PROBABILITY MASS FUNCTION (PMF)

Example: Chess Match. Anand plays Kasparov, and the first to win a game wins the match. Match is drawn if there are 10 consecutive draws.

Given: \( \Pr[\text{Anand wins}] = 0.3, \Pr[\text{Kasparov wins}] = 0.4, \Pr[\text{draw}] = 0.3 \)

Let \( L \) = duration of match. Find PMF of \( L \), i.e., \( P_L(k) \) vs. \( k \).

\[
P_L(k) = \begin{cases} 
0.3^{k-1}(0.7) & \text{for } k \leq 10 \\
0.3^k & \text{for } k \geq 0, \text{ or } k > 10 \\
0 & \text{otherwise}
\end{cases}
\]

GEOMETRIC RV: Counts time to first success (like example above)

If \( P(\text{success}) = p \) and we keep performing independent trials until first success,

\[
P_X(k) = (1-p)^{k-1}p
\]

OTHER POPULAR RVs

- BERNOULLI RV: Contains such that \( X = \begin{cases} 
1 & \text{with probability } p \\
0 & \text{with probability } (1-p)
\end{cases} \)

Example: Bit-Flip Model for Communication systems

Input Bit \( X \) \( \stackrel{\text{i.i.d.}}{\leftrightarrow} \) Output Bit \( Y \)

BINOMIAL RV: Counts successes in \( n \) trials,

\[
P_X(k) = \binom{n}{k}p^k(1-p)^{n-k} = \text{Bin}(n, p)
\]

Example: \( X \) is number of heads in \( n \) coin tosses

Dealing with combinatorics is annoying though, so consider Poisson RV instead.

POISSON RV: Good approximation of a binomial random variable

\[
P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k=0, 1, \ldots \text{ Poisson}\left(\lambda\right)
\]

Note: \( n(p) \rightarrow \text{Poisson}\left(\lambda\right) \) as \( n \rightarrow \infty \) (approximation)

Why is this true? \( \binom{n}{k}p^k(1-p)^{n-k} \geq \frac{n^k}{k!} p^k e^{-\lambda} \text{ if } n \gg k \)

\[
(\text{Recall: } e^x = 1 + x + \frac{x^2}{2} + \ldots)
\]
FUNCTIONS OF RVs

\[ Y = g(X) = aX + b \]

Note: Since \( X \) is random, \( Y \) is clearly also random.

\[ P_Y(y) = \left\{ \begin{array}{ll} \sum_{x} P_X(x) & \text{if } y \in \mathbb{R} \\ 0 & \text{elsewhere} \end{array} \right. \]

Example: Start with uniform RV such that \( P_X(x) = \frac{1}{4} \) \( x \in \{1, 2, 3, 4\} \). Let \( Y = |X| \). What is \( P_Y(y) \)?

\[ Y \]

\[ P_Y(y) \]

\[ xy \]

\[ \frac{1}{4} \]

\[ \frac{1}{4} \]

\[ \frac{1}{4} \]

\[ \frac{1}{4} \]

\[ \frac{1}{4} \]

\[ \frac{1}{4} \]

MEAN AND VARIANCE

**Expected Value:**
\[ E(X) = \sum_{x} x P(X=x) \]
where \( \Omega \) is all possible values taken by RV
\[ = \sum_{x \in \Omega} x P(x) \text{ (all \( x \) all outcomes)} \]

**Linearity of Expectation:** For any two random variables \( X \) and \( Y \) in same probability space,
\[ E(aX + bY) = aE(X) + bE(Y) \]
without requiring independence assumption!

Example: Find expected sum of two dice rolls.
\[ E \left[ \text{Sum of Rolls} \right] = E(X_1 + X_2) = E(X_1) + E(X_2) = 3 + 3 = 6 \]

Example: We turn in our homeworks, and Prof. Ramachandran mixes them up and returns them to us. What is expected number of students who get back their homework submission?

**Indicator Variable** (also a Bernoulli variable)

\[ X_i = \begin{cases} 1 & \text{if student } i \text{ gets his HW back} \\ 0 & \text{else} \end{cases} \]

\[ E[X_i] = \frac{1}{n} \cdot 1 + \frac{n-1}{n} \cdot 0 = \frac{1}{n} \]

\[ E[X] = \frac{1}{n} \cdot E[X_i] = \frac{n}{n} = \frac{1}{n} \]

(regardless of \( n \)).

Example: \( X \) ~ Binomial \((n, p)\), which means \( P(X=x) = \binom{n}{x} p^x (1-p)^{n-x} \) \( x = 0, 1, \ldots \)

\[ E(X) = \sum_{x=0}^{n} x P(X=x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \text{complicated} \]

Be lazy, apply earlier trick! Use indicator variables!

Let \( X_i = X_1 + X_2 + \ldots + X_n \) where \( X_i = \begin{cases} 1 & \text{if } i^{th} \text{ flip is } H \\ 0 & \text{otherwise} \end{cases} \)

\[ E[X] = n \cdot E[X_i] = \frac{n}{n} \]

\[ \text{Var}(X) = E[(X - E(X))^2] = E[X^2] - (E[X])^2 \]

\[ \text{Variance: } \text{Var}(X) = E[X^2] - (E[X])^2 \]

**Motivation:** Two distributions can have the same mean, but different spread.

**Standard Deviation:**
\[ \sigma_X = \sqrt{\text{Var}(X)} \]

\[ \text{Var}(X) = E[(X - E(X))^2] = E[X^2] - (E[X])^2 \]

\[ \text{Variance: } \text{Var}(X) = E[X^2] - (E[X])^2 \]

**Motivation:** Two distributions can have the same mean, but different spread.
Example: Find mean of geometric RV.  \( P_X(k) = p(1-p)^{k-1} \)

Indicator variables probably won't help much.

Geometric RVs have **memoryless property**:  \[ P_{X|X\geq k}(k+m) = \frac{P(X=k+m)}{P(X\geq k)} = p(1-p)^{m-1} = P_X(m) \]

where \( k \) starts from 1.

\[ E(X|X+1) = E(X) + 1 \]

Intuitively: has to be at least 1. Kind of "reset" for next flip.

\[ E(X) = P(x=1)E[X|X=1] + P(x>1)E[X|x>1] \]

\[ = p(1) + (1-p)(1+E[X]) \]

\[ E[X] = \frac{1}{p} \]