fun facts about 2x2 real matrices

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1 Context

These questions arose from a late-night study session for Math 113, Cal's abstract algebra course for undergraduates.

2 Definitions

2.1 Invertible

This is often denoted as the general linear group, specifically $GL_2(\mathbb{R})$, which is a group over matrix multiplication.

2.2 Determinant-1

This is often denoted as the special linear group, specifically $SL_2(\mathbb{R})$. This is a subgroup of $GL_2(\mathbb{R})$, since it is closed under inverse ($det(A^{-1}) = \frac{1}{det(A)} = 1$) and group operation ($det(AB) = det(A)det(B) = 1$). Inverses exist because the determinant is nonzero.

2.3 Rotations

Any rotation can be defined as R_θ for some $\theta \in \mathbb{R}$ ($\theta > 0$ rotates clockwise):

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

Notice that $R_\theta = R_{\theta+2\pi}$, and all rotations have determinant 1. Let's call the set of all rotations F:

$$F = \{R_\theta \mid \theta \in \mathbb{R}\} = \{R_\theta \mid \theta \in \mathbb{R}, 0 \le \theta < 2\pi\} \tag{2}$$

2.4 Reflections

Any reflection S mirrors points across some line $y = kx$ (for $k \in \mathbb{R}$) or $x = 0$. The reader can verify S_k by working out how a point $\begin{pmatrix} a \\ b \end{pmatrix}$ mirrors to $S_k \begin{pmatrix} a \\ b \end{pmatrix}$.

$$S_k = \frac{1}{k^2+1} \begin{pmatrix} 1-k^2 & 2k \\ 2k & k^2-1 \end{pmatrix} \tag{3}$$

$$S_\star = \lim_{k \to \pm\infty} S_k = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

Notice that all reflections have $det(S) = -1$. Let's call the set of all reflections G:

$$G = \{S_\star\} \cup \{S_k \mid k \in \mathbb{R}\} \tag{5}$$

2.5 Orthogonal (transpose is inverse)

This is often denoted as the orthogonal group, specifically $O_2(\mathbb{R})$. An orthogonal matrix $A \in O_2(\mathbb{R})$ satisfies $A^TA = AA^T = I$, i.e. its transpose is its inverse. This is a subgroup of $GL_2(\mathbb{R})$, since it is closed under inverse ($(A^T)^{-1} = A = (A^T)^T$) and group operation ($(AB)^T(AB) = B^TA^TAB = I$ and $(AB)(AB)^T = ABB^TA^T = I$).

2.6 Defective

Not all matrices are diagonalizable, meaning for a matrix A, there is no matrix $P \in GL_2(S)$ where $P^{-1}AP$ is diagonal. These matrices are called defective in S, where S is some field. Define D_S as the set of these defective matrices in $GL_2(\mathbb{R})$. Notice that $D_\mathbb{C} \subseteq D_\mathbb{R}$, since in $D_\mathbb{C}$, P is allowed to have complex-valued entries.

2.7 Unit-magnitude eigenvalues

Define J as the set of all matrices in $GL_2(\mathbb{R})$ with both eigenvalues having magnitude 1. Matrices in this set may have complex eigenvalues, but always have a real determinant ± 1.

2.8 Partitions of J

Let's split up J into different sets:

$$X = \{A \mid A \in J \cap D_\mathbb{C}\} \tag{6}$$
$$Y = \{A \mid A \in J, A \notin D_\mathbb{C}\} \tag{7}$$
$$K = \{A \mid A \in Y, \lambda_1 = \lambda_2 = 1\} \tag{8}$$
$$L = \{A \mid A \in Y, \lambda_1 = -\lambda_2 = 1\} \tag{9}$$
$$M = \{A \mid A \in Y, \lambda_1 = \lambda_2 = -1\} \tag{10}$$
$$N = \{A \mid A \in Y, A \notin K \cup L \cup M\} \tag{11}$$
$$\tag{12}$$

Notice that X and Y partition J, and K, L, M, N partition Y.

3 General insights

3.1 Rotations form a group

The reader can verify that $R_{\theta_1}R_{\theta_2} = R_{\theta_1+\theta_2}$. So, F is closed under group operation and inverse (as $R_\theta^{-1} = R_{-\theta}$).

