Arrow's Theorem on Single-Peaked Domains

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Abstract

For any weakly Paretian preference aggregation rule defined on the domain of all single-peaked preferences over a finite set of at least 3 alternatives, satisfaction of independence of irrelevant alternatives implies that the preference aggregation rule is neutral (*i.e.*, it does not depend upon the labels of the alternatives). The result is briefly related to the study of political institutions by pointing out several institutional features that violate neutrality, including bicameralism, gatekeeping powers, supermajority requirements, and veto power.

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1 Collective Rationality and Neutrality

It is not too strong to argue that the following theorem, Arrow's Possibility Theorem, is the foundation of the modern analytical study of political institutions.

Theorem 1 (Arrow [1951]) *If there are three or more alternatives and at least two individuals, each of whom may have any preference over the alternatives, then the only Pareto efficient preference aggregation rule that satisfies independence of irrelevant alternatives is dictatorial.*

Black's median voter theorem is as well-known as Arrow's Theorem.

Theorem 2 (Black [1948]) *If individual preferences are single-peaked, then majority rule is a non-dictatorial preference aggregation rule that satisfies independence of irrelevant alternatives.*

Obviously, the difference between the theorems of Arrow and Black is the degree of heterogeneity that individual preferences may exhibit. Of course, Black's theorem does *not* state that all preference aggregation rules make sense when preferences are known to be single-peaked. Furthermore, Black's theorem as an existence and characterization result does not go particularly far, as majority rule is a precise institutional form.[1] In this paper, we show that any weakly Paretian preference aggregation rule that is independent of irrelevant alternatives must be neutral even when preferences are known to be single-peaked. In other words, even in instances in which there is a well-defined, transitive majority preference relation, neutrality is required for collective choice to be simultaneously weakly Paretian and independent of irrelevant alternatives. The key to the result is that single-peakedness is not "enough" information about the alternatives. To know with certainty that the alternatives "can be ordered" (by the voters' preferences) is not equivalent to knowing *how* the alternatives will be ordered.

Given the generality and power of Arrow's Theorem, the novelty of our results lies in their application to *the* canonical setting for models of political institutions: the unidimensional spatial model.[2] The next Section defines the (canonical) theoretical framework and proves the paper's main result. In Section 2, we offer a brief discussion of the connections between our results and the analysis of political institutions.

[1] For example, May [1952] famously demonstrated that majority rule is equivalent to neutrality, anonymity, and a monotonicity requirement. See also Moulin [1980].

[2] A smattering of examples to justify the term "canonical" might include Downs [1957], Davis et al. [1970], McCubbins et al. [1994], Poole and Rosenthal [1997], and Krehbiel [1998] among, of course, many others.

1.1 Notation and definitions

There is a finite collection of K *alternatives* (or *policies*), X, and a finite collection of n *individuals* (or *voters*) N. We assume that $K \geq 3$ and $n \geq 2$. Individual i's preferences are represented by a reflexive, transitive and complete binary relation R_i. The notation xR_iy implies that i weakly prefers x to y, xP_iy implies that i strictly prefers x to y, and xI_iy implies i is indifferent between x and y. We write $M(R_i)$ to denote the maximal element(s) of R_i: $x \in M(R_i) \Leftrightarrow xR_iy \forall y \in X$. Without any interesting loss of generality, any element of $M(R_i)$ is referred to as i's *most-preferred policy* or *ideal point*.

