Certificates: Boundedness and Optimality

Fall 2021

Overview

LP in Symmetric Form

$$
\begin{aligned}
\min \quad & \mathbf{c}^T \mathbf{x} \\
\text{s.t.} \quad & A\mathbf{x} \geq \mathbf{b}, \\
& \mathbf{x} \geq 0.
\end{aligned}
$$

or

$$
\begin{aligned}
\min \quad & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \geq b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \geq b_2 \\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \geq b_m, \\
& x_1, x_2, \ldots, x_n \geq 0.
\end{aligned}
$$

LP transformations

$$\max(\min) \sum_{j=1}^{n} c_j x_j \iff -\min(\max) \sum_{j=1}^{n} (-c_j) x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \iff \sum_{j=1}^{n} (-a_{ij}) x_j \ge -b_i$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i \iff \sum_{j=1}^{n} (-a_{ij}) x_j \le -b_i$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \iff \left\{ \begin{array}{l} \sum_{j=1}^{n} a_{ij} x_j \le b_i \\ \sum_{j=1}^{n} a_{ij} x_j \ge b_i \end{array} \right.$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \iff \sum_{j=1}^{n} a_{ij} x_j + \underbrace{s_i}_{\text{slack variable}} = b_i, \quad s_i \ge 0$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i \iff \sum_{j=1}^{n} a_{ij} x_j - s_i = b_i, \quad s_i \ge 0$$

$$x_j \text{ free} \iff x_j = x_j^+ - x_j^-, \quad x_j^+ \ge 0, \quad x_j^- \ge 0$$

Certificate of Feasibility

Certificate of feasibility is $\mathbf{z} \in \mathbb{R}^n$ such that

$$\begin{aligned} A\mathbf{z} &\geq \mathbf{b}, \\ \mathbf{z} &\geq 0. \end{aligned}$$

or

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 + \ldots + a_{1n}z_n &\geq b_1, \\ a_{21}z_1 + a_{22}z_2 + \ldots + a_{2n}z_n &\geq b_2, \\ &\vdots \\ a_{m1}z_1 + a_{m2}z_2 + \ldots + a_{mn}z_n &\geq b_m, \\ z_1 &\geq 0, \\ &\vdots \\ z_n &\geq 0. \end{aligned}$$

Certificate of Infeasibility

Certificate of infeasibility is $\mathbf{w} \in \mathbb{R}^m$ such that

$$\begin{aligned} \mathbf{w}^T \mathbf{b} &> 0 \\ \mathbf{w}^T A &\leq 0, \\ \mathbf{w} &\geq 0. \end{aligned}$$

or

$$\begin{array}{rcl} b_1 w_1 + b_2 w_2 + \ldots + b_m w_m & > & 0, \\ a_{11} w_1 + a_{21} w_2 + \ldots + a_{m1} w_m & \leq & 0, \\ a_{12} w_1 + a_{22} w_2 + \ldots + a_{m2} w_m & \leq & 0, \\ & \vdots & \\ a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{mn} w_m & \leq & 0, \\ w_1 \geq 0, \quad \ldots, \quad w_m & \geq & 0. \end{array}$$

Certificate of Unboundedness

Certificate of unboundedness (together with the sertificate of feasibility) is $\mathbf{z} \in \mathbb{R}^n$ such that

$$\begin{aligned} \mathbf{c}^T \mathbf{z} &< 0, \\ A\mathbf{z} &\geq 0, \\ \mathbf{z} &\geq 0. \end{aligned}$$