3.2 Reflections do not form a group

The reader can verify that reflections are self-inverses. However, the product of two reflections has determinant $(-1)(-1) = 1$, so it is not a reflection. So, G is not closed under the group operation.

3.3 Orthogonal matrices have determinant ± 1

Consider an orthogonal matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\{a, b, c, d\} \subset \mathbb{R}$. Its transpose is defined $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Notice that $det(A) = ad - bc = det(A^T)$, so $1 = det(I) = det(AA^T) = det(A)det(A^T) = det(A)^2$. Since the determinant of a real matrix is real, $det(A) = \pm 1$.

3.4 Non-real eigenvalues must be different

Suppose $\lambda_1 = a + bi$ and $\lambda_2 = c + di$ with $a, b \in \mathbb{R}$ and $b \neq 0$. If the characteristic equation for a matrix is satisfied ($p(\lambda) = 0$), then $p^*(\lambda^*) = 0^* = 0$. Since p is a polynomial with real coefficients, $p = p^*$, so $p(\lambda^*) = 0$. So, $\lambda_2 = \lambda_1^* = a - bi$. The eigenvalues differ because b is nonzero (i.e. $\lambda_1 \notin \mathbb{R}$).

3.5 Matrices in $D_{\mathbb{C}}$ have $\lambda_1 = \lambda_2 \in \mathbb{R}$

In order to be defective over $\mathbb{C}$, an eigenvalue's geometric multiplicity must be strictly less than its algebraic multiplicity. This means some eigenvalue has algebraic multiplicity at least 2. For this to happen in a 2x2 matrix, there can be only one eigenvalue. By 3.4, this eigenvalue must be real.

3.6 Matrices in X have determinant 1

Since any matrix $A \in X$ is defective over $\mathbb{C}$, it has one repeated real eigenvalue. Since $A \in J$, this eigenvalue must be -1 or 1, so $det(A) = (-1)^2 = 1^2 = 1$.

3.7 Matrices in N have determinant 1

Consider a matrix $A \in N$. It has at least one unit-magnitude eigenvalue λ_1 not equal to 1 or -1; so, λ_1 is non-real. By 3.4, $\lambda_2 = \lambda_1^*$, so $det(A) = \lambda_1\lambda_2 = |\lambda_1|^2 = 1$.

3.8 K is the trivial group

A diagonalizable matrix can be written as PDP^{-1}, where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. A matrix in K can be written as $PIP^{-1} = I$, so K contains only the identity matrix, the "zero" element of the group.

3.9 $K \cup M$ is a two-element group

Similar to 3.8, a matrix in M can be written as $P(-I)P^{-1} = -I$, so M contains only the additive inverse of the identity matrix. Notice that $det(-I) = 1$. Together, I and $-I$ form a two-element subgroup of $SL_2(\mathbb{R})$, since both elements are self-inverses and $(-I)(I) = -I = (I)(-I)$.

3.10 Most rotations are in $D_{\mathbb{R}}$ and $N \subseteq D_{\mathbb{R}}$

Consider a matrix R_θ. The characteristic equation $det(R_\theta - \lambda I) = 0$ implies the following:

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0 \tag{13}$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta} \tag{14}$$

When $\theta \neq 0 \bmod \pi$, R_θ has non-real eigenvalues. These R_θ are defective over $\mathbb{R}$; there is no $P \in GL_2(\mathbb{R})$ to make $P^{-1}R_\theta P$ contain complex entries. Similarly, any matrix in N has non-real eigenvalues by 3.7, so $N \subseteq D_{\mathbb{R}}$.

4 Matrix structure

4.1 Inverse of 2x2 complex matrix

Consider an invertible matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\{a, b, c, d\} \subset \mathbb{C}$. Then, its inverse is $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$:

$$AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{pmatrix} = I \tag{15}$$

$$A^{-1}A = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{pmatrix} = I \tag{16}$$