Throughout, $\rho = (R_1, ..., R_n)$ denotes an n-dimensional preference profile describing the preferences of all individuals: the notation $\mathcal{R}^n$ represents the collection of all n-dimensional profiles of weak orders on X. Any nonempty set $\mathcal{D} \subseteq \mathcal{R}^n$ is referred to as a *preference domain*, and with strict inclusion, $\mathcal{D}$ is referred to as a *restricted domain*. We will come back to restricted domains in more detail in Section 1.3. and $\mathcal{P}^n \subset \mathcal{R}^n$ denotes the collection of all n-dimensional profiles of linear (*i.e.*, strict) orders on X. For any preference profile $\rho \in \mathcal{R}^n$, $\rho|_S$ denotes the restriction of ρ to the set of alternatives $S \subseteq X$. Similarly, for any individual preference $R \in \mathcal{R}$, $R_i|_S$ denotes the restriction of i's preference relation to the set S. For any preference profile $\rho \in \mathcal{R}^n$ and pair of alternatives $(x, y) \in X^2$, the notation $P(x, y; \rho) \equiv \{i \in N : xP_iy\}$ denotes the set of individuals who strictly prefer x to y under ρ, and $R(x, y; \rho) = \{i \in N : xR_iy\}$ denotes the set of individuals who do not strictly prefer y to x.

1.2 Preference Aggregation Rules

A *preference aggregation rule* is any function, $F : \mathcal{D} \rightarrow \mathcal{R}$, that maps preference profiles into weak orders over X. The notation $xR_F(\rho)y$ denotes weak social preference under F at profile $\rho \in \mathcal{D}$ and $xP_F(\rho)y$ denotes strict social preference. The following definitions characterize several properties of preference aggregation rules.

Definition 1 (Weakly Paretian) *A preference aggregation rule F is* weakly Paretian *if for all $\rho \in \mathcal{R}^n$ and all $(x, y) \in X^2$,*

$$P(x, y; \rho) = N \Rightarrow xP_F(\rho)y.$$

Definition 2 (Independent of Irrelevant Alternatives (IIA)) *A preference aggregation rule*

F *is* independent of irrelevant alternatives *(IIA) if, for all* $(x, y) \in X^2$ *and all* $(\rho, \rho') \in \mathcal{D}^2$*,*

$$\rho|_{\{x,y\}} = \rho'|_{\{x,y\}} \Rightarrow F(\rho)|_{\{x,y\}} = F(\rho')|_{\{x,y\}}.$$

Definition 3 (Neutrality) *A preference aggregation rule* F *is* neutral *if for every permutation* $\sigma : X \to X$*, and every profile* $\rho \in \mathcal{P} \cap \mathcal{D}$*,* [3]

$$xR_F(\rho)y \Leftrightarrow \sigma(x)R_F(\sigma(\rho))\sigma(y).$$

1.3 Restricted Domains: Single-Peakedness and Free Triples

In this section we define two restricted preference domains: single-peaked preferences and the 2-free triple domain. Restricted domains have attracted the interest of many scholars because they may lead to the existence of non-dictatorial Arrovian preference aggregation rules. Our interest, as we discuss briefly in the conclusion, is less about the existence, and more about the characterization, of such preference aggregation rules.

Single-Peaked Preferences. The domain of *single-peaked preferences* is the set of all profiles of preferences such that there exists a function $q : X \to \{1, 2, \ldots, K\}$ such that q is a bijection and every individual's preferences are consistent with a quasi-concave utility function of $\{q(x) : x \in X\}$. This preference domain is denoted by $\mathcal{S}^n \subset \mathcal{R}^n$. While this preference restriction is widely utilized and intuitively quite simple, Ballester and Haeringer [2007] prove that the set $\mathcal{S}^n$ is completely characterized by two conditions, worst-restriction (Sen [1966], Sen and Pattanaik [1969]) and α-restriction, defined below.

Definition 4 (Worst-restriction) *A profile* ρ *is* worst-restricted *if, for every triple of alternatives,* $(x, y, z) \in X^3$*,* $|W(\rho_{\{x,y,z\}})| \leq 2$*.*

Definition 5 (α-Restriction) *A preference profile* ρ *is* α-restricted *if there do not exist two agents,* $i, j \in N$*, and four alternatives* w, x, y*, and* z *such that*

1. *The preferences over* w, x*, and* z *are opposite:* wP_ixP_iz *and* zP_jxP_jw*.*
2. *The players agree about the ranking of* y *and* x*:* yP_ix *abd* yP_jx*.*

Definition 6 (Single-peakedness) *A preference profile is* single-peaked *if and only if it satisfies worst-restriction and* $\alpha-$*restriction (Ballester and Haeringer [2007]).*

[3] As with collective choice functions, we define neutrality with respect to strict preference profiles.

