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July 30, 2015

CS70, Summer 2015

Goldilocks, after the incident, has hired an Assistant A to check her food before she eats it. Every evening, they go out to dinner, and A will check if the food is "too hot", "too cold", or "just right". If A thinks it's "just right", then Goldi will eat the meal. Otherwise, no one eats the meal. Without A, Goldi would have been happy with the temperature 1/3 of the time, and A can recognize this condition 90% of the time. Still, Goldi burns her tongue 1/100 days and icily shivers in disgust 2/100 days.

a) If the food is A-ok (A has approved the food), what is the chance that Goldi is happy with the meal?

b) Goldi knows that more food is prepared hot than cold. What does that say about A?

c) Goldi pays A $3 for every good meal, $1 for each uneaten meal (he tried!), and $-9 for every bad meal (he has one job!). What is the expected daily wage of A? What is the variance?

d) Goldi is starving to death, and A is going broke. A demands that Goldi eat 2 meals a day. Goldi agrees, with the condition that the first day Goldi doesn't eat a good meal, A will be fired. What is the expected wages of A before he gets fired? (Round to the nearest dollar. A does not get paid the day he is fired.)

a. *Make sure you understand the setup. Every day they have a meal, and either the Assistant didn't like the meal, the Assistant liked the meal and Goldi liked the meal, or the Assistant liked the meal and Goldi didn't like the meal. Both have opinions about the temperature of the food.*

   Define two events:

   A: Assistant thinks food is "just right"

   G: Goldilocks thinks food is "just right"

   These are good events to define because calculating each event's probability is somewhat separate from calculating the other event's probability. The point of defining events is to translate English into math & probability. This is a skill we expect CS70 students to develop with practice.

   Then, let's translate into probability-speak.

   [Question statement] *"If A-ok, chance Goldi happy": $P(G|A)$*

   "Goldi would have been happy ⅓ of the time": $P(G) = ⅓$

   "A can recognize this condition 90% of the time": $P(A|G) = 0.9$

   "Goldi burns 1/100 days, freezes 1=2/100 days": $P(A \cap \bar{G}) = 0.03$

   If your events can't easily translate the numbers into probability-speak, try picking events specifically so the numbers translate.

   Then, use Bayes' Rule: $P(G|A) = P(A|G)P(G) / P(A) = 0.9*⅓ / (0.9*⅓ + 0.03) =$ ***10/11.***

b. *Make events G-hot, G-cold for Goldi's impression of the food being too hot or too cold. Then, $P(G\text{-hot}) > P(G\text{-cold})$, but $P(A \cap G\text{-hot}) < P(A \cap G\text{-cold})$. So,* ***$P(A|G\text{-hot}) < P(A|G\text{-cold})$****, meaning the Assistant notices if food is "too hot" more than "too cold".*

c. *Let X be a random variable corresponding to the Assistant's daily wages. The distribution of X is* $\{(3, P(A \cap G)), (1, P(\bar{A})), (-9, P(A \cap \bar{G}))\}$.

Solving for unknowns:

$P(A \cap G) = P(A|G)P(G) = 0.9 * \frac{1}{3} = 0.3$

$P(A \cap \bar{G}) = 1/100 + 2/100 = 0.03$

$P(\bar{A}) = 1 - P(A) = 1 - (P(A \cap G) + P(A \cap \bar{G})) = 1 - 0.33 = 0.67$

Plugging in numbers:

"**mean**" $= E(X) = 3(0.3) + 1(0.67) + -9(0.03) = 0.9 + 0.67 - 0.27 = \$1.30$ / day

$E(X^2) = (3)^2(0.3) + (1)^2(0.67) + (-9)^2(0.03) = 2.7 + 0.67 + 2.43 = \$5.80$ / day

"**variance**" $= Var(X) = E(X^2) - (E(X))^2 = 5.80 - (1.30^2) = 5.8 - 1.69 = \$4.11$ / day

d. *(This one is tricky!) Each day, there is a chance for the Assistant to be fired, in which case, he works no more days. Otherwise, he works another day. This is the form of a geometric distribution. We can construct events GG, GB, GN, BN, NN, BB for the six (unordered) possibilities of the day (G=Good meal, B=Bad meal, N=no food). The Assistant is fired if any of the last three events occur. (These events partition the sample space --- exactly one of the events must occur, and no two events can occur together).*

$P(GG) = 0.3*0.3 = 0.09$

$P(GB) = 2*0.3*0.03 = 0.018$

$P(GN) = 2*0.3*0.67 = 0.402$

So, the probability of the Assistant not getting fired is $p = 0.09+0.018+0.402 = 0.51$ *. On average, the Assistant will be fired after* $1/(1-p)$ *days =* $1/(0.49)$ *days, and will be paid for* $1/(1-p) - 1 = p/(1-p) = 0.51/0.49$ *days of work. Using conditional probability, we can find*

$$E(\text{wages before A is fired}) = \sum_{y=0}^{\infty} E(\text{wages} \mid \text{A is paid for y days}) * Pr(\text{A is paid for y days})$$

$$\begin{aligned} E(\text{wages} \mid \text{A is paid for y days}) &= y*E(\text{wages/day} \mid \text{A is not fired}) \\ &= y*[(3+3)*0.09 + (3-9)*0.018 + (3+1)*0.402]/0.51 \\ &= y*[0.54 - 0.108 + 1.608]/0.51 = y*2.04/0.51 \end{aligned}$$

$$\begin{aligned} E(\text{wages before A is fired}) &= \sum_{y=0}^{\infty} y*(2.04/0.51) * Pr(\text{A is paid for y days}) \\ &= (2.04/0.51) \sum_{y=0}^{\infty} y * Pr(\text{A is paid for y days}) \\ &= (2.04/0.51)\, E(\text{days before A is fired}) \\ &= (2.04/0.51)(0.51/0.49) \approx \mathbf{\$4} \end{aligned}$$

Roughly, you expect the Assistant to work 1 more day without getting fired, and if he doesn't get fired, he makes (on average) about \$4.