

# Local list decoding with a constant number of queries

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## Abstract

Recently Efremenko showed locally-decodable codes of sub-exponential length. That result showed that these codes can handle up to  $\frac{1}{3}$  fraction of errors. In this paper we show that the same codes can be locally unique-decoded from error rate  $\frac{1}{2} - \alpha$  for any  $\alpha > 0$  and locally list-decoded from error rate  $1 - \alpha$  for any  $\alpha > 0$ , with only a constant number of queries and a constant alphabet size. This gives the first sub-exponential codes that can be locally list-decoded with a constant number of queries.

## 1 Introduction

Locally decodable codes (LDCs) are codes that allow retrieving any symbol of an encoded message by reading only a constant number of symbols from its codeword, even if a large fraction of the codeword is corrupted. Formally, a code  $\mathcal{C}$  is said to be locally decodable with parameters  $(\alpha, q, \epsilon)$  if it is possible to recover any symbol  $x_i$  of a message  $x$  by making at most  $q$  queries to  $\mathcal{C}(x)$ , such that even if up to a  $1 - \alpha$  fraction of  $\mathcal{C}(x)$  is corrupted, the decoding algorithm returns the correct answer with probability at least  $1 - \epsilon$ .

The first formal definition of locally decodable codes was given by Katz and Trevisan in [KT00]. The Hadamard code is the best-known 2-query locally decodable code, and its length is  $2^n$  (where  $n$  is the message length). For 2-query LDCs tight lower bounds on the block length of  $2^{\theta(n)}$  were given in [GKST02] for linear codes and in [KdW03] for general codes. For an arbitrary constant number of queries  $q$ , there are weak polynomial bounds, see [KT00, KdW03, Woo07].

The first sub-exponential LDCs (with a constant number of queries) were obtained by Yekhanin in [Yek08]. Yekhanin obtained 3-query LDCs with sub-exponential length under a highly believable number theoretic conjecture. Later, Efremenko [Efr09] gave an unconditional construction of sub-exponential LDCs. That construction also allowed a tradeoff between the number of queries and the codeword length. Unfortunately, these constructions could handle

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only  $\frac{1}{q}$  fraction of errors (where  $q$  is the number of queries) over a large alphabet and  $\frac{1}{2q}$  over the binary alphabet. In [Woo08], Woodruff showed how to increase the handled error rate to  $\frac{1}{q}$  over binary alphabets. Dvir, Gopalan and Yekhanin [DGY10], showed how to handle  $\frac{1}{4}$  fraction of errors.

Locally decodable codes have many applications in cryptography and complexity theory, see surveys [Tre04, Gas04]. Many of these applications require LDCs that can handle high error rates. Therefore, the question of local decoding from a high error rate attracted much attention.

The goal of this paper is to construct LDCs that can handle  $1 - \alpha$  fraction of errors. Clearly, when the error rate of a code is above half its distance, it is impossible to find a unique answer. Thus, we have to consider *list-decoding*. A code  $\mathcal{C}$  is said to be  $(1 - \alpha, L)$ -list-decodable if for every word, the number of codewords within relative distance  $1 - \alpha$  from that word is at most  $L$ . The notion of list-decoding dates back to works by Elias [Eli57] and Wozencraft [Woz58] in the 50s. Roughly speaking, a code  $\mathcal{C}$  is *locally* list-decodable if it is  $(1 - \alpha, L)$ -list-decodable, and given a corrupted word  $w$ , an index  $k \in [L]$  and a target bit  $j$ , the decoder returns the  $j$ 'th message bit of the  $k$ 'th codeword that is close to  $w$ . As expected, there are some subtleties in the definition. The main issue is guaranteeing that for a fixed  $k$ , all answers for inputs  $(k, j)$  correspond to the same codeword. More formally, a local list-decoding algorithm generates  $L$  machines  $\{M_k\}$ , such that the machine  $M_k$  locally decodes one codeword that is close to  $w$ , and the machines  $\{M_k\}$  together cover all the codewords that are close to  $w$  (for a formal definition, see Section 2).

The notion of local list-decoding is a central one in the theory of computer science. It first (implicitly) appeared in the celebrated Goldreich-Levin result [GL89], that can be seen as a local list-decoding algorithm for the Hadamard code. Later on many local list-decoding algorithms were studied. Most of the currently known locally list-decodable can be divided into three categories: Reed-Muller codes [GRS00, AS03, STV01, GKZ08], direct product and XOR codes [IW97, IJK06, IJKW08] and low-rate random codes [KS09]. Many of these results play an important role in complexity theory.

**Our Results** In this paper we show how to locally list-decode the codes given in [Efr09] (and which have sub-exponential length) with only a constant number of queries. We also show that one can unique decode this code up to radius close to  $\frac{1}{2}$ . The code we work with is a linear code over a finite field  $\mathbb{F}$  of constant-size, i.e.,  $|\mathbb{F}| = f(k, \alpha) = \Theta(1)$ , where  $f$  is some function. For unique local decoding we show:

**Theorem 1** (Unique decoding). *For every  $k \geq 2, \alpha > 0$ , there exists a  $(\frac{1}{2} + \alpha, q, \epsilon)$  LDC of dimension  $n$  over  $\mathbb{F}$  of length*

$$\exp(\exp(O(\sqrt[k]{\log n (\log \log n)^{k-1}}))),$$

*with  $q = \Theta\left(\frac{k^k \log(\frac{1}{\epsilon})}{\alpha^{2+k}}\right) = \Theta(1)$  queries.*

Independent of our work Dvir, Gopalan and Yekhanin in [DGY10] show a restricted version of this theorem for  $\alpha \geq \frac{1}{4}$ .