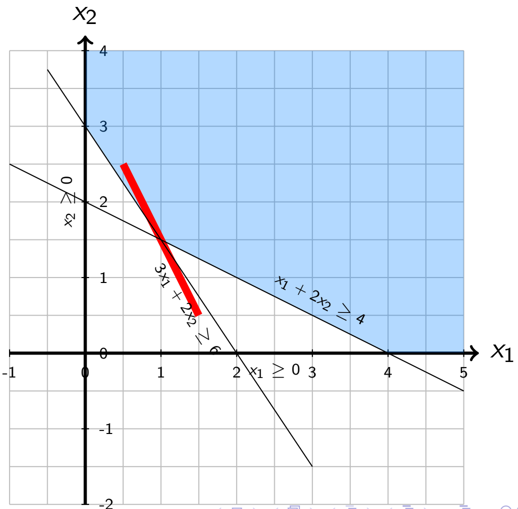
or

$$\begin{array}{rcl} c_1 z_1 + c_2 z_2 + \ldots + c_n z_n & < & 0 \\ a_{11} z_1 + a_{12} z_2 + \ldots + a_{1n} z_n & \geq & 0 \\ a_{21} z_1 + a_{22} z_2 + \ldots + a_{2n} z_n & \geq & 0 \\ & \vdots & \\ a_{m1} z_1 + a_{m2} z_2 + \ldots + a_{mn} z_n & \geq & 0, \\ z_1 \geq 0, \quad \ldots, \quad z_n & \geq & 0. \end{array}$$

Certificate of Unboundedness

How to use? - Add to the certificate of feasibility!

$$
\begin{array}{llllll}
\min & -2x_1 & -x_2 & & \\
\text{s.t.} & x_1 & +2x_2 & \geq & 4 \\
& 3x_1 & +2x_2 & \geq & 6 \\
& x_1 & & \geq & 0 \\
& & x_2 & \geq & 0
\end{array}
$$



Certificate of Boundedness

Certificate of boundedness is $\mathbf{w} \in \mathbb{R}^m$ such that

$$\begin{aligned} \mathbf{w}^T A &\leq \mathbf{c}, \\ \mathbf{w} &\geq 0. \end{aligned}$$

or

$$\begin{array}{rcl} a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m & \leq & c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m & \leq & c_2 \\ & \vdots & \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m & \leq & c_n \\ w_1 \geq 0, \quad \ldots, \quad w_m & \geq & 0. \end{array}$$

Lower bound:

$$b_1 w_1 + b_2 w_2 + \ldots + b_m w_m.$$

Certificate of Optimality

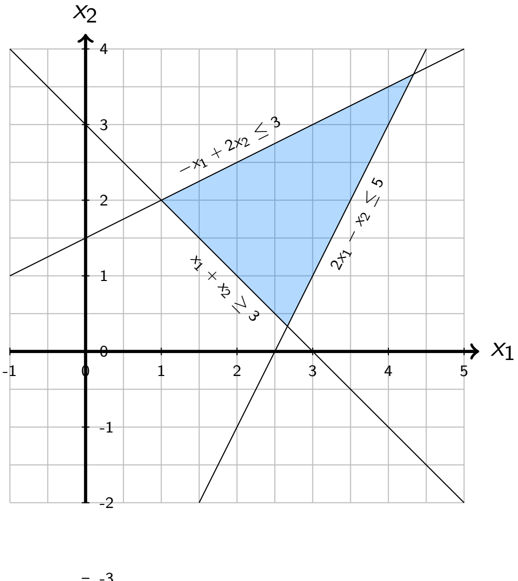
Certificate of optimality is a *Certificate of feasibility* $\mathbf{z} \in \mathbb{R}^n$ and a *Certificate of boundedness* $\mathbf{w} \in \mathbb{R}^m$ such that

$$\mathbf{c}^T\mathbf{z} = \mathbf{w}^T\mathbf{b}$$

(such $\mathbf{z}$ is the optimal solution of the LP) or

$$\begin{aligned}
a_{11}z_1 + a_{12}z_2 + \ldots + a_{1n}z_n &\geq b_1, \\
&\vdots \\
a_{m1}z_1 + a_{m2}z_2 + \ldots + a_{mn}z_n &\geq b_m \\
a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m &\leq c_1 \\
&\vdots \\
a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m &\leq c_n \\
z_1, \ldots, z_n, w_1, \ldots, w_m &\geq 0 \\
c_1z_1 + \ldots + c_nz_n - b_1w_1 - \ldots - b_mw_m &= 0.
\end{aligned}$$

How to find certificates



Example 1: Show Feasibility,

$$
\begin{array}{llllll}
\min & 2x_1 & +x_2 & & \\
\text{s.t.} & x_1 & +2x_2 & \geq & 4 \\
 & 3x_1 & +2x_2 & \geq & 6 \\
 & x_1 & & \geq & 0 \\
 & & x_2 & \geq & 0
\end{array}
$$

Certificate of Feasibility

$$
\begin{aligned}
a_{11}z_1 + a_{12}z_2 &\geq b_1, \\
a_{21}z_1 + a_{22}z_2 &\geq b_2, \\
z_1, z_2 &\geq 0.
\end{aligned}
$$

We can find the solution, i.e., $(z_1, z_2) = (2, 1)$. So the LP is feasible.