It is important to note at this point that the domain $\mathcal{S}^n$ is the set of *all* single peaked preference profiles. In other words, in *a priori* terms, any ordering of the alternatives is possible.[4]

Free Triples. Several authors have examined domain restrictions related to the heterogeneity of "triples" of preferences in any realized preference profile.[5] To be precise, the k-free-triple restriction (where $k \in \{1, 2, 3, 4, 5, 6\}$) is formally defined as follows.

Definition 7 (k-free triple domains) *For any $k \leq n$, define the k-free triple domain, $\mathcal{T}_k^n \subseteq \mathcal{R}^n$, as*

$$\mathcal{T}_k^n \equiv \{\rho \in \mathcal{R}^n : \left|R_i|_{abc} \in \rho|_{abc}\right| \leq k\} \textit{ for all } (a, b, c) \in X^3.$$

In words, $\mathcal{T}_k^n$ is the set of preference profiles such that for each triple, $(a, b, c) \in X^3$, at most k different orderings on those triples are allowable. A domain $\mathcal{D}$ satisfies the k-free triple domain restriction if $\mathcal{T}_k^n \subseteq \mathcal{D}$. The principal interest of much of the literature examining free triple restrictions is the minimal amount of preference homogeneity that one must presume to ensure that majority preference in acyclic.

Ubeda [2003] has recently used 2-free triple domain restriction, demonstrating that on any domain satisfying the 2-free triple restriction, weakly Paretian and IIA imply neutrality, a conclusion that mirrors our own (Theorem 3, below). The key distinction between Ubeda's result and Theorem 3 is that the 2-free triple domain and the single-peaked domain are not nested. Specifically, for all $n \geq 2$, $\mathcal{T}_2^n \not\subseteq \mathcal{S}^n$ and $\mathcal{S}^n \not\subseteq \mathcal{T}_2^n$. In other words, satisfaction of either the k-free triple restriction nor single-peakedness does not imply satisfaction of the other. With the preliminaries in hand, we are now in a position to state and prove our main result.

Theorem 3 *Let F be a preference aggregation rule defined on $\mathcal{S}^n$. Then F is weakly Paretian and IIA only if F is neutral.*

[4]This point is a technical one, but important for broader considerations of the results in this paper. In particular, for any given linear ordering of the alternatives, $Q \in \mathcal{P}$, one can identify the set of preferences that are single-peaked with respect to Q, this set is denoted by $\mathcal{S}_Q$, and the set of all profiles of such preferences is denoted by $\mathcal{S}_Q^n$. This space is widely discussed in the political economy literature. For a succinct and lucid overview of the power of the assumption that Q is known *a priori*, see Chapter 2.4 of Austen-Smith and Banks [2004], in particular Theorem 2.4.

[5]See, among others and in addition to those cited elsewhere in this paper, Blau [1957], Murakami [1961], Kalai et al. [1979], Bordes and Le Breton [1990], Campbell and Kelly [1993], Redekop [1993], Kelly [1994], and Bordes et al. [1995].