This inverse is unique because $GL_2(\mathbb{C})$ is a group (therefore, associative):

$$B_L A = I \implies B_L = B_L(AA^{-1}) = (B_L A)A^{-1} = A^{-1} \tag{17}$$

$$AB_R = I \implies B_R = (A^{-1}A)B_R = A^{-1}(AB_R) = A^{-1} \tag{18}$$

4.2 Structure of matrices in L

Consider a matrix $A \in L$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\{a, b, c, d\} \subset \mathbb{R}$. Since the determinant is the product of the eigenvalues, A has determinant -1. I use the characteristic equation $det(A - \lambda I) = 0$ to constrain A:

$$(a-\lambda)(d-\lambda) - bc = 0 \tag{19}$$

$$\lambda^2 - (a+d)\lambda + det(A) = 0 \qquad det(A) = ad - bc \tag{20}$$

$$(a+d)\lambda = 0 \qquad (-1)^2 = 1^2 = 1 = -det(A) \tag{21}$$

$$a = -d \implies bc = 1 - a^2 \qquad \lambda \neq 0, ad - bc = -1 \tag{22}$$

So, I can represent A in one of two forms:

$$L = \left\{ \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix} \mid a \in \mathbb{R}, b \in \mathbb{R} - \{0\} \right\} \cup \left\{ \begin{pmatrix} \pm 1 & 0 \\ c & \mp 1 \end{pmatrix} \mid c \in \mathbb{R} \right\} \tag{23}$$

The second form may seem defective, but the reader can verify that its eigenvectors are linearly independent. The reader can also verify that $A^2 = I$ for each form; thus, all $A \in L$ are self-inverses.

4.3 Structure of matrices in $D_{\mathbb{C}}$

Any matrix in $D_{\mathbb{C}}$ has a matrix $P \in GL_2(\mathbb{C})$ such that $P^{-1}XP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, the 2-dimensional Jordan block. I find the form of $A \in D_{\mathbb{C}}$ explicitly:

$$A = \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \{a, b, c, d\} \subset \mathbb{C}, ad - bc \neq 0 \tag{24}$$

$$= \lambda I + \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{25}$$

$$= \lambda I + k \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} \qquad \{a, k, c\} \subset \mathbb{C}, |a| + |c| \neq 0, k \neq 0 \tag{26}$$

$$= \lambda I + \begin{pmatrix} -lm & l^2 \\ -m^2 & lm \end{pmatrix} \qquad l, m \in \mathbb{C}, |l| + |m| \neq 0 \tag{27}$$

4.4 Structure of matrices in X

Since $X \subseteq D_{\mathbb{C}}$, any matrix has the form listed in 4.3. But matrices in X have real eigenvalues (precisely, $\lambda = \pm 1$ by 3.6), so the real Jordan form matches the complex Jordan form. In other words, $P \in GL_2(\mathbb{R})$:

$$A = \pm I + k \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} \qquad \{a, k, c\} \subset \mathbb{R}, |a| + |c| \neq 0, k \neq 0 \tag{28}$$

$$= \pm I \pm \begin{pmatrix} -lm & l^2 \\ -m^2 & lm \end{pmatrix} \qquad l, m \in \mathbb{R}, |l| + |m| \neq 0 \tag{29}$$

Since $A = \pm I + N_{lm}$, where N_{lm} is nilpotent, $A^{-1} = \pm I - N_{lm}$, as $(\pm I - N_{lm})(\pm I + N_{lm}) = I - N_{lm}^2 = I$ and $(\pm I + N_{lm})(\pm I - N_{lm}) = I - N_{lm}^2 = I$.

4.5 Structure of matrices in $O_2(\mathbb{R})$

Consider a matrix $A \in O_2(\mathbb{R})$. By 3.3, $det(A) = \pm 1$. I use the equation $A^T = A^{-1}$ to constrain A:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = A^T = A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \{a, b, c, d\} \subset \mathbb{R} \tag{30}$$

If $det(A) = 1$, then $a = d$ and $b = -c = \pm\sqrt{1 - a^2}$. Otherwise, $det(A) = -1$, so $a = -d$ and $b = c = \pm\sqrt{1 - a^2}$. Thus, any orthogonal matrix A has the following form:

$$O_2(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \mp\sqrt{1-a^2} & a \end{pmatrix} \mid a \in \mathbb{R}, |a| \leq 1 \right\} \tag{31}$$

Without loss of generality, I set $a = \cos\theta$ for $\theta \in \mathbb{R}$. Then, an orthogonal matrix looks more familiar:

$$O_2(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \pm\sin\theta \\ \mp\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \tag{32}$$

Some plus-minus signs are redundant because $\cos -\theta = \cos\theta$ and $\sin -\theta = -\sin\theta$.