For local list-decoding we show:

**Theorem 2** (List Decoding). *For every  $k \geq 2, \alpha > 0$ , there exists a code of dimension  $n$  over  $\mathbb{F}$  of length*

$$\exp(\exp(O(\sqrt[k]{\log n (\log \log n)^{k-1}}))),$$

*which is  $(\alpha, L, q, \epsilon)$  locally list-decodable code with probabilistic advice. The number of queries is  $q = O(\frac{k^k \log(\frac{1}{\epsilon})}{\alpha^{2k+1}}) = \Theta(1)$  and the list size is  $L = |\mathbb{F}|^{O(\frac{\log \frac{n}{\epsilon}}{\alpha})} = \text{poly}(n)$ .*

As we said before, the above code is a linear code over a finite field  $\mathbb{F}$  of constant-size, i.e.,  $|\mathbb{F}| = f(k, \alpha) = \Theta(1)$ , where  $f$  is some function. We can get a locally list-decodable *binary* code, by concatenating the code of Theorem 2 with a good binary code, namely,

**Theorem 3.** *For every  $k \geq 2, \alpha > 0$ , there exists a binary code of dimension at least  $n$  and length*

$$\exp(\exp(O(\sqrt[k]{\log n (\log \log n)^{k-1}}))) \cdot |\mathbb{F}|,$$

*which is  $(\alpha, L, q, \epsilon)$  locally list-decodable with probabilistic advice. The number of queries is  $q = O(\frac{k^k \log(\frac{1}{\epsilon})}{\alpha^{3(2k+1)}} \cdot \text{poly}(\frac{\log |\mathbb{F}|}{\alpha})) = \Theta(1)$  and the list size is  $L = |\mathbb{F}|^{O(\frac{\log \frac{n}{\epsilon}}{\alpha^3})} = \text{poly}(n)$ . Furthermore, if the field  $\mathbb{F}$  is of characteristic two, the binary code is linear.*

We remark that in [Efr09] it is shown that we can use a field  $\mathbb{F}$  of characteristic 2 and of size  $f(k, \alpha) \leq 2^m$  where  $m = (k/\alpha)^{O(k)}$ . With this field  $\mathbb{F}$  the resulting binary code is *linear*. Alternatively, using the Prime Number Theorem for arithmetic progressions it can be shown that we can use a field  $\mathbb{F}$  of prime order  $f(k, \alpha) \approx m \log m$  (for the above  $m$ ), which results in a shorter code, fewer queries and shorter output lists, but produces a *non-linear* binary code.

The rest of the paper is organized as follows. In Section 2 we give the necessary preliminaries. Section 3 gives the formal definitions of locally decodable and list-decodable codes. In Section 4 we revise the construction of the code and analyze its local structure. Sections 5 and 6 contains the proofs of Theorems 1 and 2, respectively. The proof of Theorem 3 will appear in the full version of this paper.

**Related work** As we said before, Dvir, Gopalan and Yekhanin [DGY10], showed how to handle  $\frac{1}{4}$  fraction of errors. Their work uses two key observations that underlie the decoder that corrects more than  $1/q$  fraction of errors. First, the observation made in [Gop09] that in the locally decodable codes of [Efr09], restrictions of codewords to (multiplicative) lines are polynomials whose exponents come from a small set  $S$ . Second, that for the specific set  $S$  used by the code (that originates from the work of Grolmusz [Gro00]), the polynomials whose exponents lie in  $S$ , do not have many roots.

Our results were obtained independently to those of [DGY10] ([DGY10] was published before our work), using the same two observations but in a way that also gives us optimal unique decoding, and local list decoding from any constant fraction of agreement.

## 2 Preliminaries

We use the following standard mathematical notation:

- $[s] = \{1, \dots, s\}$ ;

- $\mathbb{F}$  is a finite field;
- $\mathbb{F}^*$  is the multiplicative group of the field;
- $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ , the integers modulo  $m$ ;
- $\Delta(x, y)$  denotes the relative Hamming distance between vectors  $x, y \in \Sigma^n$ , i.e.  $\Delta(w, w') = \Pr_{i \in [\bar{n}]}[w_i \neq w'_i]$ ;
- $\text{Ag}(w, w') \triangleq 1 - \Delta(w, w')$ , i.e.  $\text{Ag}(w, w') = \Pr_{i \in [\bar{n}]}[w_i = w'_i]$ ;
- $A^B$  denotes the set of functions from  $B$  to  $A$ , i.e.,  $A^B = \{f : B \rightarrow A\}$ . We identify  $A^{[m]}$  with  $A^m$ .

A code is a function  $\mathcal{C} : \Sigma^n \rightarrow \Sigma^{\bar{n}}$ . We identify a code  $\mathcal{C}$  with its image  $\mathcal{C} = \{\mathcal{C}(\lambda) \mid \lambda \in \Sigma^n\}$ . The distance  $d$  of the code is the minimum distance between two codewords in  $\mathcal{C}$  and the *relative* distance is  $\delta = d/n$ . The Hamming balls of radius  $\frac{d-1}{2}$  around codewords are disjoint, and therefore one can uniquely correct up to so many errors. If we allow more than  $d/2$  errors several decodings are possible. In many cases one can allow almost up to the distance errors and still get only few possible decodings.

For  $w \in \Sigma^{\bar{n}}$  and  $\alpha > 0$ , define

$$\mathcal{L}_{\mathcal{C}}(w, \mu) = \{z \in \mathcal{C} : \Delta(w, z) \leq \mu\}.$$

When the code  $\mathcal{C}$  is implicit from the text we abbreviate  $\mathcal{L}_{\mathcal{C}}(\cdot)$  to  $\mathcal{L}(\cdot)$ .

**Definition 1.** We say that a code  $\mathcal{C}$  is  $(\mu, L)$  list-decodable if for every  $w \in \Sigma^{\bar{n}}$  there are at most  $L$  codewords within distance  $\mu$  from  $w$ , i.e.  $|\mathcal{L}(w, \mu)| \leq L$ .

**Fact 4** (The Johnson bound). Let  $\mathcal{C}$  be a code with relative distance  $\delta$ . Then, for every  $\alpha > \sqrt{1 - \delta}$ , the code  $\mathcal{C}$  is  $(1 - \alpha, \frac{\alpha - (1 - \delta)}{\alpha^2 - (1 - \delta)})$  list decodable.

### 3 Locally Decodable and List Decodable Codes

As always one can study the combinatorial properties of a code, or ask for an explicit decoding algorithm. If the decoding algorithm makes only a few queries to the corrupted word, we say it is *local*. We begin with a formal definition of local *unique* decoding:

**Definition 2.** We say a probabilistic oracle machine  $M^w$  locally outputs a string  $s$  with confidence  $1 - \epsilon$ , if

$$\forall i \Pr[M^w(i) = s_i] \geq 1 - \epsilon,$$

where the probability is taken over the randomness of  $M$ .

