Phys 112 : Lecture 19

Notes for the Fall 2017 Physics 112 Course taught by Professor Holzapfel
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1 Adiabatic Expansions

From last lecture, recall that we found that for an ideal gas, we have:

$$U = \frac{\alpha}{2} N\tau$$

where α is the number of quadratic degrees of freedom for the gas. For convenience, we usually define $\gamma = (\alpha + 2)/\alpha$, and with this constant we have that:

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

for an adiabatic, reversible (no entropy generation) process, going from (P_1, V_1) to (P_2, V_2). [1]. We also have that:

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

For a general adiabatic, reversible process, we can then calculate the change in internal energy:

$$\Delta U = U_2 - U_1 = C_V(T_2 - T_1) = \frac{\alpha}{2} Nk(T_2 - T_1) = -W$$

where W is the work done by the gas. As a prototypical example of such an adiabatic, reversible process, we can imagine a gas constrained to half a chamber, and slowly extending the piston, letting the gas do work on the piston. As the 'opposite' kind of expansion, we can imagine instead simply vanishing the piston holding the gas back instantaneously. In this case, the gas can do no work, so it's still an adiabatic process, but we find that:

$$\Delta\sigma = \sigma_2 - \sigma_1 = N \ln\left(\frac{V_2}{V_1}\right)$$

so the process is certainly not reversible (how would we put the piston back?)

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[1] Recall that an adiabatic process is one where no heat is transferred

2 Return to Gases

So far, we have been dealing in the classical limit, where

$$n << n_Q = \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}$$

and the bose-like and fermi-like gases are much the same. Now, we will more carefully study the opposite limit, where:

$$n \geq n_Q; \qquad \text{Quantum Gas}$$

Note that the quantum concentration depends on temperature; so for a system we can lower the temperature more and more, until the quantum effects actually become important. We can solve for this temperature at which the quantum effects actually become important, when $n_Q \approx n$, and we find:

$$\tau_0 = \left(\frac{2\pi\hbar^2}{m}\right)n^{2/3}$$

which is some sort of 'reference temperature' below which we need to take into serious consideration quantum effects.

Now, let's study the case of a gas of Fermions. In this case, we know by the pauli exclusion principle that each orbital can only be filled by a single particle (or two particles, spin up and spin down, depending on your definition for your orbitals). Intuitively, at low energy, the fermions aren't going to be thermally excited very much, so we expect all the lower energy levels of the system to be filled. In this case, we can consider the Fermi energy, which is essentially the difference in energy between the ground state, and the top energy state filled when every intermediate state is filled completely by fermions.

Now, we can consider the case where we have no specific potential, i.e. the fermions are all free travelling waves. In this case, our energy levels are given by:

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m}$$

and we have some fermi energy determined by the density of the fermions:

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$$

Now at $\tau = 0$, we have that for $k > k_F$, the energy states are, with certainty, empty, and for $k < k_F$ they are filled with certainty. As you increase temperature slowly, you start getting fluctuations (thermal) around the fermi energy, and this is where it gets interesting.

Now for a free travelling wave we know that:

$$n_x = \frac{L}{2\pi}k_x$$

For this case, we find that the density of states:

$$D(k)dk = \frac{L^\alpha}{(2\pi)^\alpha}d^\alpha\vec{k}$$

for low dimensions, we have that:

$$\text{1-D: } D(k)dk = \frac{L}{2\pi}d\vec{k} = \frac{L}{\pi}dk$$

$$\text{2-D: } D(k)dk = \frac{A}{(2\pi)^2}d^2\vec{k} = \frac{A}{(2\pi)^2}2\pi k dk = \frac{A}{2\pi}kdk$$

$$\text{3-D: } D(k)dk = \frac{V}{(2\pi)^3}d^3\vec{k} = \frac{V}{(2\pi)^3}4\pi k^2 dk = \frac{V}{2\pi^2}k^2dk$$

Now, we can solve for the fermi energy (in the three dimensional case), by noticing that

$$N = 2N_k = 2\int D(k)dk = \frac{2V}{2\pi^2}\int_0^{k_F} k^2dk = \frac{V}{\pi}\frac{1}{3}k_F^3$$

where the factr of 2 compensates for the degeneracy from spin up spin down. Giving:

$$k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}; \qquad \epsilon_F = \frac{k_F^2\hbar^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

which is good! Now, we can find the energy:

$$U = 2\int_0^{k_F} D(k)\epsilon_k dk = 2\int_0^{k_F} \frac{V}{2\pi^2}k^2\left(\frac{\hbar^2k^2}{2m}\right)dk = \frac{3}{5}N\epsilon_F$$

so we see that when the fermi energy is massive, the internal energy is massive, which is expected. Now, we can use our thermodynamic identity:

$$\tau d\sigma = dU + pdV - \mu dN$$

and we find that:

$$p = -\left(\frac{\partial U}{\partial V}\right)_{\sigma,N} = \frac{2}{5}\epsilon_F n = \frac{2}{3}\frac{U_0}{V}$$

This is what is known as degeneracy pressure; the force that gives shape to solids, provides support for white dwarves and neutron stars. [2] For ordinary things, the degeneracy pressure is on the order of 10^4 atmospheres, which is large compared to our small human hands.

To know when this is a good approximation, we require that $\tau << \epsilon_F$, i.e. the thermal excitations are small with respect to the fermi energy. For ordinary things, we find that the temperature required to break this assumption is $T_F \approx 50000K$, which is large compared to our cold human bodies.

Now, as a final point, we may want to express our density of states in terms of ϵ instead of k. This is just a change of variable, and we find that:

$$D(\epsilon)d\epsilon = g_s\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}d\epsilon$$

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[2] An interesting question is: what fundamental force is actually giving rise to this degeneracy pressure?

where $g_s = 2$ is the degeneracy due to spin. Now, if we want to find the expecation of some observable x, we can:

$$\langle x \rangle = \sum_n f(\epsilon_n, \tau, \mu) x(\epsilon_n) = \int d\epsilon D(\epsilon) f(\epsilon, \tau, \mu) x(\epsilon)$$

where f describes the probability of the state being filled. Note that we are no longer making the strong assumption that all the energy levels up to the fermi energy are completely filled, since we are somewhat interested in the 'dynamics' of the system. With this, we are being more accurate and complete.