Simplex method

Fall 2021

Overview

Clarification: certificate of unboundedness

Simplex-method

Farkas Lemma: Geometrical picture

Reminder: Definitions

LP Sym Form:

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \geq \mathbf{b}, \\ & \mathbf{x} \geq 0. \end{aligned}$$

- **Feasible:** exists a feasible point
- **Infeasible:** any point is not feasible
- **Bounded:** exists a feasible point and exists a number such that any feasible point has the value worse than the number
- **Unbounded:** for any number, exists a feasible point that has value better than the number

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Certificate is a point that satisfies the condition

- C. of Feasibility:

$$\begin{aligned} A\mathbf{z} &\geq \mathbf{b}, \\ \mathbf{z} &\geq 0. \end{aligned}$$

- C. of Infeasibility:

$$\begin{aligned} \mathbf{w}^T\mathbf{b} &> 0 \\ \mathbf{w}^T A &\leq 0, \\ \mathbf{w} &\geq 0. \end{aligned}$$

- C. of Boundedness:

$$\begin{aligned} \mathbf{w}^T A &\leq \mathbf{c}, \\ \mathbf{w} &\geq 0. \end{aligned}$$

- C. of Unboundedness

$$\begin{aligned} \mathbf{c}^T\mathbf{z} &< 0, \\ A\mathbf{z} &\geq 0, \\ \mathbf{z} &\geq 0. \end{aligned}$$

Reminder: Theorems

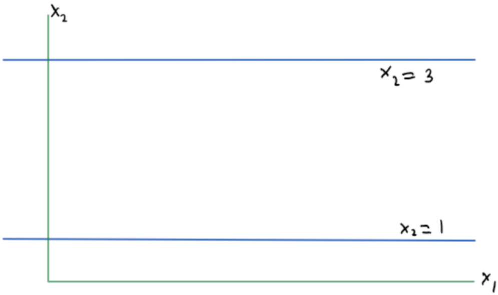
- If there exists a certificate of **feasibility**, then the problem is **feasible**
- If there exists a certificate of **infeasibility**, then the problem is **infeasible**
- If there exist a certificate of **feasibility** and a certificate of **boundedness**, then the problem is **bounded**
- If there exist a certificate of **feasibility** and a certificate of **unboundedness**, then the problem is **unbounded**

Note: the opposite are also true (by theorems of Alternatives)

Example

Feasibility certificate is crucial for unboundedness of the problem

$$
\begin{array}{llll}
\min & x_1 & & \\
\text{s.t.} & & x_2 & \geq 3 \\
 & & -x_2 & \geq -1 \\
 & x_1 & & \geq 0 \\
 & & x_2 & \geq 0
\end{array}
$$



Example

Infeasible! (cannot be unbounded)

$$
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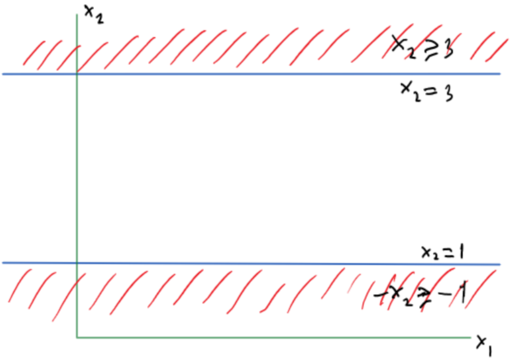


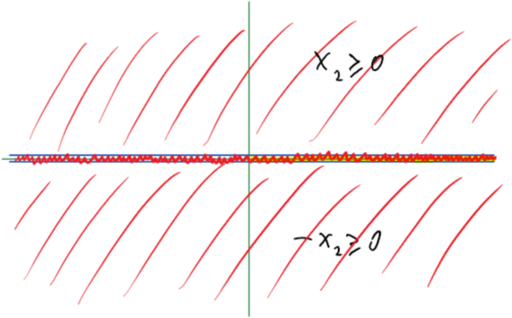
Example

Unboundedness Certificate:

$$\mathbf{c}^T\mathbf{z} < 0,$$
$$A\mathbf{z} \geq 0,$$
$$\mathbf{z} \geq 0.$$

In our case:

$$\begin{array}{rrcl} -z_1 & & < & 0 \\ & z_2 & \geq & 0 \\ & -z_2 & \geq & 0 \\ z_1 & & \geq & 0 \\ & z_2 & \geq & 0 \end{array}$$



