Physics 112 : Lecture 4

Notes for the Fall 2017 Physics 112 Course taught by Professor Holzapfel
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1 Uncertainty in a system of spin 1/2 particles

Once again, consider a system of N spin half particles, each with magnetic moment m. Suppose that we wanted to specify the spin excess of the entire system by requiring that $2s = 1$. In practice, we would accomplish this feat by applying a magnetic field to a closed system, and since the energy levels of the spin up and spin down particles are different in the magnetic field, and since there is no energy transfer with the environment, we have theoretically perfectly specified s.

But in the real world, we can never have these perfect conditions! There is always some error in our measurement of macroscopic quantities like energy and the number of particles in the system; so we should expect that there is some error δU and δN in our measurements of U and N of the system respectively. The question is, should we care about these errors?

Suppose that we do indeed impose $2s = 1$. We know from the formulas we derived in the previous lectures that

$$g(N, s = 1/2) = \frac{N!}{(N/2 + 1/2)!(N/2 - 1/2)!}$$

To be specific, let $N = 5$. Then we have:

$$g(5, 1/2) = 10$$

And we know the total number of microstates of the system (where we do not pose any restriction on the value of s) is $2^5 = 32$. Note that 10 is a considerable fraction of 32; as we saw in previous lectures as N increases the multiplicity function gets more and more peaked near $s = 0$. (relative to N) All this is to say that the errors δU and δN for single systems of spin 1/2 particles do not particularly matter in any of our calculations. [1]

[1] This last statement might seem a bit out of the blue and not particularly related to our calculations about $g(5, 1/2)$ and our discussion about the peaked nature of the multiplicity function, and I agree; I think the details of the logical chain of deductions were somewhat lost in the midst of the lecture.

2 The most probable state of two interacting systems of spin 1/2 particles

Suppose now that we have two systems S_1 and S_2 with N_1 and N_2 particles respectively, and spin excess $2s_1$ and $2s_2$ respectively. Note that we can find the total energy of the two systems as a function of $s = s_1 + s_2$:

$$U(s) = U_1(s_1) + U_2(s_2) = -2mBs$$

where m is the magnetic moment of each particle, B is the applied magnetic field. [2] If we have no energy exchange between the two systems, then the total multiplicity is just given by the product:

$$g_1(N_1, s_1)g_2(N_2, s_2)$$

But, if we allow energy exchange between the two systems, then the multiplicity is given by:

$$\sum_{s_1=-N_1/2}^{N_1/2} g_1(N_1, s_1)g_2(N_2, s - s_1)$$

since the system S_1 can have spin excess s_1 ranging from $-N_1/2$ to $+N_1/2$. Now, we can ask which of the terms in this sum is the greatest; i.e. which value of s_1 contributes the most to the multiplicity? Call this value of s_1 by $\hat{s_1}$, and the corresponding value of $s_2 = s - s_1$ by $\hat{s_2}$. To find $\hat{s_1}$, notice that:

$$g(N, s) = \sqrt{\frac{2}{\pi N}} 2^N e^{-2s^2/N}$$

for a single system, so

$$g_1(N_1, s_1)g_2(N_2, s_2) = g_1(N_1, 0)g_2(N_2, 0)e^{-2s_1^2/N_1}e^{-2(s-s_1)^2/N_2}$$

where we pull out the constant factors. To find the maximum of this function over s_1, we can solve the equivalent problem of finding the maximum of the *logarithm* of this function, since the logarithm function is strictly increasing on its real domain. So taking the logarithm we have:

$$\ln(g_1(N_1, s_1)g_2(N_2, s_2)) = \ln(g_1(N_1, 0)g_2(N_2, 0)) - \frac{2s_1^2}{N_1} - \frac{2(s - s_1)^2}{N_2}$$

Taking the derivative of this function with respect to s_1 and setting that equal to zero we find:

$$\frac{\partial}{\partial s_1} \ln(g_1(N_1, s_1)g_2(N_2, s_2)) = 0 = \frac{-4\hat{s_1}}{N_1} + \frac{4(s - \hat{s_1})}{N_2}$$

Finally, we are left with the satisfying equation:

$$\frac{\hat{s_1}}{N_1} = \frac{\hat{s_2}}{N_2} = \frac{s}{N}$$

[2]This follows from the fact that a single magnetic moment $\vec{m}$ in a magnetic field $\vec{B}$ has energy $U = -\vec{m} \cdot \vec{B}$

The last equation caused some controversy in lecture, if it's not intuitively clear than algebraically:

$$\frac{s}{N} = \frac{\hat{s_1}+\hat{s_2}}{N_1+N_2} = \frac{\hat{s_1}}{N_1}\frac{N_1}{N_1+N_2} + \frac{\hat{s_2}}{N_2}\frac{N_2}{N_1+N_2} = \frac{\hat{s_1}}{N_1} = \frac{\hat{s_2}}{N_2}$$

We can consider now the difference in multiplicity between two isolated systems, and two systems that are allowed to share energy. Consider the fraction:

$$\frac{g(N,s)}{g_1(N_1,\hat{s_1})g_2(N_2,\hat{s_2})} = \frac{2^N\sqrt{2/(\pi N)}e^{-2s^2/N}}{2^N\sqrt{4/(\pi^2 N_1 N_2)}e^{-2s^2/N}} = \frac{\sqrt{N_1 N_2}}{\sqrt{2N/\pi}}$$

For $N_1 = N_2 = N/2$, we have that:

$$\frac{g(N,s)}{g_1(N_1,\hat{s_1})g_2(N_2,\hat{s_2})} = 2\sqrt{\frac{2}{\pi}}\sqrt{N}$$

i.e. the multiplicity of the two systems when they are allowed to share energy is roughly $\sqrt{N}$ times more than the multiplicity when the two systems are isolated, and the energy has been distributed in such a way to maximise the multiplicity. [3]

Now, for this system, we can consider a small change δ in $\hat{s_1}$. Note that:

$$g_1(N_1,\hat{s_1}+\delta)g_2(N_2,\hat{s_2}-\delta) = g_1(N_1,0)g_2(N_2,0)e^{-2(\hat{s_1}+\delta)^2/N_1-2(\hat{s_2}-\delta)^2)/N_2}$$

$$= g_1(N_1,0)g_2(N_2,0)e^{-2s^2/N}e^{-2\delta^2 N/N_1N_2}$$

How large does δ have to be for the last term to become $1/e$ (i.e. something measurable?) We find that $\delta = \sqrt{N}/\sqrt{2}$.

As a final exercise (slightly removed and unrelated to the previous discussions), we can see that:

$$\begin{aligned} g(N,s) &= \sum_{s_1} g_1(N_1,s_1)g_2(N_2,s_2) \\ &= \int_{-\infty}^{+\infty} g_1(N_1,0)g_2(N_2,0)e^{-2s^2/N}e^{-2\delta^2 N/N_1N_2}d\delta \\ &= 2^N\sqrt{\frac{2}{\pi N}}e^{-2s^2/N} \end{aligned}$$

just as we expected. (The integral shown looks daunting, but it's actually just a gaussian).

[3] This factor of $\sqrt{N}$ shouldn't come as a surprise; since we usually see $\sqrt{N}$ in our calculations for the standard deviation of the multiplicity function