5 Finding all orthogonal matrices

5.1 Rotations are special orthogonal matrices

By 4.5, matrices in $O_2(\mathbb{R}) \cap SL_2(\mathbb{R})$ have a specific structure; namely, $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_\theta$ for some $\theta \in \mathbb{R}$. The set of rotations F is exactly the set of determinant-1 orthogonal matrices! F is also known as $SO_2(\mathbb{R})$, the special orthogonal group of real 2x2 matrices.

5.2 Reflections are orthogonal matrices

Any reflection matrix $A \in G$ is symmetric and its own inverse. So, $A^T = A = A^{-1}$, thus $A \in O_2(\mathbb{R})$.

5.3 Orthogonal matrices are either reflections or rotations

I first transform the reflection S_k into polar coordinates. If $k = y/x = \arctan\phi$:

$$\cos 2\phi = \cos^2\phi - \sin^2\phi = \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - k^2}{1 + k^2} \tag{33}$$

$$\sin 2\phi = 2\sin\phi\cos\phi = \frac{2xy}{x^2 + y^2} = \frac{2k}{1 + k^2} \tag{34}$$

Then, S_k can be defined by the angle ϕ between the line $y = 0$ and the reflection axis $y = kx$:

$$S_k = \frac{1}{k^2 + 1} \begin{pmatrix} 1 - k^2 & 2k \\ 2k & k^2 - 1 \end{pmatrix} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \qquad \phi = \arctan k \tag{35}$$

$S_\star$ also has this form with $\phi = \pi/2$. So, any matrix in G is described by ϕ (plus any integer multiple of π):

$$G = \left\{ \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \mid \phi \in \mathbb{R}, -\frac{\pi}{2} < \phi \leq \frac{\pi}{2} \right\} = \left\{ \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \mid \phi \in \mathbb{R} \right\} \tag{36}$$

By 4.5, this is exactly the form of any orthogonal matrix with determinant -1 (where $\theta = 2\phi$). Thus, $G = \{A | A \in O_2(\mathbb{R}), A \notin SL_2(\mathbb{R})\}$, and $F \cup G = O_2(\mathbb{R})$.

5.4 Two reflections make a rotation

Consider $A, B \in G$. By 3.2, $AB \notin G$. By 2.5 and 5.3, $O_2(\mathbb{R}) = F \cup G$ is a group, so $A, B \in F \cup G$ implies $AB \in F \cup G$. Thus, $AB \in F$. The interested reader can explicitly verify that the product of two reflections has the matrix structure of a rotation.

5.5 Any rotation is the product of two reflections

Via 4.5 and 5.3, the reader can verify $R_\theta = S_0 S_\star$ when $\theta = \pi \bmod 2\pi$ and $R_\theta = S_0 S_{\tan(\theta/2)}$ otherwise. Any rotation clockwise by θ can be achieved by reflecting over the axis at angle $\theta/2$ counterclockwise from $y = 0$, then reflecting over $y = 0$. Together with 5.4, the set of rotations equals the set of two reflections, i.e. $F = \{AB \mid A, B \in G\}$.

6 Exploring J

6.1 Rotations are in $K \cup M \cup N$

By 3.10, a rotation R_θ has unit-magnitude eigenvalues $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$, so $F \subseteq J$. I diagonalize R_θ over $\mathbb{C}$:

$$P = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{37}$$

$$P^{-1} = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{38}$$

$$P^{-1}R_\theta = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} e^{i\theta}/\sqrt{2} & -ie^{i\theta}/\sqrt{2} \\ -ie^{-i\theta}/\sqrt{2} & e^{-i\theta}/\sqrt{2} \end{pmatrix} \tag{39}$$

$$P^{-1}R_\theta P = \begin{pmatrix} e^{i\theta}/\sqrt{2} & -ie^{i\theta}/\sqrt{2} \\ -ie^{-i\theta}/\sqrt{2} & e^{-i\theta}/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{40}$$

Thus, no matrix R_θ is in $D_\mathbb{C}$, so $F \subseteq Y$. Since $det(R_\theta) = 1$, $R_\theta \notin L$ by 4.2, so $F \subseteq K \cup M \cup N$.

