Duality and Sensitivity

Fall 2021

Overview

Dual Linear Program

Sensitivity Analysis

AMPL

Reminder: Definitions

LP Sym Form:

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \geq \mathbf{b}, \\ & \mathbf{x} \geq 0. \end{aligned}$$

Certificate of Boundedness:

$$\begin{aligned} & \mathbf{y}^\top A \leq \mathbf{c}^\top, \\ & \mathbf{y} \geq 0. \end{aligned}$$

Theorem 4:

$$\mathbf{c}^\top \mathbf{x} \geq \mathbf{y}^\top \mathbf{b}$$

Reminder: Definitions

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Certificate of Boundedness:

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Theorem 4:

$$\mathbf{c}^\top\mathbf{x} \geq \mathbf{y}^\top\mathbf{b}$$

Dual problem aims to find *the certificate of boundedness that gives the highest lower bound* $\mathbf{y}^\top\mathbf{b}$:

$$\begin{aligned} \max \quad & \mathbf{y}^T\mathbf{b} \\ \text{s.t.} \quad & \mathbf{y}^\top A \leq \mathbf{c}^\top, \\ & \mathbf{y} \geq 0. \end{aligned}$$

Example: Find the Dual

Consider a Linear Program (not in the Symmetric Form)

$$
\begin{array}{llllll}
\min & x_1 & +2x_2 & +3x_3 & & \\
s.t. & -x_1 & +3x_2 & & = & 5 \\
 & 2x_1 & -x_2 & +3x_3 & \geq & 6 \\
 & & & x_3 & \leq & 4 \\
 & x_1 & & & \geq & 0 \\
 & & x_2 & & \leq & 0 \\
 & & & x_3 & \text{free} &
\end{array}
$$

Normal course of actions:

$$
\begin{array}{c}\text{Primal}\\\text{Problem}\end{array} \rightarrow \begin{array}{c}\text{Primal in}\\\text{Sym Form}\end{array} \rightarrow \begin{array}{c}\text{Dual}\\\text{Problem}\end{array} \xrightarrow{(optional)} \begin{array}{c}\text{Dual in}\\\text{Sym Form}\end{array}
$$

Example: Find the Dual

Normal course of actions:

$$\text{Primal Problem} \rightarrow \text{Primal in Sym Form} \rightarrow \text{Dual Problem} \xrightarrow{(optional)} \text{Dual in Sym Form}$$

Example: Find the Dual

Normal course of actions:

$$\begin{matrix}\text{Primal}\\\text{Problem}\end{matrix} \rightarrow \begin{matrix}\text{Primal in}\\\text{Sym Form}\end{matrix} \rightarrow \begin{matrix}\text{Dual}\\\text{Problem}\end{matrix} \xrightarrow{(\textit{optional})} \begin{matrix}\text{Dual in}\\\text{Sym Form}\end{matrix}$$

	Primal	**Dual**	
Objective	min	max	**Objective**
Constraints	$\geq$ $=$ $\leq$	$\geq$ *free* $\leq$	**Decision variables**
Decision variables	$\geq$ *free* $\leq$	$\leq$ $=$ $\geq$	**Constraints**

A faster course of actions:

$$\begin{matrix}\text{Primal}\\\text{Problem}\end{matrix} \xrightarrow[\textit{table}]{\textit{using}} \begin{matrix}\text{Dual}\\\text{Problem}\end{matrix} \xrightarrow{(\textit{optional})} \begin{matrix}\text{Dual in}\\\text{Sym Form}\end{matrix}$$

Example: Find the Dual

$$
\begin{array}{llllll}
\min & x_1 & +2x_2 & +3x_3 & & & \\
s.t. & -x_1 & +3x_2 & & = & 5 & (y_1) \\
 & 2x_1 & -x_2 & +3x_3 & \geq & 6 & (y_2) \\
 & & & x_3 & \leq & 4 & (y_3) \\
 & x_1 & & & \geq & 0 & \\
 & & x_2 & & \leq & 0 & \\
 & & & x_3 & \text{free} & &
\end{array}
$$