Show Boundedness and give lower bounds.

$$
\begin{array}{llcll}
\min & 2x_1 & +x_2 & & \\
\text{s.t.} & x_1 & +2x_2 & \geq & 4 \\
 & 3x_1 & +2x_2 & \geq & 6 \\
 & x_1 & & \geq & 0 \\
 & & x_2 & \geq & 0
\end{array}
$$

Show Boundedness and give lower bounds.

$$
\begin{array}{llllll}
\min & 2x_1 & +x_2 & & \\
\text{s.t.} & x_1 & +2x_2 & \geq & 4 \\
 & 3x_1 & +2x_2 & \geq & 6 \\
 & x_1 & & \geq & 0 \\
 & & x_2 & \geq & 0
\end{array}
$$

Certificate of Boundedness

$$
\begin{array}{rcl}
a_{11}w_1 + a_{21}w_2 & \leq & c_1 \\
a_{12}w_1 + a_{22}w_2 & \leq & c_2 \\
w_1, w_2 & \geq & 0
\end{array}
$$

Show Boundedness and give lower bounds.

$$\begin{array}{llllll} \min & 2x_1 & +x_2 & & \\ \text{s.t.} & x_1 & +2x_2 & \geq & 4 \\ & 3x_1 & +2x_2 & \geq & 6 \\ & x_1 & & \geq & 0 \\ & & x_2 & \geq & 0 \end{array}$$

Certificate of Boundedness

$$\begin{aligned} a_{11}w_1 + a_{21}w_2 &\leq c_1 \\ a_{12}w_1 + a_{22}w_2 &\leq c_2 \\ w_1, w_2 &\geq 0 \end{aligned}$$

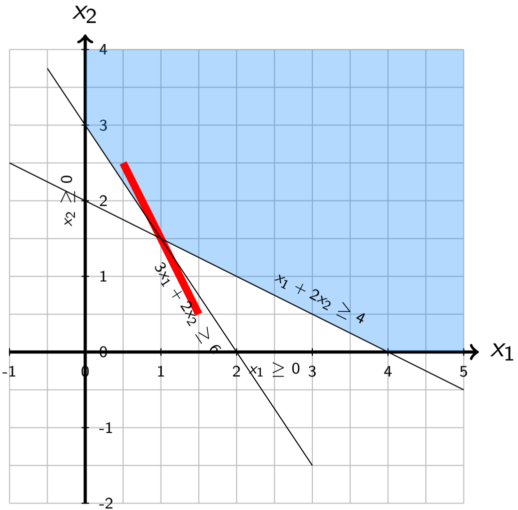
and

$$\begin{aligned} w_1 + 3w_2 &\leq 2 \\ 2w_1 + 2w_2 &\leq 1 \\ w_1, w_2 &\geq 0. \end{aligned}$$

We can find the solution, i.e., $(w_1, w_2) = (0.5, 0)$, and the corresponding lower bound $4 \cdot 0.5 = 2$.

Solve Graphically to Optimality

$$
\begin{array}{llllll}
\min & 2x_1 & +x_2 & & \\
\text{s.t.} & x_1 & +2x_2 & \geq & 4 \\
& 3x_1 & +2x_2 & \geq & 6 \\
& x_1 & & \geq & 0 \\
& & x_2 & \geq & 0
\end{array}
$$



Example 2: Bounded?

Certificate of Boundedness

$$
\begin{aligned}
\min \quad 5x_1 - 6x_2 + 4x_3 \\
x_1 - x_2 - 4x_3 \geq 1 \\
-x_1 + 3x_2 - x_3 \geq 2 \\
x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
a_{11}w_1 + a_{21}w_2 &\leq c_1 \\
a_{12}w_1 + a_{22}w_2 &\leq c_2 \\
a_{13}w_1 + a_{23}w_2 &\leq c_3 \\
w_1, w_2 &\geq 0
\end{aligned}
$$

Example 2: Bounded?

