**Sum of Independent Random Variables**

Let \( Z = X + Y \) and assume \( X, Y \) are continuous PDFs and independent. Then:

\[
f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx = \int_{-\infty}^{\infty} f_X(x) f_Z(z|x) \, dx
\]

Apply the conditional:

\[
F_Z(z|x) = P[Z = z | X = x] = P[Y = z-x] = F_Y(z-x)
\]

\[
f_Z(z|x) = f_Y(z-x) \quad (\text{Differentiate with respect to } z)
\]

\[
f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx = (f_X * f_Y)(z) \quad \text{(Convolution)}
\]

Example of convolution: \( X \) and \( Y \) are independent. How is \( Z = X + Y \) distributed?

\( (X, Y \sim U[0,1]) \) with and without

![Graph](image)

Example: \( X, Y \sim N(0,1) \) are independent. \( Z = X + Y \). Find \( f_Z(z) \)

\[
f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx = \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{x^2}{2\sigma_1^2}} \right) \left( \frac{1}{\sqrt{2\pi} \sigma_2} e^{-\frac{(z-x)^2}{2\sigma_2^2}} \right) \, dx
\]

\[
= \frac{1}{2\pi \sigma_1 \sigma_2} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} e^{-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}} \, dx
\]

\[
= \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{(z-\mu_1)^2}{2\sigma_1^2}} \left( \frac{1}{\sqrt{2\pi} \sigma_2} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}} \, dx \right)
\]

\[
\Rightarrow Z \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)
\]

Note: Start with any two forms and convolve repeatedly. Eventually will get a Gaussian.

**Covariance and Independence**

- Start by looking at variance: \( \text{Var}(X+Y) = \text{E}[(X+Y-M_X-M_Y)^2] = \text{E}[(X-M_X)^2 + (Y-M_Y)^2] = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y) \)

\[
\text{Cov}(X,Y) = \text{E}[(X-M_X)(Y-M_Y)]
\]

- **Covariance:** \( \text{cov}(X,Y) = \text{E}[(X-M_X)(Y-M_Y)] = \text{E}(XY) - \text{E}(X)\text{E}(Y) \)

- \( \text{cov}(X,Y) = 0 \) when \( \text{E}(XY) = \text{E}(X)\text{E}(Y) \) \( (X \text{ and } Y \text{ are uncorrelated}) \)

- If \( X \) and \( Y \) are independent, they are uncorrelated.

\( X, Y \) are independent \( \Rightarrow f_{XY}(x,y) = f_X(x)f_Y(y) \quad \forall x,y \)

\( X, Y \text{ uncorrelated} \neq X, Y \text{ are independent!} \) (See HW 3 Q.6)
So what is relationship between uncorrelated vs independent?

\[ \text{E}[XY|Y=y] = \text{E}[X] \quad \Rightarrow \quad \text{Uncorrelated} \]

\[ \text{f}_{XY}(x,y) = f(x) \quad \forall x \quad \Rightarrow \quad \text{Independent} \quad \text{[Independence is more stringent!]} \]

**Correlation Coefficient Between X and Y:**

\[ \rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} -1 \leq \rho \leq 1 \]

(normalizes \(\text{cov}(X,Y)\))

True by Cauchy-Schwarz

This is a measure of linearity, NOT independence!

**Example:**

\[ Y = ax + b \]

What is \(\rho(X,Y)\)?

\[ \rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{E}[X(ax+b)] - \text{E}[X] \text{E}[ax+b]}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{a \text{Var}(X)}{\sqrt{\text{Var}(X)a^2}} = \frac{a}{\sqrt{a^2}} = \frac{a}{|a|} \]

\[ \rho(X,Y) = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases} \]

\[ \frac{\text{E}[x^2+bx] - \text{E}[x] \text{E}[ax+b]}{\sqrt{\text{Var}(x)\text{Var}(ax+b)}} \]

**Example:**

\[ Z = X^2 \]

(Note that \(Z\) is fully determined by \(X\))

We assume this is highly correlated, but it is not. Let \(X = \begin{cases} P_X(1) = 0.2 \\ P_X(2) = 0.8 \end{cases} \)

\[ \text{Cov}(X,Z) = 0.28 \]

\[ \rho(X,Z) = 0.582 \]

**Iterated Expectation:**

\[ \text{E}_Y[\text{E}_X(X|Y=y)] = \text{E}[X] \quad \text{Read 4.3.} \]

**Law of Total Variance:**

\[ \text{Var}[X] = \text{E}[\text{Var}(X|Y)] + \text{Var}[\text{E}(X|Y)] \]

**Moment Generating Functions (M.G.F.s) (Transforms)**

Start with Taylor Series expansion. \(s\) is scalar, \(X\) is random variable.

\[ e^X = 1 + sX + \frac{s^2X^2}{2} + \ldots \]

Each \(X^n\) is the \(n^{th}\) moment.

**Using linearity of expectation:**

\[ \text{E}[e^{sX}] = e + s \text{E}[X] + \frac{s^2}{2!} \text{E}[X^2] + \ldots \]

We have generated all expected values of moments as terms in a Taylor series!

Observe:

\[ \frac{d}{ds} \text{E}[e^{sX}] = \text{E}[X] + s \text{E}[X^2] + s^2 \text{E}[X^3] + \ldots \]

Evaluating \(s=0\):

\[ \frac{d}{ds} \text{E}[e^{sX}] = \text{E}[X] \]

Want the \(n^{th}\) moment? Take \(n^{th}\) derivative! Avoid integrals!