Econ 140 - Spring 2016
Section 2

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1 Normal Distribution

Exercise 1.1. (From Section 2 Notes, Question 2a, 2b, 2c) In a population, $X \sim N(50, 100)$. Calculate:

(a) $P(X \leq 60)$

$$P(X \leq 60) = P\left(\frac{X-50}{\sqrt{100}} \leq \frac{60-50}{\sqrt{100}}\right) = P\left(Z \leq \frac{10}{10}\right) = P(Z \leq 1) = \boxed{0.8413}$$

from the normal table (Appendix Table 1 of textbook)

(b) $P(X > 90)$

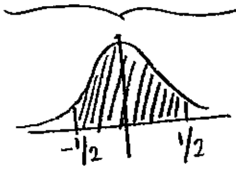
$$P(X > 90) = P\left(\frac{X-50}{\sqrt{100}} > \frac{90-50}{\sqrt{100}}\right) = P\left(Z > \frac{40}{10}\right) = P(Z > 4) = 1 - \underbrace{P(Z \leq 4)}_{\approx 1}$$

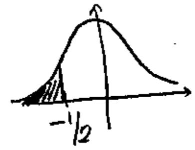
$$\approx 1 - 1 = \boxed{0}$$

(c) $P(45 \leq X \leq 55)$

$$P(45 \leq X \leq 55) = P\left(\frac{45-50}{\sqrt{100}} \leq \frac{X-50}{\sqrt{100}} \leq \frac{55-50}{\sqrt{100}}\right) = P\left(-\frac{5}{10} \leq Z \leq \frac{5}{10}\right)$$

$$= \underbrace{P\left(-\frac{1}{2} \leq Z \leq \frac{1}{2}\right)}_{} = \underbrace{P\left(Z \leq \frac{1}{2}\right)}_{} - \underbrace{P\left(Z \leq -\frac{1}{2}\right)}_{} = 0.6915 - 0.3085 = \boxed{0.383}$$





2 Sample Average (a.k.a. Sample Mean)

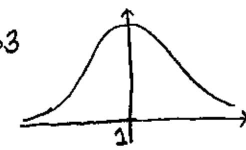
Exercise 2.1. (Stock & Watson, Review the Concepts, Question 2.4) An econometrics class has 80 students, and the mean student weight is 145 lb. A random sample of 4 students is selected from the class, and their average weight is calculated. Will the average weight of the students in the sample equal 145 lb? Why or why not? Use this example to explain why the sample average, $\overline{Y}$, is a random variable.

The average weight of students in the sample will not necessarily be equal to 145 lbs, because of sampling variation. Different groups of 4 students may have different sample averages, hence $\overline{Y}$ is a random variable.

3 Law of Large Numbers (LLN)

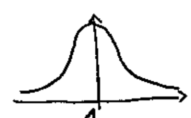
Exercise 3.1. (Stock & Watson, Review the Concepts, Question 2.5) Suppose that $Y_1, \ldots, Y_n$ are i.i.d. random variables, with a $N(1,4)$ distribution. Sketch the probability density of $\overline{Y}$ when $n = 2$. Repeat this for $n = 10$ and $n = 100$. In words, describe how the densities differ. What is the relationship between your answer and the law of large numbers?

- when $n=2$: $E(\overline{Y})=1$, $SD(\overline{Y})=\frac{2}{\sqrt{10}} \approx 0.63$



$\overline{Y}$ is normally distributed because each Y_i is normally distributed

- when $n=100$: $E(\overline{Y})=1$, $SD(\overline{Y})=\frac{2}{\sqrt{100}} = 0.2$



- Essentially, the difference in the densities is that it is tighter around the mean 1 when $n=100$, since $SD(\overline{Y})$ is smaller when n gets larger.
- As n gets large, $SD(\overline{Y})$ gets closer to zero, so the density collapses around the mean. Another way of saying this is that the distribution of $\overline{Y}$ gets more & more concentrated around μ_Y as n gets large, so the probability that $\overline{Y}$ is close to μ_Y goes to 1, which is what LLN says.

4 Central Limit Theorem (CLT)

Exercise 4.1. (Stock & Watson, Exercise 2.17a) Y_i, $i = 1, \ldots, n$ are i.i.d. Bernoulli random variables with $p = 0.4$. Let $\overline{Y}$ denote the sample mean. Use the central limit theorem to compute approximations for (i) $P(\overline{Y} \geq 0.43)$ when $n = 100$, (ii) $P(\overline{Y} \leq 0.37)$ when $n = 400$.

- First, let's solve for $E(Y_i)$ and $Var(Y_i)$.

$E(Y_i) = 1 \cdot 0.4 + 0 \cdot 0.6 = 0.4$, $Var(Y_i) = E(Y_i^2) - E(Y_i) = 0.4^2 - 0.4 = 0.4(1-0.4) = 0.24$

$E(Y_i^2) = 1^2 \cdot 0.4 + 0^2 \cdot 0.6 = 0.4$

- Next, note that CLT assumptions hold because Y_i are iid, $var(Y_i) = 0.24 < \infty$, and n is large.

