Random tensor networks and holographic duality

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Outline

• Background: tensor networks and holographic duality
• Definition of random tensor networks
• Entanglement properties of random tensor networks
• Consequences in holography
  -- Ryu-Takayanagi formula with quantum corrections
  -- Quantum error correction properties of the holographic mapping
  -- Correspondence in correlation functions
• Finite D corrections

Reference

• Patrick Hayden, Sepehr Nezami, Xiao-Liang Qi, Nathaniel Thomas, Michael Walter, Zhao Yang, arxiv: 1601.01694
Tensor networks

- Building many-body entangled states from few-qubit building blocks.
- Projected Entangled Pair States (PEPS)  

\[
T_{1\mu\nu\tau}|\mu\nu\tau\rangle = |V_1\rangle \\
T_{2\alpha\beta\gamma}|\alpha\beta\gamma\rangle = |V_2\rangle \\
|L\rangle = g^{\mu\beta}|\mu\beta\rangle
\]

\[
|\Psi\rangle = \langle V_1|\langle V_2|L_{12}\rangle|L_{13}\rangle|L_{14}\rangle|L_{25}\rangle|L_{26}\rangle
\]

F. Verstraete, J.I. Cirac, 04'}
Tensor networks

- Advantage of tensor networks: geometrical manifestation of the entanglement structure
- $S_A \leq \log D_{\text{min}}(A)$
- or $S_A \leq |\gamma_A| \log D$. $|\gamma_A|$ is the minimal surface area bounding A region.

$|\Psi\rangle = \sum_{a=1}^{D} |\Psi^A_a\rangle|\Psi^B_a\rangle$

$\Rightarrow \text{rank}(\rho_A) \leq D, S_A \leq \log D$
Tensor networks and holographic duality

Tensor networks
\[ S \leq |\gamma_A| \log D \]

Holographic duality
Ryu-Takayanagi formula
\[ S = \frac{1}{4\pi G} |\gamma_A| \]

Ryu&Takayanagi ‘06, Swingle ‘09, Nozaki et al ‘12, XLQ ‘13, Pastawski et al ‘15, Yang et al ‘15
• The tensor network proposal: geometry emerges from the entanglement structure of quantum states

• How to proceed with this idea?
Random tensor networks

- Random tensor networks turns out to have “optimal” entanglement properties.
- A random tensor $V_{\mu\nu\tau}$ corresponds to a (Haar) random state in the Hilbert space $|V\rangle = V_{\mu\nu\tau}|\mu\rangle|\nu\rangle|\tau\rangle$.
- A random tensor network = EPR pairs with random projections on each site. (Similar to Gutzwiller projection)

|\Psi\rangle = \prod_x \langle V_x | |P\rangle

- The link state can be EPR pairs $|P\rangle = \prod_{xy} |L_{xy}\rangle$ but can also be more general
Entanglement properties of random tensor networks

- \( \rho = |\Psi\rangle\langle\Psi| = tr(\prod_x |V_x\rangle\langle V_x| \rho_P) \) is a linear function of \( |V_x\rangle\langle V_x| \).

- Renyi entropies \( S_A = \frac{1}{1-n} \log \frac{tr(\rho_A^n)}{tr(\rho)^n} \)

- For any quantity that is polynomial in \( \rho \), such as \( tr(\rho_A^n) \), the random average can be easily obtained.

- For example,
\[
tr(\rho_A^2) = tr(\rho \otimes \rho X_A) \\
= tr(\rho_P \otimes \rho_P [X_A \otimes \prod_x |V_x\rangle\langle V_x| \otimes |V_x\rangle\langle V_x|])
\]

- Random average \( |V_x\rangle\langle V_x| \otimes |V_x\rangle\langle V_x| = \frac{1}{D_x^2 + D_x} (I_x + X_x) \)

- Random average \( \Leftrightarrow \) sum over an Ising variable at each \( x \)
Summary of the key results

• For a random tensor network

\[ \text{tr} \left( \rho_A^2 \right) = Z_A = \sum_{\{\sigma_x=\pm 1\}} e^{-\mathcal{A}[\{\sigma_x\}]} \]

\[ \mathcal{A}[\{\sigma_x\}] = S(\{\sigma_x = -1\}; \rho_P) \text{ “the second Renyi entropy of } \sigma_x = -1 \text{ domain for state } \rho_P = |P\rangle\langle P| \]

• Boundary condition: spin $\downarrow$ in $A$ and $\uparrow$ elsewhere

• The second Renyi entropy $S_A \approx -\log \frac{Z_A}{Z_\emptyset}$ is the “cost of free energy” of flipping spins in $A$ from $\uparrow$ to $\downarrow$. 
RT formula

- If $|P\rangle = \prod_{xy} |L_{xy}\rangle$ consists of maximally entangled EPR pairs with rank $D$,
  \[ \mathcal{A}[\{\sigma_x\}] = -\frac{1}{2} \log D \sum_{xy} \sigma_x \sigma_y \]

- Boundary cond. $\sigma_x = \begin{cases} -1, & x \in A \\ +1, & x \in \overline{A} \end{cases}$

- The action is proportional to the domain wall area.
  \[ \text{tr}(\rho_A^2) = \sum_{\gamma \sim A} e^{-\log D |\gamma|} \]

- $D \to \infty \Rightarrow$
  - low T limit of Ising model
  \[ S \approx -\log \text{tr}(\rho_A^2) \approx \log D |\gamma_A| \] (RT formula)
Quantum corrections to the RT formula

- To understand holographic duality, we need a tensor network description of the mapping between the two systems, rather than only a prescription of certain states.