Example: Find the Dual

$$
\begin{array}{lrrrcll}
\min & x_1 & +2x_2 & +3x_3 & & & \\
s.t. & -x_1 & +3x_2 & & = & 5 & (y_1) \\
 & 2x_1 & -x_2 & +3x_3 & \geq & 6 & (y_2) \\
 & & & x_3 & \leq & 4 & (y_3) \\
 & x_1 & & & \geq & 0 & \\
 & & x_2 & & \leq & 0 & \\
 & & & x_3 & \text{free} & &
\end{array}
$$

$$
\begin{array}{lrrrcl}
\max & 5y_1 & +6y_2 & +4y_3 & & \\
s.t & -y_1 & +2y_2 & & \leq & 1 \\
 & 3y_1 & -y_2 & & \geq & 2 \\
 & & 3y_2 & +y_3 & = & 3 \\
 & y_1 & & & \text{free} & \\
 & & y_2 & & \geq & 0 \\
 & & & y_3 & \leq & 0
\end{array}
$$

Example: Certify Optimality

Consider the following linear program:

$$
\begin{array}{lrcrcrcrcr}
\min & 2x_1 & & & & & + & 2x_4 & & \\
 & & & 3x_2 & + & x_3 & - & x_4 & \leq & 1, \\
 & 2x_1 & & & + & 5x_3 & & & = & 4, \\
 & & & x_2 & + & 2x_3 & & & \geq & 2. \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}
$$

- Write the dual problem.
- Write down the complementary slackness conditions.
- Prove that $(x_1, x_2, x_3, x_4) = (0, 0.4, 0.8, 1)$ is optimal for the primal without solving the linear program.

Example: Certify Optimality

The dual problem is

$$
\begin{array}{lrcrcrcr}
\max & -y_1 & + & 4y_2 & + & 2y_3 & & \\
 & & & -y_2 & & & \geq & -1 \\
 & 3y_1 & & & - & y_3 & \geq & 0 \\
 & y_1 & - & 5y_2 & - & 2y_3 & \geq & 0 \\
 & -y_1 & & & & & \geq & -2 \\
 & y_1 & & & & & \geq & 0 \\
 & & & & & y_3 & \geq & 0
\end{array}
$$

Example: Certify Optimality

The complementary slackness conditions are

$$\begin{aligned}
x_1 \cdot (1 - y_2) &= 0 \\
x_2 \cdot (3y_1 - y_3) &= 0 \\
x_3 \cdot (y_1 - 5y_2 - 2y_3) &= 0 \\
x_4 \cdot (2 - y_1) &= 0 \\
y_1 \cdot (1 - 3x_2 - x_3 + x_4) &= 0 \\
y_3 \cdot (x_2 + 2x_3 - 2) &= 0
\end{aligned}$$

Plugging $(x_1, x_2, x_3, x_4) = (0, 0.4, 0.8, 1)$ into the above system of equations yields that $(y_1, y_2, y_3) = (2, -2, 6)$. This is the certificate of optimality.

Terminology

- **Sensitivity analysis**
  - **General definition**
    How the uncertainty in the output of a mathematical model or system (numerical or otherwise) can be apportioned to different sources of uncertainty in its inputs
  - **Linear program**
    How "small" changes in parameter c_j, b_i, a_{ij} affect the current optimal solution x^* and optimal objective value $\sum_{i=1}^{n} c_j x_j^*$

Terminology

Given a feasible solution $\bar{x}_1, \ldots, \bar{x}_n$:

- Decision variable $\bar{x}_j$ is **basic** if $\bar{x}_j \neq 0$
- Decision variable $\bar{x}_j$ is **non-basic** if $\bar{x}_j = 0$
- Constraint $\sum_{j=1}^n a_{ij} x_j \geq b_i$ is **binding** if $\sum_{j=1}^n a_{ij} \bar{x}_j = b_i$
- Constraint $\sum_{j=1}^n a_{ij} x_j \geq b_i$ is **not binding** if $\sum_{j=1}^n a_{ij} \bar{x}_j > b_i$

Terminology

- **Shadow price** y_i = Dual variable
  Change of the objective function from one unit **increase** in its right-hand side b_i
- **Reduced cost** r_j = Dual slack = $(\mathbf{c}^\top - \mathbf{y}^\top A)_j$
  Amount by which the cost coefficient of non-basic variable c_j must be **lowered** for that variable to become basic
- **Allowable increase/decrease**
  - Optimal solution x^* and objective $\sum_{j=1}^{n} c_j x_j^*$ may change
  - Whether a decision variable is basic or non-basic stays unchanged
  - Whether a constraint is binding or non-binding stays unchanged

Complementary Slackness

If x^* is optimal for the Primal problem and y^* is optimal for the Dual problem, then:

1. $[\mathbf{c}^\top - (\mathbf{y}^*)^\top A]_j x_j^* = r_j^* x_j^* = 0 \quad \forall j = 1 \ldots n$
2. $[A\mathbf{x}^* - \mathbf{b}]_i y_i^* = s_i^* y_i^* = 0 \quad \forall i = 1 \ldots m$

Where r_j^* is the Reduced Cost and s_i^* is the slack of a Primal constraint.