Exists $\mathbf{z} = (1, 0)$ — an unboundedness certificate

Idea: certificate of optimality, expanded

(Primal) feasibility

$$\begin{aligned} A\mathbf{x} &\geq \mathbf{b}, \\ \mathbf{x} &\geq 0. \end{aligned}$$

Boundedness (aka Dual feasibility)

$$\begin{aligned} \mathbf{y}^T A &\leq \mathbf{c}, \\ \mathbf{y} &\geq 0. \end{aligned}$$

Tightness:

$$\mathbf{c}^\top \mathbf{x} = \mathbf{y}^\top \mathbf{b}$$

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Tightness can be replaced with the **Complementary Slackness Condition**

$$[\mathbf{c}^\top - \mathbf{y}^\top A]_i x_i = 0 \text{ for all } i \in \{1 \dots n\} \text{ (a)}$$

$$[Ax - \mathbf{b}]_j y_j = 0 \text{ for all } j \in \{1 \dots m\} \text{ (b)}$$

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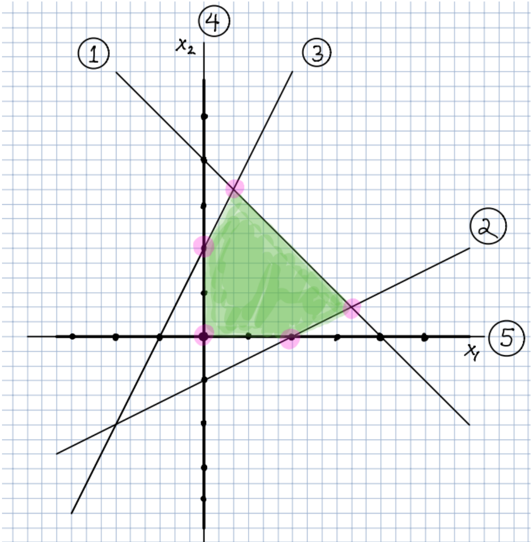
Fact: The optimum can be achieved only in an extreme feasible point. (This is a general fact for any concave objective)
Idea behind Simplex Method: Walk on (x, y) of the form (feasible extreme point, dual point) satisfying complementary slackness, looking for y that satisfies dual feasibility

Example



$$
\begin{array}{llllll}
\min & -x_1 & -2x_2 & & \\
\text{s.t.} & -x_1 & -x_2 & \geq & -4 \\
 & -x_1 & +2x_2 & \geq & -2 \\
 & 2x_1 & -x_2 & \geq & -2 \\
 & x_1 & & \geq & 0 \\
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\end{array}
$$

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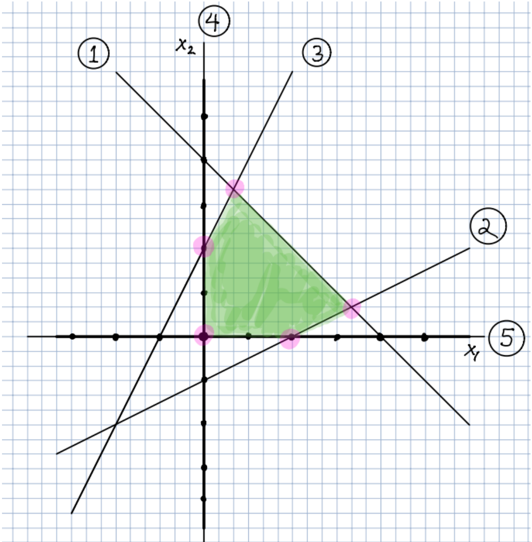
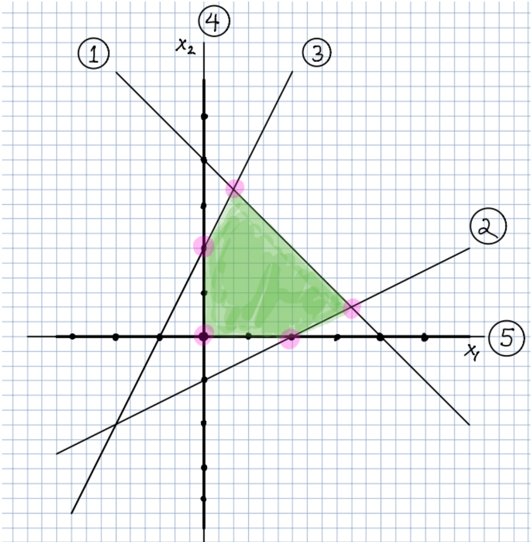


Point $(2, 0)$.