**Definition 3** (Local Unique Decoding). A code  $\mathcal{C}$  over a field  $\mathbb{F}$ ,  $\mathcal{C} : \mathbb{F}^n \mapsto \mathbb{F}^{\bar{n}}$  is said to be  $(q, \alpha, \epsilon)$  locally decodable if there exists a probabilistic oracle machine  $M^w$  (the decoding algorithm) with oracle access to a received codeword  $w$  such that:

1. For every message  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{F}^n$  and for every  $w \in \mathbb{F}^{\bar{n}}$  such that  $\text{Ag}(\mathcal{C}(\lambda), w) \geq \alpha^1$ , it holds that  $M^w$  locally outputs  $\lambda$  with confidence  $1 - \epsilon$ .

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<sup>1</sup>Note that here  $\alpha$  denotes agreement and not distance.

2.  $M^w(i)$  makes at most  $q$  queries to  $w$  for all  $i \in [n]$ .

Recall that a code  $\mathcal{C}$  is list decodable if for every codeword  $w$  there are a few codewords near  $w$ . Let  $\mathcal{C}(y_1), \mathcal{C}(y_2), \dots, \mathcal{C}(y_L)$  be the list of codewords near  $w$ . Roughly speaking, a code  $\mathcal{C}$  is locally list decodable if there exists a machine  $M$  that given  $i, j$  and an oracle access to the received word  $w$ , outputs the  $j$ th symbol of  $y_i$ . The locality property requires that the machine  $M$  makes a few queries to  $w$ . To make this formal:

**Definition 4.** Let  $\mathcal{C} : \Sigma^n \rightarrow \Sigma^{\bar{n}}$  be a code. We say a set of probabilistic oracle circuits  $M_1 \dots M_L$  with oracle queries to  $w$ ,  $(\alpha, L, q, \epsilon)$  local list-decodes  $\mathcal{C}$  at the word  $w \in \Sigma^{\bar{n}}$ , if,

- Every oracle circuit  $M_j$  makes at most  $q$  queries to the input word  $w$ .
- For every codeword  $c \in \mathcal{C}$  with  $\text{Ag}(c, w) \geq \alpha$ , there exists some  $k \in [L]$ , such that  $M_k^w$  locally outputs  $c$  with confidence  $1 - \epsilon$ .

**Definition 5** (Locally List Decodable Codes with deterministic advice). We say a deterministic algorithm  $A$   $(\alpha, L, q, \epsilon)$  local list-decodes the code  $\mathcal{C} : \Sigma^n \rightarrow \Sigma^{\bar{n}}$ , if on input  $1^n$ ,  $A$  outputs probabilistic oracle circuits  $M_1 \dots M_L$  which  $(\alpha, L, q, \epsilon)$  local list-decode  $\mathcal{C}$  at every word  $w \in \Sigma^{\bar{n}}$ .

The code  $\mathcal{C}$  is  $(\alpha, L)$  list-decodable and therefore every  $w \in \Sigma^{\bar{n}}$  has at most  $L$   $\alpha$ -close codewords  $c_1, \dots, c_L$ . Each such codeword  $c_i = \mathcal{C}(\lambda_i)$  is represented by a probabilistic circuit  $M_i$  such that  $\forall j M_i(j) = \lambda_j$  (recall that  $M_i$  is a probabilistic circuit, and therefore  $M_i(j) = \lambda_j$  means that  $M_i$  outputs  $\lambda_j$  with probability at least  $1 - \epsilon$ ). The algorithm  $A$  outputs  $L$  machines that are good for every  $w \in \Sigma^{\bar{n}}$ . One way to think about it, is that  $i \in [L]$  is an advice telling which of the  $L$  solutions corresponds to the codeword we are interested in.

For instance, in [STV01] the advice string is a position in the codeword and a value of the codeword at this position. Since the code considered in [STV01] is of polynomial length, there is a polynomial number of advice strings. Therefore, the local list decoding algorithm of [STV01] works with deterministic advice. However, for codes with super-polynomial length (such as the Hadamard code and the code considered in this paper) one cannot have a position in a codeword as part of the advice string and still maintain a polynomial number of advice strings. Indeed, the local list decoding for the Hadamard code [GL89] has *probabilistic* advice, as we define now:

**Definition 6** (Locally List Decodable Codes with probabilistic advice). We say a probabilistic algorithm  $A$   $(\alpha, L, q, \epsilon)$  local list-decodes the code  $\mathcal{C} : \Sigma^n \rightarrow \Sigma^{\bar{n}}$ , if on input  $1^n$ ,  $A$  outputs probabilistic oracle circuits  $M_1 \dots M_L$  such that for every word  $w \in \Sigma^{\bar{n}}$ , with probability  $2/3$  over the random coins of  $A$ ,  $M_1 \dots M_L$  local list-decode  $\mathcal{C}$  at  $w$ , i.e.,

$$\forall w \in \Sigma^{\bar{n}} \Pr_A \left[ \forall \lambda \left( \text{Ag}(\mathcal{C}(\lambda), w) \geq \alpha \Rightarrow \exists i \forall j \Pr[M_i(j) = \lambda_j] \geq 1 - \epsilon \right) \right] \geq 2/3.$$

In the [GL89] local list decoding algorithm, the algorithm  $A$  (which generates the probabilistic circuits  $M_1, \dots, M_L$ ) picks a random subset of evaluation points, and takes as advice the value of the codeword on these evaluation points. Therefore, the [GL89] local list decoding algorithm works with *probabilistic* advice. Notice the order of the quantifiers: for every  $w \in \Sigma^{\bar{n}}$  most sets (i.e., random coins of  $A$ ) are good for  $w$ , however, it is not the case that most choices of  $A$  are good for every  $w$ .

## 4 The Code

In this section we will define the code and study its local properties.

### 4.1 Definition of the Code

We first review the definition of the code from [Efr09]. Fix a composite number  $m = p_1 \cdot p_2 \dots p_k$  which is a product of  $k$  distinct primes. The definition of the code will depend only on  $m$ .