Certificate of Boundedness

$$\begin{aligned} w_1 - w_2 &\leq 5 \\ -w_1 + 3w_2 &\leq -6 \\ -4w_1 - w_2 &\leq 4 \\ w_1, w_2 &\geq 0. \end{aligned}$$

However, no solution exists. Why?

$$w_1 - w_2 \leq 5, \; -w_1 + 3w_2 \leq -6$$
$$\implies \quad w_2 \leq -0.5,$$

but $w_2 \geq 0$.

Example 2: Bounded?

$$\begin{aligned} \min \quad 5x_1 - 6x_2 + 4x_3 \\ x_1 - x_2 - 4x_3 \geq 1 \\ -x_1 + 3x_2 - x_3 \geq 2 \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{aligned}$$

Certificate of Unboundedness

$$\begin{aligned} c_1z_1 + c_2z_2 + c_3z_3 &< 0 \\ a_{11}z_1 + a_{12}z_2 + a_{13}z_3 &\geq 0 \\ a_{21}z_1 + a_{22}z_2 + a_{23}z_3 &\geq 0 \\ z_1, z_2, z_3 &\geq 0. \end{aligned}$$

Example 2: Bounded?

Certificate of Unboundedness

$$\begin{aligned} 5z_1 - 6z_2 + 4z_3 &< 0 \\ z_1 - z_2 - 4z_3 &\geq 0 \\ -z_1 + 3z_2 - z_3 &\geq 0 \\ z_1, z_2, z_3 &\geq 0. \end{aligned}$$

We can find the solution, i.e.,

$$(z_1, z_2, z_3) = (1, 1, 0)$$

So we can conclude that this LP is unbounded.

Extension: LP in Standard Form

$$
\begin{aligned}
\min \quad & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m, \\
& x_1, x_2, \ldots, x_n \geq 0.
\end{aligned}
$$

Certificate of Feasibility

Certificate of feasibility is the solution to the following system of inequalities,

$$
\begin{aligned}
a_{11}z_1 + a_{12}z_2 + \ldots + a_{1n}z_n &= b_1, \\
a_{21}z_1 + a_{22}z_2 + \ldots + a_{2n}z_n &= b_2, \\
&\vdots \\
a_{m1}z_1 + a_{m2}z_2 + \ldots + a_{mn}z_n &= b_m, \\
z_1 &\geq 0, \\
&\vdots \\
z_n &\geq 0.
\end{aligned}
$$

More specifically, if there exists $z_1, \ldots, z_n$ such that the above inequality holds true, then the LP is feasible.

Certificate of Infeasibility

Certificate of infeasibility is the solution to the following system of inequalities,

$$
\begin{aligned}
b_1 w_1 + b_2 w_2 + \ldots + b_m w_m &> 0, \\
a_{11} w_1 + a_{21} w_2 + \ldots + a_{m1} w_m &\leq 0, \\
a_{12} w_1 + a_{22} w_2 + \ldots + a_{m2} w_m &\leq 0, \\
&\vdots \\
a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{mn} w_m &\leq 0.
\end{aligned}
$$

More specifically, if there exists $w_1, \ldots, w_m$ such that the above inequality holds true, then the LP is infeasible.

Certificate of Unboundedness

Certificate of unboundedness is the solution to the following system of inequalities,

$$
\begin{aligned}
c_1 z_1 + c_2 z_2 + \ldots + c_n z_n &< 0 \\
a_{11} z_1 + a_{12} z_2 + \ldots + a_{1n} z_n &= 0 \\
a_{21} z_1 + a_{22} z_2 + \ldots + a_{2n} z_n &= 0 \\
&\vdots \\
a_{m1} z_1 + a_{m2} z_2 + \ldots + a_{mn} z_n &= 0, \\
z_1 &\geq 0, \\
&\vdots \\
z_n &\geq 0.
\end{aligned}
$$

More specifically, if there exists $z_1, \ldots, z_n$ such that the above inequality holds true, then the LP is unbounded.