Example

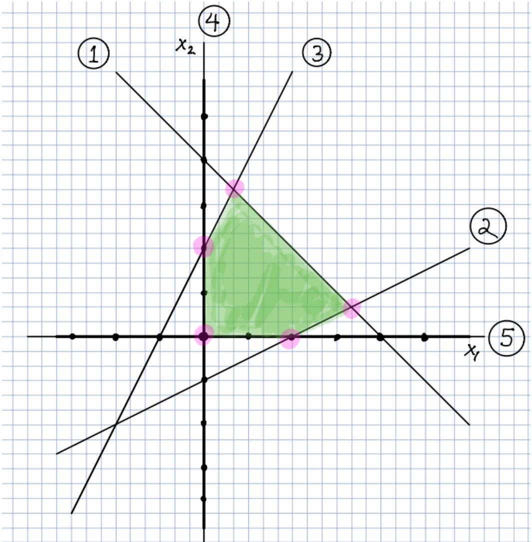


Point $(2, 0)$.
Complementary slackness (a):

$$[-1+y_1+y_2-2y_3]x_1 = 0$$

$$[-2+y_1-2y_2+y_3]x_2 = 0$$

Example



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$$[-1+y_1+y_2-2y_3]x_1 = 0$$

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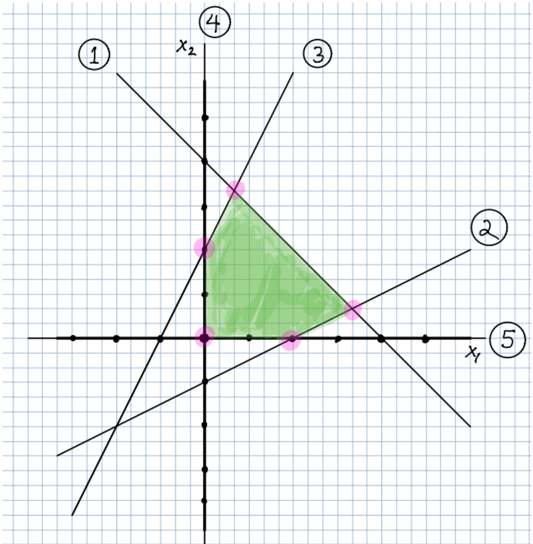
Complementary slackness (b):

$$[-x_1 - x_2 + 4]y_1 = 0$$

$$[-x_1 + x_2 + 2]y_2 = 0$$

$$[2x_1 - x_2 + 2]y_3 = 0$$

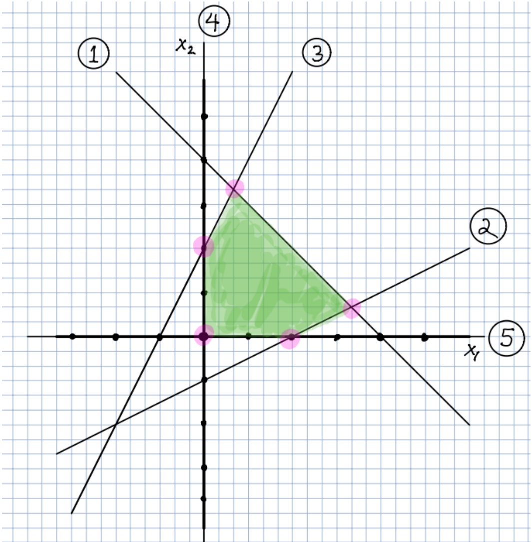
Example



From (a):

$$y_1 + y_2 - 2y_3 = 1$$

Example



From (a):