AMPL setup for sensitivity analysis

Type the following commands into the **console**:

1. Set the solver to be CPLEX:

   ```
   option solver cplex;
   ```

2. Enable sensitivity analysis:

   ```
   option cplex_options 'sensitivity';
   ```

3. Turn off presolve (needed for sensitivity analysis):

   ```
   option presolve 0;
   ```

4. Load model and solve as usual.

   ```
   model paint.mod;
   ```

   ```
   solve;
   ```

AMPL sensitivity analysis output

- Display the objective function, constraint or variable:
  ```
  display <name>;
  ```
  For example:
  ```
  display totalProfit;
  ```
- Display all variables:
  ```
  display _varname, _var, _var.rc, _var.down, _var.current, _var.up;
  ```
- Display all constraints:
  ```
  display _conname, _con, _con.slack, _con.up, _con.current, _con.down;
  ```

Example: continuous knapsack

You want to set up an emergency bag in case of an earthquake. Four items can be packed: gold, water, pillow, brick, with the following data

	Gold	Water	Pillow	Brick
Value	24	5	2	3
Volume	3	8	14	6
Weight	20	10	2	15

and you want to pack at least 5 units of water. Suppose your pack has the maximum volume of 60 and you can bear at most 100 weight. Find how much each item to pack to maximize the value.

Example: continuous knapsack

$$
\begin{array}{lrrrrcr}
\max & 24x_1 & +5x_2 & +2x_3 & +3x_4 & & \\
s.t. & 3x_1 & +8x_2 & +14x_3 & +6x_4 & \leq & 60 \\
 & 20x_1 & +10x_2 & +2x_3 & +15x_4 & \leq & 100 \\
 & & x_2 & & & \geq & 5 \\
 & x_1 & & & & \geq & 0 \\
 & & x_2 & & & \geq & 0 \\
 & & & x_3 & & \geq & 0 \\
 & & & & x_4 & \geq & 0
\end{array}
$$

Optimal solution $x_1 = 2.5$ and $x_2 = 5$

Example: continuous knapsack

```
ampl: include cont_knapsack.run;
CPLEX 12.6.1.0: sensitivity
CPLEX 12.6.1.0: optimal solution; objective 85
1 dual simplex iterations (1 in phase I)

suffix up OUT;
suffix down OUT;
suffix current OUT;
:    x          x.rc       x.current   x.down     x.up      :=
1   2.5    -3.55271e-15      24          20     1e+20
2   5      -1.77636e-15       5      -1e+20        12
3   0      -0.4               2      -1e+20         2.4
4   0     -15                 3      -1e+20        18
;

: _conname   _con _con.slack _con.current _con.down     _con.up      :=
1   volume    0       12.5          60        47.5     1e+20
2   weight    1.2      0           100        50         183.333
3   water    -7        0             5         0           6.92308
;
```

Example: continuous knapsack

$$
\begin{array}{lrrrcl}
\min & 60y_1 & +100y_2 & +5y_3 & & \\
s.t. & 3y_1 & +20y_2 & & \geq & 24 \\
 & 8y_1 & +10y_2 & +y_3 & \geq & 5 \\
 & 14y_1 & +2y_2 & & \geq & 2 \\
 & 6y_1 & +15y_2 & & \geq & 3 \\
 & y_1 & & & \geq & 0 \\
 & & y_2 & & \geq & 0 \\
 & & & y_3 & \leq & 0
\end{array}
$$

Optimal solution $y_1 = 0, y_2 = 1.2, y_3 = -7.$

Diet example

All the necessary files are on bCourses.

How to derive sensitivity analysis: Key Idea

In order for a change to be withing the allowable range, both of these must be true:

- Whether a decision variable is basic or non-basic stays unchanged.
- Whether a constraint is binding or non-binding stays unchanged.

