RTN) ↔ Quantum gravity in $(d+1)$-dimension



Open questions

- How to reconcile the difference between AdS/CFT and tensor networks (e.g. n dependence of Renyi entropies)
- How to describe dynamics? (Zhao Yang & XLQ, on-going work)
- Back reaction. Derivation of Einstein equation
- Holography in non-AdS geometries
- Applications of random tensor network states in condensed matter.