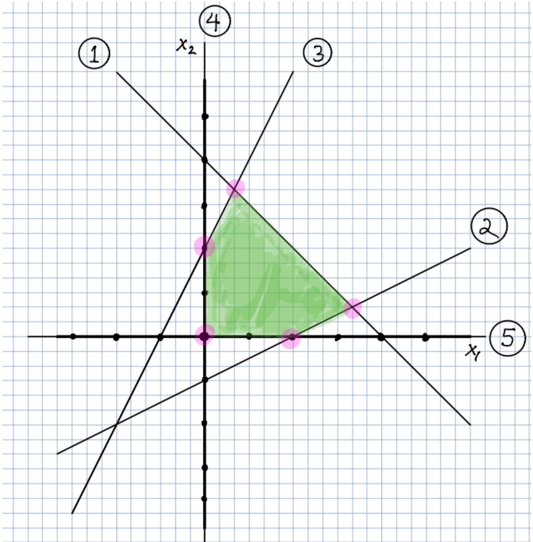
$$y_1 + y_2 - 2y_3 = 1$$

From (b):

$$y_1 = 0; \quad y_3 = 0$$

Therefore, $y_2 = 1$

Example



From (a):

$$y_1 + y_2 - 2y_3 = 1$$

From (b):

$$y_1 = 0; \quad y_3 = 0$$

Therefore, $y_2 = 1$ Dual feasibility:

$$\begin{array}{ccccc} -y_1 & -y_2 & +2y_3 & \leq & -1 \\ -y_1 & +y_2 & -y_3 & \leq & -2 \\ y_1, & y_2, & y_3 & \geq & 0 \end{array}$$

Is violated, so $(2, 0)$ is not optimal

Finding a basic solution

Example:

$$A = \begin{pmatrix} 3 & -2 & 1 & 4 \\ 0 & 1 & 3 & -6 \\ -1 & 2 & 2 & 0 \\ 3 & -1 & -1 & 2 \\ 6 & 0 & 3 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 8 \\ 3 \\ 1 \\ 15 \end{pmatrix}, \quad c = \begin{pmatrix} 5 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

$\min c^T x$	$\max b^T y$
$Ax \geq b$	s.t $A^T y \leq c$
s.t $x \geq 0$	$y \geq 0$

Finding a basic solution

$I = \{1, 3\}$, $J = \{2, 3\}$, $\hat{I} = \{2, 4, 5\}$, $\hat{J} = \{1, 4\}$

$$A_{IJ} = \begin{pmatrix} -2 & 1 \\ 2 & 2 \end{pmatrix} \qquad \bar{x}_J = \begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\bar{x}_{\hat{J}} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\bar{y}_I = \begin{Bmatrix} \bar{y}_1 \\ \bar{y}_3 \end{Bmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \qquad \bar{y}_{\hat{I}} = \begin{pmatrix} \bar{y}_2 \\ \bar{y}_4 \\ \bar{y}_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{s}_I = \begin{pmatrix} \bar{s}_1 \\ \bar{s}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \bar{s}_{\hat{I}} = \begin{pmatrix} 1 & 3 \\ -1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} - \begin{pmatrix} 8 \\ 1 \\ 15 \end{pmatrix}$$

$$\bar{r}_J = \begin{pmatrix} \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \bar{r}_{\hat{J}} = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} \bar{y}_2 \\ \bar{y}_3 \end{pmatrix}$$

Finding a basic solution

Can use Gauss-Jordan method to calculate the inverse

$$A_{IJ}^{-1} = \begin{pmatrix} -2 & 1 \\ 2 & 2 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} -2 & 1 \\ 2 & 2 \end{pmatrix}$$

		5
		3
-2	1	-7
2	2	16

$$\bar{x}_J = \begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = A_{IJ}^{-1}\underbrace{\begin{pmatrix} b_1 \\ b_3 \end{pmatrix}}_{= b_I} = \frac{1}{6}\begin{pmatrix} -2 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -7 \\ 16 \end{pmatrix}$$

Another way: $\begin{cases} -2x_2 + x_3 = 5 \\ 2x_2 + 2x_3 = 3 \end{cases} \Rightarrow 3x_3 = 8 \Rightarrow -6x_2 + 8 = 15$

$$\Downarrow \qquad \qquad \Downarrow$$

$$6x_3 = 16 \qquad -6x_2 = 7$$

$$\Downarrow$$

$$\begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -7 \\ 16 \end{pmatrix}$$