(i) By CLT ($n=100$ is sufficiently large), $\overline{Y} \sim \mathcal{N}\left(0.4, \frac{0.24}{100}\right)$

$$\Rightarrow P(\overline{Y} \geq 0.43) = 1 - P(\overline{Y} < 0.43) = 1 - P\left(\frac{\overline{Y}-0.4}{\sqrt{\frac{0.24}{100}}} \leq \frac{0.43-0.4}{\sqrt{\frac{0.24}{100}}}\right) = 1 - P(Z < 0.6124) \approx 1 - 0.7291$$

$$= \boxed{0.27}$$

(ii) By CLT, ($n=400$ is sufficiently large), $\overline{Y} \sim^{2} \mathcal{N}\left(0.4, \frac{0.24}{400}\right)$

$$\Rightarrow P(\overline{Y} \leq 0.37) = P\left(\frac{\overline{Y}-0.4}{\sqrt{\frac{0.24}{400}}} \leq \frac{0.37-0.4}{\sqrt{\frac{0.24}{400}}}\right) = P(Z \leq -1.22) = \boxed{0.1112}$$

Note: $\overline{Y}$ is normally distributed for large N by CLT, even if each Y_i is a Bernoulli r.v.

Exercise 4.2. (Stock & Watson, Exercise 2.18) In any year, the weather can inflict storm damage to a home. From year to year, the damage is random. Let Y denote the dollar value of damage in any given year. Suppose that in 95% of the years, $Y = \$0$, but in 5% of the years, $Y = \$20,000$.

(a) What are the mean and standard deviation of the damage in any year?

$E(Y) = 0.95 \cdot 0 + 0.05 \cdot 20{,}000 = \boxed{\$1{,}000}$

$E(Y^2) = 0.95 \cdot 0 + 0.05 \cdot (20{,}000)^2 = \$^2 20\,\text{M}$ "Million"

$\text{Var}(Y) = E(Y^2) - E(Y)^2 = \$^2 20\text{M} - (1{,}000)^2 = \$^2 20\text{M} - \$^2 1\text{M} = \$^2 19\text{M}$

$SD(Y) = \sqrt{\$^2 19\text{M}} = \boxed{\$4359}$

(b) Consider an "insurance pool" of 100 people whose homes are sufficiently dispersed so that, in any year, the damage to different homes can be viewed as independent distributed random variables. Let $\overline{Y}$ denote the average damage to these 100 homes in a year. (i) What is the expected value of the average damage $\overline{Y}$? (ii) What is the probability that $\overline{Y}$ exceeds \$2000?

(i) $E(\overline{Y}) = E\left(\frac{1}{n}\sum Y_i\right) = \frac{1}{n}\sum E(Y_i)$ where $n = 100$ and $E(Y_i) = \$1{,}000$ from part (a)

$\Rightarrow E(\overline{Y}) = \frac{1}{n} \cdot n\, E(Y_i) = E(Y_i) = \boxed{\$1{,}000}$

Also note: $SD(\overline{Y}) = \frac{\sigma_Y}{\sqrt{n}} = \frac{\$4359}{\sqrt{100}} = \$435.9$

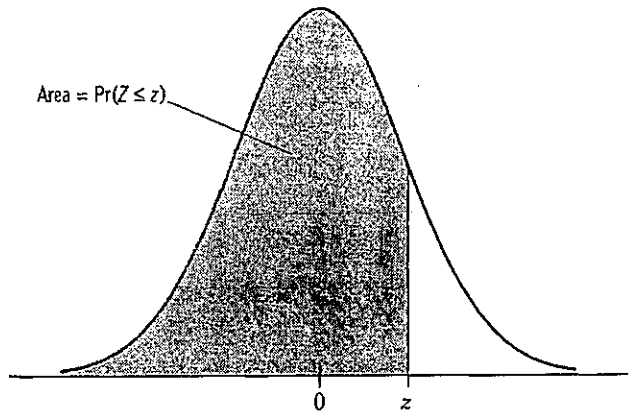
(ii) CLT assumptions hold b/c Y_i are iid, $\text{var}(Y_i) = 19\text{M} < \infty$, and $n = 100$ is sufficiently large. Hence, by CLT, $\overline{Y} \sim \mathcal{N}\left(1{,}000,\ \underbrace{\frac{19\text{M}}{\sqrt{100}}}_{=435.9}\right)$

$\Rightarrow P(\overline{Y} \geq 2000) = 1 - P(\overline{Y} < 2000) = 1 - P\left(\frac{\overline{Y} - 1000}{435.9} < \frac{2000-1000}{435.9}\right)$

$= 1 - P(Z < 2.29) = 1 - 0.9890 = \boxed{0.011}$

Appendix

TABLE 1 The Cumulative Standard Normal Distribution Function, $\Phi(z) = \Pr(Z \le z)$



Second Decimal Value of z

z	0	1	2	3	4	5	6	7	8	9
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611

(Table 1 continued)

(Table 1 continued)

	Second Decimal Value of z									
z	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

This table can be used to calculate $\Pr(Z \leq z)$ where Z is a standard normal variable. For example, when $z = 1.17$, this probability is 0.8790, which is the table entry for the row labeled 1.1 and the column labeled 7.