Holographic mapping, or holographic code (XLQ ’13, Pastawski et al ‘15, Yang, et al ‘15)

\[ |\Psi_\partial\rangle = \prod_x \langle V_x | \left( |\Psi_b\rangle \otimes \prod_{xy} |L_{xy}\rangle \right) = M |\Psi_b\rangle \]
Quantum corrections to the RT formula

- Applying our general formula
  \[ \text{tr} \left( \rho_A^2 \right) = Z_{\text{Ising}} = \sum \{ \sigma_x = \pm 1 \} \ e^{-A[\{ \sigma_x \}]} \]
- For state \( |P\rangle = |\Psi_b\rangle \otimes \prod_{xy} |L_{xy}\rangle \)
- \( \mathcal{A}(\gamma) = \log D \ |\gamma| + S(\Sigma, |\Psi_b\rangle\langle\Psi_b|) \)

- Large \( D \) limit, \( S_A \approx \log D \ |\gamma_A| + S(E_A, |\Psi_b\rangle\langle\Psi_b|) \)
- Consistent with AdS/CFT results \( (\text{Faulkner, Lewkowycz & Maldacena'13}) \)
Example: Hawking-Page transition

- Entanglement of bulk matter field modifies the minimal surface. This provides a way to see “back reaction” of matter to geometry.
- Example: consider $|\Psi_b\rangle$ to be a random state in a bulk region. The tensor network is a discretization of hyperbolic space.
- Minimize $S = \log D |\gamma| + S_{bulk}(\Sigma)$
- Random state

\[ S_{bulk}(\Sigma) = \log \frac{D_b^{VT+1}}{D_b^{\Sigma} + D_b^{VT-\Sigma}}. \]

(E Lubkin & T Lubkin Int J of Theo Phys 32 933 (1993))
Example: Hawking-Page transition

- Topology of the minimal surfaces change upon increase of bulk state entanglement

- Black-hole formation driven by entanglement

- Black-hole phase ↔ cusp in the entropy curve $S(l)$ of the boundary
Some general comments

- In the RT formula with quantum corrections
  \[ S_A \approx \frac{1}{4G_N} |\gamma_A| + S(E_A, \rho_{bulk}) \]

- In random tensor networks, both terms come from quantum entanglement in the "parent state" \( S_A \approx S(E_A, \rho_P) \) which reduces to the two terms when \( \rho_P = \rho_b \otimes \prod_{xy} |L_{xy}\rangle \langle L_{xy}| \).

- The separation of the two terms is artificial

- Short-range entanglement \( \Rightarrow \) geometry
  Longer-range entanglement \( \Rightarrow \) matter
Generalization to higher Renyi entropies

- $n$-th Renyi entropy $S_n(A) = \frac{1}{1-n} \log \text{tr}(\rho_A^n)$

- For random tensor network state $\rho = \text{tr}_{\text{bulk}}(\prod_x |V_x\rangle\langle V_x| \rho_P)$. Using $|V_x\rangle\langle V_x| \otimes^n = C_n^{-1} \sum_{g_x \in S^n} g_x$

- $\text{tr}(\rho_A^n) = C^{-1} \sum_{\{g_x \in S^n\}} \text{tr} (\rho_P^{\otimes n} \prod_x g_x)$

  with the boundary condition $g_x = C_n$ (cyclic permutation) in $A$ and $g_x = I$ (identity) elsewhere.

- These terms are generally multipartite entanglement measures. (LU invariants Leifer, Linden, Winter PRA 69 052304 (2004))
Generalization to higher Renyi entropies

- If $\rho_P = |P\rangle\langle P| = \prod_{xy} |L_{xy}\rangle\langle L_{xy}|$ is made by EPR pairs,

- $tr(\rho_A^n) = \sum_{\{g_x\in S^n\}} e^{-A[\{g_x\}]}$

- $A[\{g_x\}] = -\log D \sum_{xy} \chi(g_x^{-1}g_y)$

- $\chi(g_x^{-1}g_y) = \# \text{ of loops in permutation } g_x^{-1}g_y$.