Finding a basic solution

$$\bar y_I = \begin{pmatrix}\bar y_1\\ \bar y_3\end{pmatrix} = (A_{IJ}^{-1})^T \underbrace{\begin{pmatrix}c_2\\ c_3\end{pmatrix}}_{c_J} = \frac{1}{6}\begin{pmatrix}-2 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}0\\ -2\end{pmatrix} = \frac{1}{6}\begin{pmatrix}-4\\ -4\end{pmatrix} = -\frac{2}{3}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\bar s_{\hat I} = \begin{pmatrix}\bar s_2\\ \bar s_4\\ \bar s_5\end{pmatrix} = A_{\hat I J}\bar x_J - b_{\hat I} = \frac{1}{6}\begin{pmatrix}1 & 3\\ -1 & -1\\ 0 & 3\end{pmatrix}\begin{pmatrix}-7\\ 16\end{pmatrix} - \begin{pmatrix}8\\ 1\\ 15\end{pmatrix} = \frac{1}{6}\begin{pmatrix}41\\ -9\\ 48\end{pmatrix} - \frac{1}{6}\begin{pmatrix}48\\ 6\\ 90\end{pmatrix} = -\frac{1}{6}\begin{pmatrix}7\\ 15\\ 42\end{pmatrix}$$

$$\bar r_{\hat J} = \begin{pmatrix}\bar r_1\\ \bar r_4\end{pmatrix} = c_{\hat J} - A_{I\hat J}^T \bar y_I = \begin{pmatrix}5\\ 1\end{pmatrix} - \left(-\frac{2}{3}\right)\begin{pmatrix}3 & -1\\ 4 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}15\\ 3\end{pmatrix} + \frac{1}{3}\begin{pmatrix}4\\ 8\end{pmatrix} = \frac{1}{3}\begin{pmatrix}19\\ 11\end{pmatrix}$$

Farkas Lemma formulation (for Standard form)

Standard form LP:

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq 0. \end{aligned}$$

Feasibility certificate:

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Infeasibility certificate:

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Farkas Lemma: Exactly one out of two exists: x or y
(Equivalent to Theorem 2 of alternatives from LPMATH)

Reminder from Linear Algebra

Consider $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$. The matrix $[\mathbf{a}_1 \ldots \mathbf{a}_n] = A \in \mathbb{R}^{m \times n}$

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- Columns of the matrix in multiplication on the left:

$$\mathbf{y}^\top A = \mathbf{y}^\top [\mathbf{a}_1 \ldots \mathbf{a}_n] = [\mathbf{y}^\top \mathbf{a}_1 \ \ \ldots \ \ \mathbf{y}^\top \mathbf{a}_n]$$

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- Columns of the matrix in multiplication on the right:

$$A\mathbf{x} = [\mathbf{a}_1 \ldots \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{a}_1 + \ldots + x_n \mathbf{a}_n = \sum_{j=1}^{n} x_j \mathbf{a}_j$$

Conic combination of vectors

For $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$, a *linear combination* is a vector $\mathbf{v} \in \mathbb{R}^n$ that can be represented as

$$\mathbf{v} = \sum_{i=1}^{m} w_i \mathbf{a}_i \text{ or } v = \mathbf{v}^\top A$$

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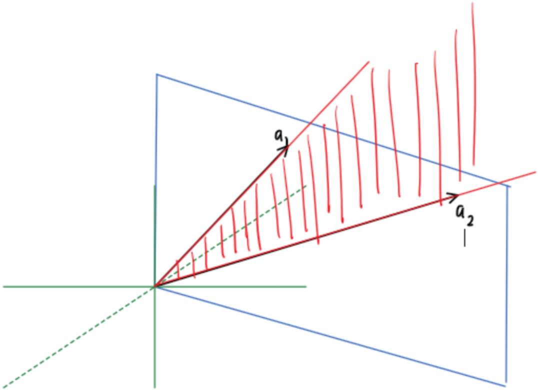
for some $w_1, \ldots, w_m \geq 0$

Conic combination of vectors

As the set of all linear combinations produces a linear span, the set of all conic combinations produces a **cone**.