6.2 Reflections are in L

Any matrix $A \in G$ has determinant -1, so it cannot have non-real eigenvalues (otherwise its determinant would be 1 by 3.4 and 3.7). Since there is no $\lambda \in \mathbb{R}$ such that $\lambda^2 = -1$, A must have two distinct eigenvalues, and therefore two distinct eigenvectors $\hat{x}_1, \hat{x}_2$. Since A is its own inverse:

$$\hat{x}_1 = I\hat{x}_1 = A^2\hat{x}_1 = \lambda_1^2\hat{x}_1 \implies \lambda_1^2 = 1 \implies |\lambda_1| = 1 \tag{41}$$

$$\hat{x}_2 = I\hat{x}_2 = A^2\hat{x}_2 = \lambda_2^2\hat{x}_2 \implies \lambda_2^2 = 1 \implies |\lambda_2| = 1 \tag{42}$$

Since both eigenvalues are unit-magnitude, $A \in J$. By 3.5, matrices in $D_\mathbb{C}$ have repeated eigenvalues, so $A \notin D_\mathbb{C}$, therefore $A \in Y$. Since $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$, $A \in L$. So, all reflections $G \subseteq L$.

6.3 Orthogonal matrices are in Y

Since $F \subseteq K \cup M \cup N \subseteq Y$ (by 6.1) and $G \subseteq L \subseteq Y$ (by 6.2), $O_2(\mathbb{R}) = F \cup G \subseteq Y$. All 2x2 orthogonal matrices are diagonalizable over $\mathbb{C}$ and have unit-magnitude eigenvalues.

6.4 X is not a group

Although X is closed over inverses, X is not closed over matrix multiplication:

$$(I + N_{lm})(I + N_{no}) = I + N_{lm} + N_{no} + N_{lm}N_{no} \tag{43}$$

$$= I + \begin{pmatrix} -lm & l^2 \\ -m^2 & lm \end{pmatrix} + \begin{pmatrix} -no & n^2 \\ -o^2 & no \end{pmatrix} + \begin{pmatrix} -lm & l^2 \\ -m^2 & lm \end{pmatrix} \begin{pmatrix} -no & n^2 \\ -o^2 & no \end{pmatrix} \tag{44}$$

$$= I + \begin{pmatrix} -lm - no & l^2 + n^2 \\ -m^2 - o^2 & lm + no \end{pmatrix} + \begin{pmatrix} lmno - l^2o^2 & -lmn^2 + l^2no \\ m^2no - lmo^2 & -m^2n^2 + lmno \end{pmatrix} \tag{45}$$

$$= I + \begin{pmatrix} -lm - no + lmno - l^2o^2 & l^2 + n^2 - lmn^2 + l^2no \\ -m^2 - o^2 + m^2no - lmo^2 & lm + no - m^2n^2 + lmno \end{pmatrix} \tag{46}$$

Comparing the diagonal terms of the nilpotent matrix, $l^2o^2 + m^2n^2 - 2lmno = (lo - mn)^2 \notin \{0, 4\}$ in general, so this matrix product is not always in X.

6.5 Every matrix in J has an inverse in J

For $A \in X$, $A = \pm I \pm N_{lm}$, and $A^{-1} = \pm I \mp N_{lm} \in X$. For $A \in K \cup L \cup M$, A is its own inverse. For $A \in N$, $A = P\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}P^{-1}$ for some $P \in GL_2(\mathbb{C})$ and $\theta \in \mathbb{R}, \theta \neq 0 \bmod \pi$, so $A^{-1} = P\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}P^{-1} \in N$. Since $J = K \cup L \cup M \cup N \cup X$, every $A \in J$ also has $A^{-1} \in J$.