In order to define the code we need the following definition:

**Definition 7.** A family of vectors  $\{u_i\}_{i=1}^n \subseteq \mathbb{Z}_m^h$  is said to be  $S$ -matching if the following conditions hold:

1.  $\langle u_i, u_i \rangle = 0$  for every  $i \in [n]$ .
2.  $\langle u_i, u_j \rangle \in S$  for every  $i \neq j$ .

Grolmusz [Gro00] showed how to construct a large set of  $S$ -matching vectors  $\{u_i\}_{i=1}^n$ ,  $u_i \in \mathbb{Z}_m^h$ , for

$$S = \{x \in \mathbb{Z}_m \setminus \{0\} \mid \forall i, x \bmod p_i \in \{0, 1\}\}.$$

Let  $\mathbb{F}$  be a field that contains an element  $\gamma \in \mathbb{F}$  of order  $m$ , i.e.  $\gamma^m = 1$  and  $\gamma^i \neq 1$  for  $i < m$ . We define a code  $\mathcal{C} : \mathbb{F}^n \mapsto \mathbb{F}^{m^h}$ , where we think of a codeword as a function from  $\mathbb{Z}_m^h$  to  $\mathbb{F}$ . The encoding of the message  $\lambda_1, \lambda_2 \dots \lambda_n$  is the function:

$$\mathcal{C}(\lambda_1, \lambda_2, \dots, \lambda_n)(x) \triangleq \sum_{i=1}^n \lambda_i \gamma^{\langle u_i, x \rangle}.$$

Equivalently, we can write

$$\mathcal{C}(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i f_i, \tag{1}$$

where  $f_i(x) \triangleq \gamma^{\langle u_i, x \rangle}$ . We will denote the codeword length by  $\bar{n} = m^h$ . An asymptotic relation between  $n$  and  $\bar{n}$  is:

$$\bar{n} = \exp(\exp(O(\sqrt[k]{\log n (\log \log n)^{k-1}}))).$$

Note that the asymptotic rate of the code depends only on  $k$ , the number of different primes dividing  $m$ .

For simplicity sometimes we denote  $G \triangleq \mathbb{Z}_m^h$ .

### 4.2 Local Properties of the Code

In this subsection we study local properties of the code. Specifically we study the restriction of the code to lines.

**Definition 8.** (line) Let  $v, u \in G$ . The line through  $v$  in direction  $u$  is the function  $\ell = \ell_{v,u} \in G^{[m]}$  defined by  $\ell(t) = v + tu$ .

**Definition 9** (restriction). Let  $\ell \in G^{[m]}$  be a line.

- For a function  $f \in \mathbb{F}^G$ , the restriction of  $f$  to  $\ell$ , denoted by  $f|_\ell \in \mathbb{F}^{[m]}$  is defined by  $f|_\ell(t) = f(\ell(t))$ .
- For a code  $\mathcal{C} : \mathbb{F}^n \rightarrow \mathbb{F}^G$ , the restriction of  $\mathcal{C}$  to  $\ell$ , denoted by  $\mathcal{C}|_\ell : \mathbb{F}^n \rightarrow \mathbb{F}^m$ , is the vector space  $\{\mathcal{C}(\lambda)|_\ell \mid \lambda \in \mathbb{F}^n\}$ .

Now, we analyze the restriction of the code in direction  $u_j$ . Observe that

$$\begin{aligned} \mathcal{C}(\lambda_1, \dots, \lambda_n)(v + tu_j) &= \sum_i \lambda_i \gamma^{\langle u_i, v + tu_j \rangle} = \sum_i \lambda_i \gamma^{\langle u_i, v \rangle} (\gamma^{\langle u_i, u_j \rangle})^t \\ &= \sum_{b \in S \cup \{0\}} \left[ \sum_{i: \langle u_i, u_j \rangle = b} \lambda_i \gamma^{\langle u_i, v \rangle} \right] (\gamma^t)^b. \end{aligned}$$

Define  $p : \mathbb{F} \rightarrow \mathbb{F}$  by  $p(x) = \sum_{b \in S \cup \{0\}} a_b x^b$ , where  $a_b = \sum_{i: \langle u_i, u_j \rangle = b} \lambda_i \gamma^{\langle u_i, v \rangle}$ , then  $\mathcal{C}|_{\ell_{v, u_j}}(\lambda)(t) = p(\gamma^t)$ . Furthermore,  $a_0 = \lambda_j \gamma^{\langle u_j, v \rangle}$ , and so when  $\lambda_j \neq 0$ ,  $p$  is a non-zero polynomial.

The observation that a codeword restricted to a line is a polynomial whose free coefficient encodes  $\lambda_j$  appears in [Gop09]. We now prove (Lemma 5) that this polynomial does not have too many roots and therefore the code restricted to the line has a large distance. This Lemma was also independently found by Dvir, Gopalan and Yekhanin [DGY10].

**Lemma 5.** Let  $\mathcal{C}$  be the code above. For every  $v \in G$  and  $j \in [n]$ , the code  $\mathcal{C}|_{\ell_{v, u_j}} : \mathbb{F}^n \rightarrow \mathbb{F}^m$  is of dimension at most  $2^k$  and distance  $\delta \geq 1 - \sum_{i=1}^k \frac{1}{p_i}$ .

**Proof:** In order to prove the lemma we need to show that the polynomial  $p(x) = \sum_{b \in S \cup \{0\}} a_b x^b$  does not have too many roots in the set  $H \triangleq \{\gamma^i \mid 0 \leq i < m\}$ . Recall that the set  $S$  is

$$S = \{x \in \mathbb{Z}_m \setminus \{0\} \mid \forall i \ x \bmod p_i \in \{0, 1\}\}.$$

Notice that  $p$  might have a large degree, and therefore might have a large number of roots in  $\mathbb{F}$ . Nevertheless, we show that the number of roots  $p$  has in  $H$  is at most  $\sum_i \frac{m}{p_i}$ . To see that denote  $\tilde{p}(x) = p(x^{\sum_i \frac{m}{p_i}})$ . We show that  $\tilde{p}$  has the same number of roots as  $p$ . Let  $s = \sum_i \frac{m}{p_i}$ . Then,

$$s \pmod{p_i} = \frac{m}{p_i} \pmod{p_i} \neq 0.$$

Therefore,  $\gcd(s, m) = 1$ , that is,  $s$  is invertible in  $\mathbb{Z}_m$ . This implies the mapping  $\psi : H \rightarrow H$ ,  $\psi(x) = x^s$  is a bijection.

