QFT Lecture 1

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1 Outline

Classical Field Theory
- Quantum Scalar Theory
- Quantum Dirac Field (spinor)
- Interacting Fields
- Perturbation Theory
- A little on normalisation

2 Motivation

If we actually wanted to describe the universe, we would need a Quantum Gravity theory. This would theoretically describe the universe perfectly; however in real life we don't have access to very high energies, or very small scales. So as an *effective theory*, we have relativistic quantum field theory, to describe the universe. At even lower energies, and even larger time scales, we have nonrelativistic QFT. To probe both these QFT theories, we don't have direct access, we only have access to *decoherence*, which clues us in to the nature of reality. At even larger amounts of decoherence, we find ourselves with classical field theory, which is an effective model for our theory. Physics is about going 'backwards', we start with classical field theory, trying to undo the approximations, go back to QFT, relativistiv QFT, and maybe one day quantum gravity.

Let's start depacking what we mean by this. Firstly, what does it mean for a quantum theory to be relativistic? As a first answer, we can say that the theory is symmetric under the Poincare group. Recall that if (x_0, x_1, x_2, x_3) are coordinates (where x_0 is the temporal coordinate) in an inertial frame, then in any other reference frame, we have:

$$\eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}$$

(under the einstein summation notation) or

$$\eta_{\mu\nu} \frac{dx'^{\mu}}{dx^{\rho}} \frac{dx'^{\nu}}{dx^{\sigma}} = \eta_{\rho\sigma}$$

where η is a diagonal matrix such that $\eta_{0,0} = 1$, $\eta_{1,1} = \eta_{2,2} = \eta_{3,3} = -1$. We know from our early days in special relativity that any coordinate transformation satisfying the above constraints, is linear and is of the form:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$$

where Λ is thought of as a rotation, and a^{μ} is a constant 4-vector, where Λ obeys the equation:

$$\eta_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = \eta_{\rho\sigma}$$

which can be thought of as:

$$\Lambda^T \eta \Lambda = \eta$$

These transformations form a group, P_4, when written in the form (Λ, a), then its easy to find the product rule:

$$(\bar{\Lambda}, \bar{a}) \cdot (\Lambda, a) = (\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a})$$

So now we know what it means for something to be 'relativistic'. From our early undergraduate quantum days, we know that symmetries are represented by either a unitary or an antiunitary operator. So, to summarise, to build a relativistic QM theory, we need:

- A (separable) Hilbert space $\mathcal{H}$
- To every $(\Lambda, a) \in P_4$ we can find a unitary operator

$$U(\Lambda, a) : \mathcal{H} \to \mathcal{H}$$

such that the identity poincare transformation is associated to the identity operator (multiplied by a phase), and the group structure of the Poincare transformations is preserved (i.e. we are really looking for a homomorphism from P_4 into the group of unitary operations on the hilbert space (up to a phase factor). If we do happen to find a set of unitary operators that obey these axioms, they are called 'Projective Unitary Representations'.

As a 'trivial' example, we can choose all the U to be the identity operator, but this is not *really* what we are looking for. As a more 'nontrivial' example, we might set $U = \Lambda$, which seems to preserve group structure, but sadly the Λ are not necessarily unitary. You might go on to think more about finding examples, but as a warning:

There are no nontrivial finite-dimensional unitary representations of P_4

This is in stark contrast with rotations, $SO(3)$, where there are loads of unitary representations of it (in simple spin systems, for example). The mathematical reason why $SO(3)$ is easy but P_4 is hard, is that $SO(3)$ is a compact group whereas P_4 is not compact!

Now, for some even more bad news. We can consider the 'time translation subgroup' of P_4 given by the collection of poincare transformations $(I, (t, 0, 0, 0))$, i.e. just shifts in time. Call this subgroup V_t, the time translation subgroup. Suppose we found a unitary representation of P_4, then $V(t) = U(I, (t, 0, 0, 0))$ is a one parameter family of unitary operators, $V(s + t) = V(s)V(t)$, which solves the shrodinger equation:

$$\frac{dV(t)}{dt} = iHV(t)$$

for some H self adjoint. That is to say, if we really did find the whole family U, we would have found all the $V(t)$, but this is essentially solving all of quantum mechanics! Relativistic QFT is a strictly more difficult problem than ordinary quantum mechanics, and we already know how hard that was! Note that our H here doesn't have much requirements placed on it; which may be bad news, imagine if H had eigenvalues that approached negative infinity! Then our system could just emit energy, and approach lower and lower energy levels, and become completely unstable! So we have an additional requirement on our unitary representation (which makes life even harder), that our U are "positive energy", i.e. that the spectrum of H is a subset of $\mathbb{R}^+$.

Now for some good news! All the representations of the poincare group have been classified! (all such *single particle* systems, that is). They were classified by Wigner, and are labelled by two numbers, mass m and helicity/spin s.

The bad news is; the lecturer will *not* cover this; because even though it is beautiful, nature chooses not to be a single of single particles. Nature chooses to be consisting of many particles.