Types of analysis

- Case 1: Change b_i
  - Case 1a: Change b_i of non-binding constraint
  - Case 1b: Change b_i of binding constraint
  - Case 1c: Find g if Case 1b.
- Case 2: Change c_j
  - Case 2a: Change c_j of non-basic variable
  - Case 2b: Change c_j of basic variable
- Case 3: Change a_{ij}
  - Case 3a: Change a_{ij} of non-basic variable
  - *Case 3b: Change a_{ij} of basic variable*
- Case 4: Add a new constraint
- Case 5: Add a new decision variable

Case 2a: Change c_j of non-basic variable

Change c_j of non-basic variable

- Reduced cost $r_j \neq 0$ [1]

$$r_j = c_j - \sum_{i=1}^{m} a_{ij} y_i$$

- Consider c_3 which has reduced cost $r_3 = -0.4$
  - Allowable increase: $-r_j = 0.4$
  - Allowable decrease: $+\infty$
- Consider changing c_3 from 2 to 2.1
  - New optimal solution: Unchanged
  - New optimal objective value: Unchanged

[1] $r_j = 0$ for non-basic variable means multiple optimal solutions

Case 2b: Change c_j of basic variable

Change c_j of basic variable

$$
\begin{array}{rrrrrrcl}
\max & 24x_1 & +(5+\delta)x_2 & +2x_3 & +3x_4 & & & \\
s.t. & 3x_1 & +8x_2 & +14x_3 & +6x_4 & \leq & 60 \\
 & 20x_1 & +10x_2 & +2x_3 & +15x_4 & \leq & 100 \\
 & & x_2 & & & \geq & 5 \\
 & x_1 & & & & \geq & 0 \\
 & & x_2 & & & \geq & 0 \\
 & & & x_3 & & \geq & 0 \\
 & & & & x_4 & \geq & 0
\end{array}
$$

Case 2b: Change c_j of basic variable

Consider the Dual problem:

$$
\begin{array}{llllcll}
\min & 60y_1 & +100y_2 & +5y_3 & & & \\
s.t. & 3y_1 & +20y_2 & & \geq & 24 & (1) \\
 & 8y_1 & +10y_2 & +y_3 & \geq & 5+\delta & (2) \\
 & 14y_1 & +2y_2 & & \geq & 2 & (3) \\
 & 6y_1 & +15y_2 & & \geq & 3 & (4) \\
 & y_1 & & & \geq & 0 & (5) \\
 & & y_2 & & \geq & 0 & (6) \\
 & & & y_3 & \leq & 0 & (7)
\end{array}
$$

Optimal solution $y_1 = 0, y_2 = 1.2, y_3 = -7$

Case 2b: Change c_j of basic variable

$$
\begin{array}{lrrcll}
\min & +100y_2 & +5y_3 & & & \\
s.t. & +20y_2 & & = & 24 & (1) \\
 & +10y_2 & +y_3 & = & 5+\delta & (2) \\
 & +2y_2 & & \geq & 2 & (3) \\
 & +15y_2 & & \geq & 3 & (4) \\
 & y_2 & & \geq & 0 & (6) \\
 & & y_3 & \leq & 0 & (7)
\end{array}
$$

Case 2b: Change c_j of basic variable

From (1) we get $y_2 = \frac{24}{20}$ (satisfies (3), (4), (6)), substitute in (2)

$$y_3 = -7 + \delta.$$

From (7)

$$\delta \leq 7$$

Case 2b: Change c_j of basic variable

Change c_j of basic variable

- Reduced cost $r_j = 0$
- Consider c_2
  - Allowable increase: 7
  - Allowable decrease: $+\infty$
- Consider changing c_2 from $5 \rightarrow 10$
  - New optimal solution: Unchanged
  - New optimal objective value:

$$\sum_{j=1}^{n} c_j^{new} x_j^* = \sum_{j=1}^{n} c_j x_j^* + \delta x_2^* = 110$$

Case 4: Add a new constraint

Add a new constraint

- If current solution satisfies the new constraint
  - New optimal solution: Unchanged
  - New optimal objective value: Unchanged
- If current solution does not satisfy the new constraint
  - Dual simplex method (but don't worry about this for now)

Note: the problem might become infeasible

Note

Sensitivity analysis lets you simultaniously think about a continious set of instances of LP for which δ is within the range. The other instances still have to be considered individually.

Thank you for your attention !