Thus, in order to show  $p$  has few roots in  $H$ , it suffices to show that  $\tilde{p}$  is a low-degree polynomial. Each monomial of  $\tilde{p}$  is of degree  $b \cdot s \pmod{m}$  for some  $b \in S \cup \{0\}$ . Notice that for every  $1 \leq i, j \leq k$ ,

$$\frac{m}{p_i} \cdot b \bmod p_j = \begin{cases} 0 & j \neq i \\ 0 & (j = i) \wedge (b \bmod p_i = 0) \\ \frac{m}{p_i} \bmod p_i & (j = i) \wedge (b \bmod p_i = 1) \end{cases}$$

This implies that for every  $i$ ,

$$\frac{m}{p_i} \cdot b \bmod m = \begin{cases} 0 & b \bmod p_i = 0 \\ \frac{m}{p_i} & b \bmod p_i = 1 \end{cases}$$

Hence,  $b \cdot s \pmod{m} \leq \sum_i \frac{bm}{p_i} \pmod{m} \leq \sum_i \frac{m}{p_i}$ . We conclude that  $\tilde{p}$  has at most  $\sum_i \frac{m}{p_i}$  roots in  $H$  and therefore so does  $p$ .

For a polynomial  $p : \mathbb{F} \rightarrow \mathbb{F}$  define the vector  $\bar{p} \in \mathbb{F}^m$  by  $\bar{p}(t) = p(\gamma^t)$ . Then,  $\mathcal{C}|_{\ell_{v,u_j}}$  is a linear subspace of the vector-space  $\text{Span}\{\bar{x}^b : b \in S \cup \{0\}\}$ , which is a dimension  $2^k$   $\mathbb{F}$ -subspace. Every non-zero codeword corresponds to a non-zero polynomial that can have at most  $\sum_i \frac{m}{p_i}$  roots. As the elements  $\gamma^t$  are distinct for  $1 \leq t \leq m$ , every codeword has at most that many zeroes. ■

Let  $\mathcal{C}$  be the code above. Let  $v \in G$  and  $j \in [n]$ . Then every codeword of  $\mathcal{C}|_{\ell_{v,u_j}}$  corresponds to a polynomial with  $2^k$  monomials, where the free coefficient is  $\lambda_j \gamma^{\langle u_j, v \rangle}$ . Thus, any restricted codeword  $z \in \mathcal{C}|_{\ell_{v,u_j}}$  contains information about  $\lambda_j$ .

**Definition 10.** Using the above notation, we denote  $D_{v,j}(z) = \lambda_j$ .

In particular,

**Corollary 6.** Let  $\mathcal{C}$  be the code above. Let  $v \in G$  and  $j \in [n]$ . If  $z, z' \in \mathcal{C}|_{\ell_{v,u_j}}$  and  $D_{v,j}(z) \neq D_{v,j}(z')$  then  $z \neq z'$  and therefore  $\Delta(z, z') \geq \delta$ .

Another corollary is,

**Corollary 7.** The distance of the code  $\mathcal{C}$  is at least  $\delta$ , where  $\delta = 1 - \sum_{i=1}^k \frac{1}{p_i}$ .

**Proof:** Look at two different codewords  $\mathcal{C}(\lambda)$  and  $\mathcal{C}(\tilde{\lambda})$  for some  $\lambda \neq \tilde{\lambda}$ . Then, there exists some  $j \in [n]$  such that  $\lambda_j \neq \tilde{\lambda}_j$ . We can now partition  $G$  to disjoint lines in direction  $u_j$ . From Corollary 6 it follows that on each of these lines the restrictions of  $\mathcal{C}(\lambda)$  and  $\mathcal{C}(\tilde{\lambda})$  are different. From Lemma 5 we know that the distance on each of these lines is at least  $\delta$ . It follows that the distance between  $\mathcal{C}(\lambda)$  and  $\mathcal{C}(\tilde{\lambda})$  is at least  $\delta$ . ■

*Remark 8.* By taking all  $p_i$ 's of the same order we get that  $\delta = 1 - O(\frac{k}{\sqrt[k]{m}})$ . In this paper we assume that  $m$  is such a product.

## 5 Local unique decoding

We are given some word  $w \in \mathbb{F}^G$  that has agreement  $\frac{1}{2} + \alpha$  with some codeword  $\mathcal{C} = \mathcal{C}(\lambda)$ . We are also given some  $j \in [n]$ . Our goal is to recover (with a good probability)  $\lambda_j$ . A first attempt at local decoding is restricting the code to a random line  $\ell_{v,u_j}$  in direction  $u_j$ . Intuitively, this is a good step because we restrict the global code to a small fragment of constant size  $m$ , while still keeping information about  $\lambda_j$ . Specifically, by Lemma 5,  $\mathcal{C}|_{\ell_{v,u_j}}$  is a linear code with a large distance, and by Corollary 6, a codeword  $z = \mathcal{C}(\lambda)|_{\ell_{v,u_j}} \in \mathcal{C}|_{\ell_{v,u_j}}$  corresponds to a polynomial with  $2^k$  monomials, where the free coefficient is  $\lambda_j \gamma^{\langle u_j, v \rangle}$ .



As  $\mathcal{C}(\lambda)$  has  $\frac{1}{2} + \alpha$  agreement with  $w$ , when we pick a random line in direction  $u_j$ , the expected agreement between  $w|_{\ell_{v,u_j}}$  and  $\mathcal{C}(\lambda)|_{\ell_{v,u_j}}$  is  $\frac{1}{2} + \alpha$ . The problem is that it may still happen that with high probability the agreement between  $w|_{\ell_{v,u_j}}$  and  $\mathcal{C}(\lambda)|_{\ell_{v,u_j}}$  is less than  $\frac{1}{2}$  and we will decode wrong value. In order to overcome this problem we can sample  $K = O(\frac{\log(\frac{1}{\epsilon})}{\alpha^2})$  independent lines. Then with high probability the agreement between  $w$  and  $\mathcal{C}(\lambda)$  is at least  $\frac{1}{2} + \frac{\alpha}{2}$  on the sampled lines. Note that the code  $\mathcal{C}(\lambda)$  restricted to the union of independent lines in direction  $u_j$  may not have a good distance, as two different codewords may coincide on a restriction to a line. However, for any two codewords  $\mathcal{C}(\lambda)$  and  $\mathcal{C}(\tilde{\lambda})$ , where  $\lambda_j \neq \tilde{\lambda}_j$ , the distance between the restrictions of these two codewords on *each line* must be large because of Corollary 6.