- In large $D$ limit, $tr(\rho_A^n) \approx e^{-A_{\text{min}}} = e^{(1-n)\log D||\gamma_A||}$

- $S_n \approx \log D ||\gamma_A||$

- In large $D$ limit, $S_n$ is independent from $n$

- Thus we expect $S_{vN} = \log D ||\gamma_A||$
Error correction properties of the holographic mapping

- The holographic mapping defined by a random tensor network also defines a mapping between bulk and boundary operators

\[ |\Psi_\partial\rangle = M |\Psi_b\rangle \]

\[ \Omega |\Psi_\partial\rangle, \Omega M = M \phi \]

\[ \phi(x) |\Psi_b\rangle \]
Error correction properties of the holographic mapping

- If we have a small bulk matter field Hilbert space $D_b$, the correspondence between bulk operator $\phi(x)$ and boundary operator is not one-to-one.
- For region $A$, there exists $O_A$ supported on $A$ such that $O_A M = M \phi(x)$ if and only if $x \in E_A$ is in the entanglement wedge.
- Proof: the equivalent statement is that the mutual information $I_{x\bar{A}} = S_x + S_{\bar{A}} - S_{x\bar{A}} = 0$
- In the Ising model, this means the boundary fields at $x$ and $\bar{A}$ are uncorrelated.
Error correction properties of the holographic mapping

- Example: at the center point $x$, $\phi(x)$ can be represented on any boundary region with size $> \frac{1}{2}$ system size.
- The further away $x$ is from the boundary, the more nonlocal are the operators $O$ corresponding to $\phi(x)$.
- This property means the holographic mapping is a quantum error correction code. (Almheiri, Dong, Harlow ‘14)

When we measure the state at the center point $x$, errors made in any region $< \frac{1}{2}$ system size can be corrected.

- This error correction property is important for the holographic duality to be both local (in entanglement wedge) and isotropic.
Bulk-boundary correspondence of correlation functions

- Our approach applies to more generic entanglement properties (LU invariants) beyond Renyi entropies.
- Boundary two-point functions \( \langle \Psi_\partial | O_1 O_2 | \Psi_\partial \rangle \)
- We want a LU invariant that characterizes two-point functions
- Define the correlation matrix
  \[
  M_{AB}^{\alpha\beta} = \left\langle \Psi_\partial | O_A^\alpha O_B^\beta | \Psi_\partial \right\rangle
  \]
- \( O_A^\alpha \) is an orthonormal basis of operators in \( A \) region.
- \( tr \left( (M_{AB}^+ M_{AB})^n \right) \) are LU invariants
Bulk-boundary correspondence of correlation functions

\[ C_{2n}(AB) = tr\left( (M_{AB}^+ M_{AB})^n \right) = tr(\lvert \Psi_\partial \rangle \langle \Psi_\partial \lvert \otimes^{2n} X_A Y_B) \]

\[ X_A = \bigotimes X \bigotimes X \bigotimes X \bigotimes X \quad Y_B = \]

\[ For \ random \ tensor \ networks, \ C_{2n}(AB) \ is \ a \ partition \ function \ of \ S_{2n} \ spin \ model \ with \ boundary \ conditions \ X_A \ and \ Y_B. \]

\[ C_{2n}(AB) = I_{2n} + \]

\[ + \]

\[ + \cdots \]
Bulk-boundary correspondence of correlation functions

- Large $D$ limit

\[ C_{2n}(AB) = \begin{vmatrix} I_{2n} & \end{vmatrix} = D^{-n|\gamma_A| - n|\gamma_B|} C_{2n}(E_A E_B) \]

- Singular values of correlation matrices $M_{AB}$ are identical to those of $M_{E_A E_B}$ of the bulk state. i.e. in suitable basis,

\[ \langle O_A^n O_B^n \rangle_{\text{boundary}} = D^{-\frac{1}{2}(|\gamma_A| + |\gamma_B|)} \times \langle \phi_{E_A}^n \phi_{E_B}^n \rangle_{\text{bulk}} \]

- “Scaling dimension gap” of the boundary theory (in the limit of $D \to \infty$, $D_b$ finite.)
Finite $D$ corrections

• We can bound the deviation of $S_n(A)$ away from the RT value

\[ \forall \delta, \exists D_c(\delta, V) \text{ such that for } D \gg D_c \]
\[ |S_n(A) - S_n^{RT}(A)| < \delta \text{ holds with probability } 1 - \frac{D_c}{D} \]

• This conclusion can be proved rigorously for $D_c = \alpha \delta^{-2} e^{c2nV}$

• Under reasonable physical conditions (behavior of Free energy in local spin models), $D_c$ can be improved to

\[ D_c = \alpha' \delta^{-2} V^{\frac{2}{\Delta 2n}} \]
Conclusion

- Random tensor networks are a large class of interesting highly entangled states.

- Random tensor networks are (best so far) candidates of tensor network holographic mapping.
Open questions

• How to reconcile the difference between AdS/CFT and tensor networks (e.g. $n$ dependence of Renyi entropies)

• How to describe dynamics? (Zhao Yang & XLQ, on-going work)

• Back reaction. Derivation of Einstein equation

• Holography in non-AdS geometries

• Applications of random tensor network states in condensed matter.