Proof: The proof is adapted from Ubeda [2003] to the case of domain $\mathcal{S}^n$. Consider two strict profiles ρ_1 and $\rho_2 \in \mathcal{S}^n$ with $\rho_2 = \sigma(\rho_1)$. We can get from one ρ to any $\sigma(\rho)$ by switching alternatives one at a time, and so we may limit ourselves to considering permutations that only switch one pair of alternatives. Thus, ρ_1 and ρ_2 are such that for one pair x, y with $x \neq y$, $\rho_1|_{ab} = \rho_2|_{ab}$ for all $a, b \neq x, y$, and $\rho_1|_{xz} = \rho_2|_{yz}$, for all z. In other words, ρ_1 is identical to ρ_2 up to a permutation of x, y. By IIA we know that if $xR_F(\rho)y$ then $xR_F(\rho|_{xy})y$. It suffices to show that for all distinct triples $a, x, y \in X$, $xR_F(\rho_1)a \Rightarrow yR_F(\rho_2)a$ and $aR_F(\rho_1)x \Rightarrow aR_F(\rho_2)y$. Construct a new profile $\rho' \in \mathcal{S}^n$ with $P(x, a; \rho') = P(y, a; \rho') = P(x, a; \rho_1) = P(y, a; \rho_2)$. Suppose $aR_F(\rho_1)x$. Let $P(x, y; \rho') = N$. F IIA implies $aR_F(\rho')x$, because $\rho'|_{ax} = \rho_1|_{ax}$. F weakly Paretian and transitive implies that $aR_F(\rho')xR_F(\rho')y$. F IIA again implies $aR_F(\rho_2)y$, because $\rho'|_{ay} = \rho_2|_{ay}$. Thus, $aR_F(\rho_1)x \Rightarrow aR_F(\rho_2)y$. The case where $xR_F(\rho_1)a$ follows similarly, with $P(x, y; \rho') = \emptyset$.

We last need to check that the constructed ρ' is indeed an element of $\mathcal{S}^n$. We required that $P(x, a; \rho') = P(y, a; \rho') = P(x, a; \rho_1)$, and that $P(x, y; \rho') = N$ or $\emptyset$. ρ' clearly satisfies $\alpha-$restriction because it only specifies preferences over three elements of X. And ρ' satisfies worst-restriction because $P(x, y; \rho') = N$ or $\emptyset$ implies that either x or y is never ranked last by any individual. $\square$

2 Conclusions

A theory of political institutions necessarily must deal with the possibility that policy choices in the future may have no natural structure that is known *a priori* – even if some such ordering is presumed to structure all political choices. For example, while the ordering of preferences over the marginal rate of a flat income tax may be presumed (with at least some implicit heroism) to be single-peaked according to the usual ordering of the real line, the general presumption that preferences are single-peaked with respect to a set of political alternatives does not provide enough information to declare what the "median most-preferred alternative" is, even after individuals have informed you of their most-preferred alternatives. So long as this ordering is not known when the institution is designed, the mere fact that some such ordering will exist does not obviate the need to be careful in one's choice of institutional details. In particular, if *ex post* interpretation/rationalizability of collective preference is a desideratum, then the institution in question *must* be neutral with respect to the alternatives of political choice. This is an important point particularly once one acknowledges that many features of policymaking institutions, such as bicameral requirements, supermajoritarian quotas, separation-of-powers systems, and gatekeeping institutions such as legislative committees necessarily lead to violations of neutrality.[6]

Theorem 3 implies that one must be careful in interpreting collective will (even insofar as being "well-behaved") in any institution that is non-neutral. This point is highly relevant for those scholars who insist that majority rule cycles are infrequent and/or untroubling (*e.g.*, Mackie [2003]) and either explicitly or implicitly then rely upon appeals to aggregate outcomes such as vote totals and the passage or failure of proposed legislation as being indicative of collective will. The point of this paper, at some level, is that even in the realm of well-defined *majority will*, many institutions that are simultaneously and unambiguously democratic and relevant (*i.e.*, extant) must lead one to question whether the behaviors and/or outcomes produced within are necessarily "representative." To be even more forceful – the normative, prescriptive, descriptive, and inferential issues raised by Arrow's theorem (among others) are more than simple mathematical curiosities dreamed up for the purpose of scholarly debate. In very precise and simple terms, a democratic institution may be defended as truly "representative" – even with the presumption of single-peaked individual preferences only if its rules are themselves invariant to the alternatives under consideration.

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[6]This is particularly true of the equilibrium policy outcomes predicted to occur within such systems by non-cooperative game theoretic analysis. This linkage is a deep one that is beyond the scope of the current project. An excellent discussion of this relationship is offered in Chapter 8 of Schofield [2006]. Related recent models of political institutions with one or more of these features include Tsebelis and Money [1997], Krehbiel [1998], Epstein and O'Halloran [1999], McCarty [2000], Cameron [2000], and Crombez et al. [2006].

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