Let  $\alpha \geq 2(1-\delta)$  (where  $\delta$  is the distance of the code, and by Lemma 5 is at least  $1 - \sum_i \frac{1}{p_i}$ ). The unique decoding algorithm for  $\frac{1}{2} + \alpha$  agreement is as follows:

**Input:**

- $w \in \mathbb{F}^G$  that has agreement  $\frac{1}{2} + \alpha$  with some codeword  $\mathcal{C}$ ,
- $j \in [n]$ ,
- $\epsilon > 0$

**Randomness:** A set of  $K = \Theta(\frac{\log(\frac{1}{\epsilon})}{\alpha^2})$  random elements in  $G$ ,  $\bar{v} = (v_1, \dots, v_K) \in G$ .

**Queries:** For each  $k \in [K]$ , the algorithm queries all points on the line  $\ell_{v_k, u_j}$ .

**Algorithm:** For every  $k \in [K]$  and for every symbol  $\sigma \in \mathbb{F}$ , the algorithm computes

$$\text{weight}_k(\sigma) = \max \left\{ \text{Ag}(w, z) : z \in \mathcal{C}|_{\ell_{v_k, u_j}}, D_{v_k, j}(z) = \sigma \right\}.$$

The algorithm then computes  $\text{weight}(\sigma) = \frac{1}{K} \sum_{k=1}^K \text{weight}_k(\sigma)$ . The output of the algorithm is the symbol  $\sigma$  with the highest weight.

**Theorem 9.** Assume  $\alpha \geq 2(1-\delta)$ . For every  $\lambda \in \mathbb{F}^n$ ,  $w \in \mathbb{F}^G$  with  $\text{Ag}(w, \mathcal{C}(\lambda)) \geq \frac{1}{2} + \alpha$  and every  $j \in [n]$ ,

$$\Pr_{\bar{v}}[\text{The algorithm outputs } \lambda_j] \geq 1 - \epsilon.$$

The algorithm uses  $\Theta(\frac{\log(\frac{1}{\epsilon})}{\alpha^2} \cdot m)$  queries.

**Proof:** Suppose that  $\mathcal{C}(\lambda)$  is a codeword which has  $\frac{1}{2} + \alpha$  agreement with the received word  $w$ . Then

$$\mathbb{E}_{v \in G} \left[ \text{Ag}(w|_{\ell_{v, u_j}}, \mathcal{C}(\lambda)|_{\ell_{v, u_j}}) \right] = \frac{1}{2} + \alpha.$$

We say  $\bar{v} = (v_1, \dots, v_K)$  is *good*, if

$$\frac{1}{K} \sum_{k=1}^K \left[ \text{Ag}(w|_{\ell_{v_k, u_j}}, \mathcal{C}(\lambda)|_{\ell_{v_k, u_j}}) \right] \geq \frac{1 + \alpha}{2}.$$

By a standard application of the Chernoff bound,

$$\Pr_{\bar{v}}[\bar{v} \text{ is not good}] \leq 2^{-\Omega(\alpha^2 K)} = \epsilon.$$

We now prove that if  $\bar{v}$  is good the algorithm outputs the correct answer.

Denote  $ag_k = \text{Ag}(w|_{\ell_{v_k, u_j}}, \mathcal{C}(\lambda)|_{\ell_{v_k, u_j}})$ . Then,

- For every  $k$ ,  $\text{weight}_k(\lambda_j) \geq \text{Ag}(w|_{\ell_{v_k, u_j}}, \mathcal{C}(\lambda)|_{\ell_{v_k, u_j}}) \geq ag_k$  and so  $\text{weight}(\lambda_j) \geq \mathbb{E}_k[ag_k] \geq \frac{1+\alpha}{2}$ .
- Fix any  $\sigma \neq \lambda_j$  and  $k \in [K]$ . Let  $z \in \mathcal{C}|_{\ell_{v_k, u_j}}$  be such that  $D_{v_k, j}(z) = \sigma$ . Then, by the triangle inequality,

$$\delta \leq \Delta(z, \mathcal{C}(\lambda)|_{\ell_{v_k, u_j}}) \leq \Delta(\mathcal{C}(\lambda)|_{\ell_{v_k, u_j}}, w|_{\ell_{v_k, u_j}}) + \Delta(w|_{\ell_{v_k, u_j}}, z).$$

Thus,  $\Delta(w|_{\ell_{v_k, u_j}}, z) \geq \delta + ag_k - 1$ , and  $\text{weight}_k(\sigma) \leq 1 - ag_k + 1 - \delta$ . In particular,

$$\text{weight}(\sigma) \leq 1 - \delta + \mathbb{E}_k[1 - ag_k] \leq \frac{1}{2} + 1 - \delta - \frac{\alpha}{2} \leq \frac{1}{2}.$$

Thus, whenever  $\bar{v}$  is good the algorithm outputs  $\lambda_j$ . ■

We are now ready to prove Theorem 1.

**Proof of Theorem 1:** The code  $\mathcal{C}$  has distance at least  $\delta = 1 - O(\frac{k}{m^{1/k}})$  and the code length is

$$\exp(\exp(O(\sqrt[k]{\log n (\log \log n)^{k-1}}))).$$

We take  $m$  to be a product of  $m$  almost equal primes. From Theorem 9, for every  $\alpha \geq 2(1 - \delta) = O(\frac{k}{m^{1/k}})$ , the code is  $(\frac{1}{2} + \alpha, q, \epsilon)$  locally decodable with  $q = \Theta(\frac{\log(\frac{1}{\epsilon})}{\alpha^2} \cdot m)$  queries. We think of  $k$  as a constant, and  $m$  as depending on  $\alpha$ , growing to accommodate the required error rate. Thus  $\alpha = 2(1 - \delta) \approx \frac{2k}{m^{1/k}}$ , or equivalently,  $m \approx (\frac{2k}{\alpha})^k$ . Thus, the number of queries is  $\Theta(\frac{m \log(\frac{1}{\epsilon})}{\alpha^2}) = \Theta(k^k \cdot \alpha^{-(k+2)} \cdot \log(\frac{1}{\epsilon}))$ . For  $k = 2$  the number of queries is  $\Theta(\alpha^{-4} \cdot \log(\frac{1}{\epsilon}))$ . ■