Certificate of Boundedness

Certificate of boundedness is the solution to the following system of inequalities,

$$\begin{aligned} a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m &\leq c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m &\leq c_2 \\ &\vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m &\leq c_n. \end{aligned}$$

More specifically, if there exists $w_1, \ldots, w_m$ such that the above inequality holds true, then the LP is bounded with a lower bound:

$$b_1w_1 + b_2w_2 + \ldots + b_mw_m.$$

Certificate of Optimality

Certificate of optimality is the solution to the unified system of *Certificate of feasibility* and *Certificate of boundedness* with an additional equality:

$$\begin{aligned}
a_{11}z_1 + a_{12}z_2 + \ldots + a_{1n}z_n &= b_1, \\
&\vdots \\
a_{m1}z_1 + a_{m2}z_2 + \ldots + a_{mn}z_n &= b_m \\
a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m &\leq c_1 \\
&\vdots \\
a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m &\leq c_n \\
z_1, \ldots, z_n &\geq 0 \\
c_1 z_1 + \ldots + c_n z_n - b_1 w_1 - \ldots - b_m w_m &= 0.
\end{aligned}$$

Also, $z_1, \ldots, z_n$ is an optimal solution to the LP problem.

LP in Inequality Form

$$\begin{aligned}
\min \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m.
\end{aligned}$$

Certificate of Feasibility

Certificate of feasibility is the solution to the following system of inequalities,

$$
\begin{aligned}
a_{11}z_1 + a_{12}z_2 + \ldots + a_{1n}z_n &\geq b_1, \\
a_{21}z_1 + a_{22}z_2 + \ldots + a_{2n}z_n &\geq b_2, \\
&\vdots \\
a_{m1}z_1 + a_{m2}z_2 + \ldots + a_{mn}z_n &\geq b_m.
\end{aligned}
$$

More specifically, if there exists $z_1, \ldots, z_n$ such that the above inequality holds true, then the LP is feasible.

Certificate of Infeasibility

Certificate of infeasibility is the solution to the following system of inequalities,

$$
\begin{aligned}
b_1 w_1 + b_2 w_2 + \ldots + b_m w_m &> 0,\\
a_{11} w_1 + a_{21} w_2 + \ldots + a_{m1} w_m &= 0,\\
a_{12} w_1 + a_{22} w_2 + \ldots + a_{m2} w_m &= 0,\\
&\vdots\\
a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{mn} w_m &= 0,\\
w_1 &\geq 0,\\
&\vdots\\
w_m &\geq 0.
\end{aligned}
$$

More specifically, if there exists $w_1, \ldots, w_m$ such that the above inequality holds true, then the LP is infeasible.

Certificate of Unboundedness

Certificate of unboundedness is the solution to the following system of inequalities,

$$\begin{aligned} c_1 z_1 + c_2 z_2 + \ldots + c_n z_n &< 0 \\ a_{11} z_1 + a_{12} z_2 + \ldots + a_{1n} z_n &\geq 0 \\ a_{21} z_1 + a_{22} z_2 + \ldots + a_{2n} z_n &\geq 0 \\ &\vdots \\ a_{m1} z_1 + a_{m2} z_2 + \ldots + a_{mn} z_n &\geq 0. \end{aligned}$$

More specifically, if there exists $z_1, \ldots, z_n$ such that the above inequality holds true, then the LP is unbounded.

Certificate of Boundedness

Certificate of boundedness is the solution to the following system of inequalities,

$$\begin{aligned} a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m &= c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m &= c_2 \\ &\vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m &= c_n \\ w_1 &\geq 0 \\ &\vdots \\ w_m &\geq 0. \end{aligned}$$

More specifically, if there exists $w_1, \ldots, w_m$ such that the above inequality holds true, then the LP is bounded with a lower bound:

$$b_1 w_1 + b_2 w_2 + \ldots + b_m w_m.$$

Certificate of Optimality

Certificate of optimality is the solution to the unified system of *Certificate of feasibility* and *Certificate of boundedness* with an additional equality:

$$
\begin{aligned}
a_{11}z_1 + a_{12}z_2 + \ldots + a_{1n}z_n &\geq b_1, \\
&\vdots \\
a_{m1}z_1 + a_{m2}z_2 + \ldots + a_{mn}z_n &\geq b_m \\
a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m &= c_1 \\
&\vdots \\
a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m &= c_n \\
w_1, \ldots, w_m &\geq 0 \\
c_1z_1 + \ldots + c_nz_n - b_1w_1 - \ldots - b_mw_m &= 0.
\end{aligned}
$$

Also, $z_1, \ldots, z_n$ is an optimal solution to the LP problem.

Thank you for your attention !