LAW OF ITERATED EXPECTATION

\[ E\left[E_X(X|Y)\right] = E[X] \]

\[ E[X|Y=y] \text{ is a function of random variable } Y \]

Proof: \[
LHS = \sum_y Pr(y) E[X|Y] = \sum_y Pr(y) \left[ \sum_x P_{X|Y}(x|y) x \right] = \sum_y P_Y(y) \sum_x P_{X|Y}(x|y) x
\]

\[ = \sum_x \left( \sum_y P_{X|Y}(x|y) \right) x = \sum_x P_X(x) x = E[X] \]

Example: Biased coin. Probability of getting a head is itself a random variable Y. Toss the coin n times. Let X = number of heads. Find E[X].

Intuitively, if Y is uniform over \{0,1\}, \( E[Y] = \frac{n}{2} \). To show formally:

\[ E[X|Y=y] = ny \]

\[ E[X] = E[E[X|Y]] = E[nY] = n \; E[Y] \]

Law of Iterated Expectation

If \( Y \sim U(0,1) \):

\[ E[X] = n \; E[Y] = \frac{n}{2} \]

ESTIMATION

Given an observation Y, we want to estimate another quantity of interest X. In particular, let's find estimate \( \hat{x} \) of X that minimizes the expected mean square error, i.e., find \( \hat{x} \) such that \( E[(X-\hat{x})^2] \) is minimized.

Effectively, we want to find \( \hat{x} = E[X|Y] \). Why? \( \hat{x} \) is function of Y, given Y it is a constant.

Proof:

\[ E[(X-\hat{x})^2|Y] = E[x^2 - 2x\hat{x} + \hat{x}^2|Y] = E[x^2|Y] - 2\hat{x} E[x|Y] + \hat{x}^2 \]

\[ = E[x^2|Y] - \left( E[x|Y] \right)^2 + \left( E[x|Y] \right)^2 - 2\hat{x} E[x|Y] + \hat{x}^2 \]

\[ = \text{Var}(x|Y) + \left( E[x|Y] - \hat{x} \right)^2 \]

\[ \Rightarrow \left( E[x|Y] - \hat{x} \right)^2 = 0 \Rightarrow E[x|Y] = \hat{x} \]

\[ \text{Proof:} \text{ From nature, the CLT controls. To minimize, make this term 0.} \]
Define $\Delta = X - \hat{x}$. What is $E(\Delta)$? (Assume $\hat{x} = E(X|Y)$)

$E(\Delta) = E(E(\Delta|Y)) = E(E[X - E(X|Y)|Y]) = E(E[X|Y]) - E(X|Y) = 0$

Given $Y$, $E(X|Y)$ is fully determined.

Whenever we have $E(\Delta) = 0$, we have an **UNBIASED ESTIMATOR**.

What is $E[A \cdot \hat{x}]$?

$E[A \cdot \hat{x}] = E[E[A \cdot \hat{x}; Y]] = E(E[A \cdot \hat{x}|Y]) = 0$

$\Rightarrow A$ and $\hat{x}$ are uncorrelated.

$LAW \ OF \ TOTAL \ VARIANCE$

From above diagram, $X = \hat{x} + \Delta$. 

$Var[X] = Var[\hat{x}] + Var[\Delta] + 2 \cdot Cov(X, \Delta) = Var[\hat{x}] + Var[\Delta]$

$\Rightarrow Convince \ yourself$.

The **LAW OF TOTAL VARIANCE**: $Var[X] = Var[E(X|Y)] + E(Var(X|Y))$

Example: Back to the biased coin, where $X$ was the number of heads. Find $Var(X)$. (Assume that $Y = Pr(H)$ is uniformly distributed between 0 and 1.)

Remember: we know $Y$! (since we condition.) This is just coin flipping with a biased coin - binomial distribution.

$E[\text{Var}(X|Y)] = E[n\text{Var}(Y)] = n \cdot E[Y] - n \cdot E[Y^2] = \frac{n}{6}$

$Var \text{ of } Bin(n, Y)$

$Var[E(X|Y)] = Var(nY) = n^2 \cdot Var[Y] = \frac{n^2}{12}$

$\therefore Var[X] = Var(E[X|Y]) + E(Var(X|Y)) = \frac{n^2}{12} + \frac{n}{6}$

Price of not knowing bias is higher variance. (Go away with it on mean side)

**MOMENT GENERATING FUNCTION** (TRANSFORMS)

$X \rightarrow E[e^{sX}] \equiv M_X(s)$

By Taylor Series: $e^{sX} = 1 + sX + \frac{s^2X^2}{2!} + \frac{s^3X^3}{3!} + ...$

$E(e^{sX}) = 1 \cdot E(X) + \frac{s^2}{2!} \cdot E(X^2) + \frac{s^3}{3!} \cdot E(X^3) + ...$ *This is a snapshot of $X$. Captures all interesting moments of $X$.*

$\frac{d^n}{ds^n} E(e^{sX})|_{s=0} = E(X^n)$

$\frac{d^n}{ds^n} E(e^{sX})|_{s=0} = E(X^n)$

$E(e^{sX}) = \int_{-\infty}^{\infty} e^{sx} f_X(x) \, dx \quad \ast \text{Laplace transform of PDF.} \quad (F(s) = \int_{0}^{\infty} e^{-xt} dt)$
Recall the sum of independent random variables: \( Y = X_1 + X_2 \)

\[ f_Y(y) = (f_{X_1} \ast f_{X_2})(y) \quad \leftrightarrow \quad M_Y(s) = M_{X_1}(s) \cdot M_{X_2}(s) \]

Prove from properties of Laplace transforms.

Example: Let \( X_1, X_2 \) be i.i.d. \( \sim N(0, 1) \). Find the PDF of \( Y = X_1 + X_2 \).

Since \( X_1 \sim N(0, 1) \),

\[ f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{compute the MGF:} \]

\[ M_X(s) = E[e^{sX}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx} e^{-\frac{x^2}{2}} \, dx = e^{\frac{s^2}{2}} \]

\[ M_Y(s) = M_{X_1}(s) \cdot M_{X_2}(s) = e^{\frac{s^2}{2}} \quad \leftrightarrow \quad f_Y(y) \sim N(0, 2) \]