## 6 Local list decoding with probabilistic advice

We first remind the reader of the setting. A probabilistic algorithm  $A$  has to produce  $L$  probabilistic circuits  $M_1, \dots, M_L$  that  $(\alpha, L, q, \epsilon)$  local list-decode  $\mathcal{C}$ .  $A$  uses its internal random coins to sample a random subset  $\Lambda \subseteq G$  of cardinality  $\Theta(\frac{\log \frac{n}{\epsilon}}{\alpha})$ . The list size  $L$  is  $|\mathbb{F}^\Lambda|$  and corresponds to all possible values a codeword may take on  $\Lambda$ . We identify an index of a machine  $i \in [L]$  with a function  $\text{ad} : \Lambda \mapsto \mathbb{F}$  of values of a codeword on the set  $\Lambda$ . The machine  $M_{\text{ad}}^w$  locally outputs a message  $\lambda$  such that  $\mathcal{C}(\lambda)$  has  $\alpha$  agreement with  $w$  and  $\text{ad} = \mathcal{C}(\lambda)|_\Lambda$ .

Given a corrupted word  $w \in \mathbb{F}^G$  and a value  $j \in [n]$ ,  $M_{\text{ad}}$ 's goal is to find (the hopefully unique) codeword  $c \in \mathcal{C}$  that is  $\alpha$  close to  $w$ , and that is consistent with the given advice  $\text{ad} \in \mathbb{F}^\Lambda$ . To do so,  $M_{\text{ad}}$  does the following:  $M_{\text{ad}}$  picks  $K$  (and  $K$  will turn out to be constant) random lines in direction  $u_j$  that pass through some point in  $\Lambda$ . For each such line,  $M_{\text{ad}}$  queries  $w$  on the line, and finds all the restricted codewords that are close to the  $w$  (on the line). We say that a line is good if among all those codewords, *exactly* one matches the value  $\text{ad}$  gives on the point from  $\Lambda$ . For each good line,  $M_{\text{ad}}$  extracts from this unique codeword the value  $\lambda_j$  and adds it to the candidates list. The output of  $M_{\text{ad}}$  is the most common value in the candidates list. More formally, the algorithm  $M_{\text{ad}}$  is defined as follows.

**A's random coins:** A random subset  $\Lambda$  of of cardinality  $\Theta(\frac{\log \frac{n}{\epsilon}}{\alpha})$ .

**Advice:** Values of some codeword  $c$  on  $\Lambda$ .

**Input:**  $w \in \mathbb{F}^G$ ,  $j \in [n]$ .

**M's randomness:** A random subset  $\{s_1, \dots, s_K\}$  of  $\Lambda$  of cardinality  $K = \Theta(\frac{\log(\frac{1}{\epsilon})}{\alpha})$ .

**Queries:** For each  $k \in [K]$ ,  $M$  queries the values of  $w$  on the  $K$  lines  $\ell_{s_k, u_j}$ .

**Algorithm:** For every  $k$ , the algorithm goes over all codewords of  $\mathcal{C}' = \mathcal{C}|_{\ell_{s_k, u_j}}$ . For every such  $k$ , if there exists *exactly* one codeword  $z$  of  $\mathcal{C}'$  with:

- $\text{Ag}(z, w|_{\ell_{s_k, u_j}}) \geq \frac{\alpha}{2}$ , and,
- $z(s_k) = \text{ad}(s_k)$

then the algorithm adds the value  $D_{v_k, j}(z)$  to the candidates list.

**Output:** The most common value in the candidates list.

**Theorem 10.** For any  $\alpha \geq 8\sqrt{1-\delta}$ ,  $\epsilon > 0$  and  $L = |\mathbb{F}^\Lambda| = q^{O(\frac{\log \frac{n}{\epsilon}}{\alpha})}$ ,  $q = Km = O(\frac{m \log(\frac{1}{\epsilon})}{\alpha}) = O(\log(\frac{1}{\epsilon}))$ . The above algorithm is a probabilistic polynomial-time  $(\alpha, L, q, \epsilon)$  local list-decoding algorithm.

Theorem 2 follows immediately from Theorem 10.

**Proof of Theorem 2:** We take  $m$  a product of  $k$  distinct almost equal primes. From Theorem 10 we know that for any  $\alpha > 8\sqrt{1-\delta} = O(\frac{\sqrt{k}}{2k/m})$  the code is  $(\alpha, L, q, \epsilon)$  local list-decodable with  $q = O(\frac{m \log(\frac{1}{\epsilon})}{\alpha})$ . Therefore,  $m = O(\frac{k^k}{\alpha^{2k}})$  and  $q = O(k^k \cdot \alpha^{-(2k+1)} \cdot \log(\frac{1}{\epsilon}))$  with a codeword length:

$$\exp(\exp(O(\sqrt[k]{\log n (\log \log n)^{k-1}})))$$

■

We are left to prove Theorem 10.

## 6.1 Proof of correctness

We need to show that for every received word  $w$ , with high provability over the choice of the set  $\Lambda$ , for every codeword  $c = \mathcal{C}(\lambda)$  that has  $\alpha$  agreement with  $w$ , when the advice is  $\text{ad} = c|_\Lambda$ , it holds that for every  $j \in [n]$ ,  $\Pr[M_{\text{ad}}^w(j) = \lambda_j] \geq 1 - \epsilon$ , where the probability is over the randomness of  $M$ .

Fix  $w \in \mathbb{F}^G$ , a codeword  $c = \mathcal{C}(\lambda)$  and  $j \in [n]$ . For  $v \in G$  the machine  $M_{\text{ad}}$  considers the set

$$U_j(v) = \left\{ z \in \mathcal{C}|_{\ell_{v, u_j}} : (\text{Ag}(z, w|_{\ell_{v, u_j}}) \geq \alpha/2) \wedge (z(v) = \text{ad}(v)) \right\}.$$

In the  $k$ th iteration,  $M_{\text{ad}}$  adds  $\lambda_j$  to the candidates list if  $U_j(s_k) = \left\{ c|_{\ell_{s_k, u_j}} \right\}$ .