Geometric interpretation

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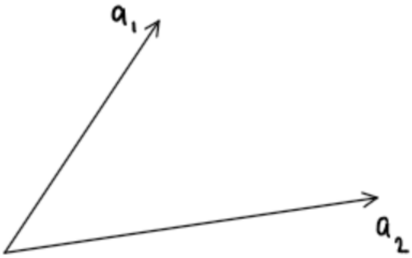

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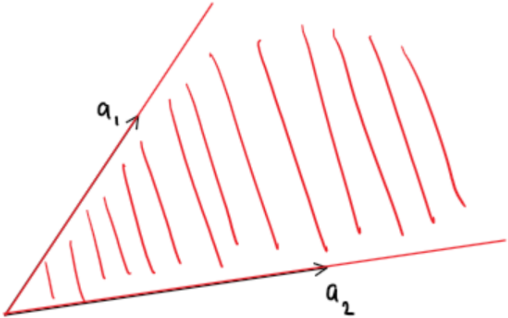

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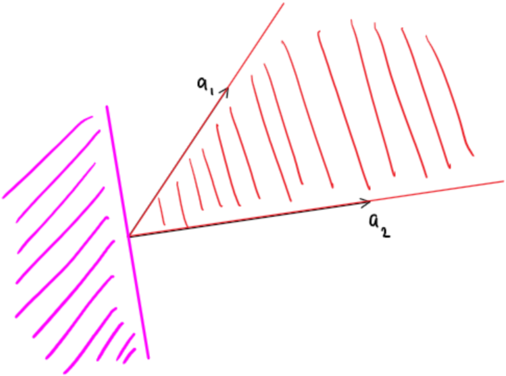

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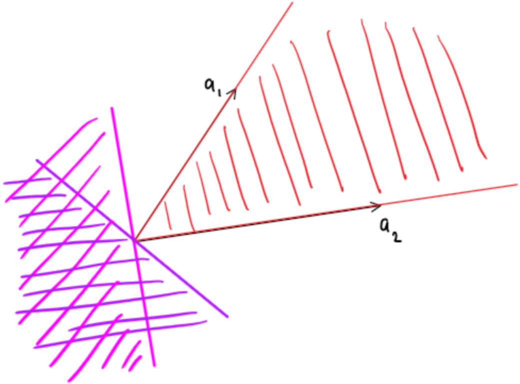

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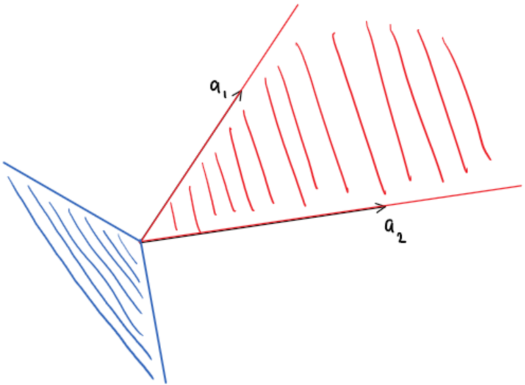

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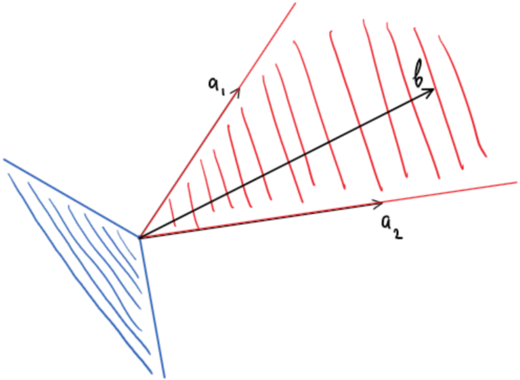

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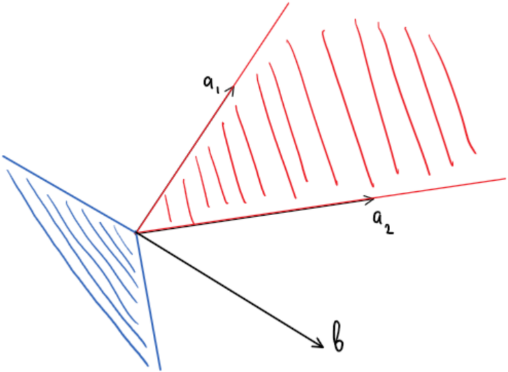
$$\mathbf{y}^T \mathbf{b} > 0$$
$$\mathbf{y}^T A \leq 0.$$



Thank you for your attention !