We say that  $v$  is *useful* if  $c|_{\ell_{v,u_j}} \in U_j(b)$ . Notice that  $c|_{\ell_{v,u_j}}(v) = \text{ad}(v)$ , hence for  $v$  to be useful we only need a high agreement between  $v$  and  $w$  on the line  $\ell_{v,u_j}$ . We say that  $v$  *filters* if  $U_j(v) \subseteq \{c|_{\ell_{v,u_j}}\}$ , i.e., for any codeword in the restricted code  $z \in \mathcal{C}|_{\ell_{v,u_j}}$  such that  $z \neq c$  it holds that  $z \notin U_j(v)$ .

**Lemma 11.** *For any  $\alpha \geq 8\sqrt{1-\delta}$  it holds that*

- $\Pr_{v \sim G}[v \text{ is useful}] \geq \frac{\alpha}{2}$
- $\Pr_{v \sim G}[v \text{ does not filter}] \leq \frac{4}{\alpha} \cdot (1 - \delta) \leq \frac{\alpha}{16}$ .

**Proof:** Since  $\mathbb{E}_v[\text{Ag}(w|_{\ell_{v,u_j}}, c|_{\ell_{v,u_j}})] = \alpha$ , an averaging argument implies that the probability  $v \in G$  is useful is at least  $\alpha/2$ .

We turn to the second item. A point  $v$  does not filter if there is a restricted codeword  $z \in \mathcal{C}|_{\ell_{v,u_j}}$  such that  $z \neq c|_{\ell_{v,u_j}}$  and  $z \in U_j(v)$ . A restricted codeword  $z$  is in  $U_j(v)$  if it is in the list  $\mathcal{L}(w|_{\ell_{v,u_j}}, \alpha/2)$  and  $z(v) = c(v)$ . One way to choose  $v$  uniformly from  $G$  is by first choosing a random line  $\ell$  in direction  $u_j$ , and then choosing a random point  $v$  on the line. For any line  $\ell$  in direction  $u_j$ ,  $\mathcal{C}' = \mathcal{C}|_{\ell}$  has distance  $\delta$ . Therefore, for any  $z \neq c$  the probability that  $z(v) = c(v)$  is at most  $1 - \delta$ . By the Johnson bound (see Fact 4), the number of codewords with  $\alpha/2$  agreement with  $w|_{\ell}$  satisfies

$$|\mathcal{L}(w|_{\ell_{v,u_j}}, \alpha/2)| \leq \frac{\alpha/2 - (1 - \delta)}{\alpha^2/4 - (1 - \delta)} < \frac{4}{\alpha},$$

when  $\alpha \geq 2\sqrt{2(1-\delta)}$ . The probability that such a codeword  $z$  agrees with  $c$  at  $v$  is at most  $1 - \delta$ . The lemma follows from the union bound. ■

**Definition 11.** *For  $w \in \mathbb{F}^G$ , a set  $\Lambda \subseteq G$  is good for  $w$ , if for every  $c \in \mathcal{L}(w, \alpha/2)$  and every  $j \in [n]$ ,*

- $\Pr_{v \in \Lambda}[v \text{ is useful and filters for } (w, c, j)] \geq \frac{\alpha}{4}$ .
- $\Pr_{v \in \Lambda}[v \text{ does not filter } (w, c, j)] \leq \frac{\alpha}{8}$ .

**Lemma 12.** *Fix  $w \in \mathbb{F}^G$ . Pick a set  $\Lambda$  uniformly at random from  $G$ . The probability  $\Lambda$  is not good for  $w$  is at most  $\frac{n}{\alpha} \cdot 2^{-\Omega(\alpha|\Lambda|)}$ .*

**Proof:** For any  $w, j$  and  $c \in \mathcal{L}(w, \alpha)$ , the probability a single  $v$  is useful and filters, by Lemma 11, is at least  $\frac{\alpha}{3}$ . By the Chernoff bound, the probability we do not have  $\frac{\alpha}{4}$  fraction of good vectors in the sample  $\Lambda$  is at most  $2^{-\Omega(\alpha|\Lambda|)}$ .

Similarly, by Lemma 11, for any  $w, j$  and  $c \in \mathcal{L}(w, \alpha)$ , the probability a single  $v$  does not filter  $(w, c, j)$ , is at most  $\frac{\alpha}{16}$ . By the Chernoff bound, the probability we have more than  $\frac{\alpha}{8}$  fraction of vectors that do not filter  $(w, c, j)$  in the sample  $\Lambda$  is at most  $2^{-\Omega(\alpha|\Lambda|)}$ .

The lemma follows from a union bound over  $j$  and  $c \in \mathcal{L}(w, \alpha)$ . ■

Assume  $\Lambda$  is good for  $w$ . The probability that at the  $i$ th iteration,  $M_{\text{ad}}$  adds the correct value  $\lambda_j$  to the candidates list is at least the probability that  $v$  is useful and filters. By Definition 11 this probability is at least  $\frac{\alpha}{4}$ . The probability that  $M_{\text{ad}}$  adds a wrong value to the candidates

list is bounded by the probability that  $v$  does not filter, which is at most  $\frac{\alpha}{8}$ . Therefore, by the Chernoff bound, it follows that after  $\Theta(\frac{\log(\frac{1}{\epsilon})}{\alpha})$  iterations the probability that  $\lambda_j$  is the most common value in the candidates list is at least  $1 - \epsilon$ . Theorem 10 follows from the above lemma, since for every  $w$ ,  $\Lambda$  is good for  $w$  with probability at least  $\epsilon$  (by the choice of the cardinality of  $\Lambda$ ).

*Remark 13.* If  $\Lambda$  is chosen to be a random subset  $G$  of size  $K = \Theta(\frac{\log(\frac{1}{\epsilon})}{\alpha})$  (which is constant) then  $M_{\text{ad}}$  becomes deterministic. In this case, the above lemma shows that for any *fixed*  $j$ , the machine  $M_{\text{ad}}$  outputs the correct answer with probability at least  $1 - \epsilon$ . However, the definition requires that  $M_{\text{ad}}$  outputs the correct answer for *every*  $j$ .

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