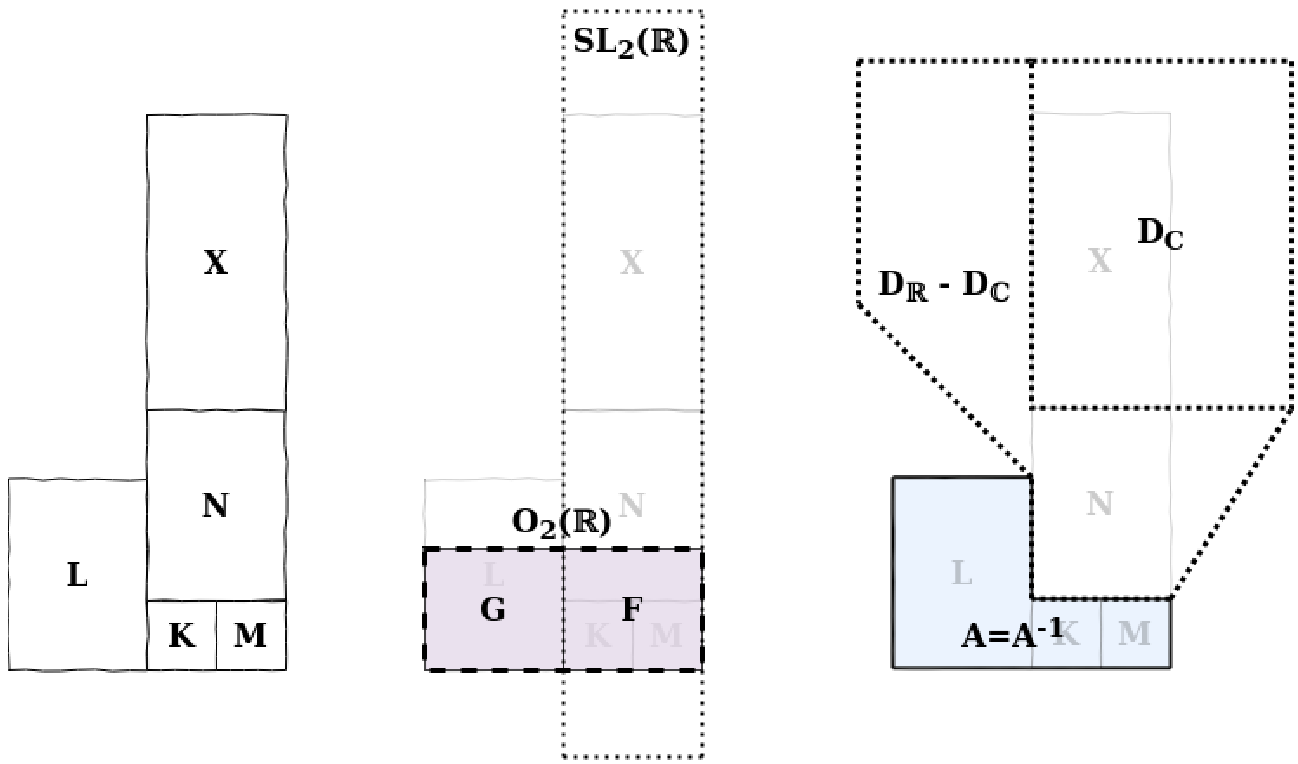

Figure 1: This image relates partitions of J with other subsets of $GL_2(\mathbb{R})$: orthogonal matrices $O_2(\mathbb{R})$, rotations F and reflections G, determinant-1 matrices $SL_2(\mathbb{R})$, self-inverses, and defective matrices $D_{\mathbb{C}} \subset D_{\mathbb{R}}$.

6.6 $K \cup L \cup M$ is the set of all self-inverses

Consider a matrix in $GL_2(\mathbb{R})$ with $A = A^{-1}$. Then, since $1 = det(I) = det(AA^{-1}) = det(A^2) = det^2(A)$, the determinant of A is ± 1. I constrain A by inspecting its matrix structure:

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1} = A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \{a, b, c, d\} \subset \mathbb{R} \tag{47}$$

$$det(A) = 1 \implies a = d, b = -b, c = -c \implies A = aI \qquad a^2 = 1 \tag{48}$$

$$det(A) = -1 \implies a = -d \implies A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad bc = 1 - a^2 \tag{49}$$

When $det(A) = 1$, $A = \pm I$, the two matrices in $K \cup M$. When $det(A) = -1$, A has the same constraints as described in 4.2 for a matrix in L. The reader can verify $A \in L$ with an argument similar to 6.2. So, $A \in K \cup L \cup M$. Every matrix in $K \cup L \cup M$ is self-inverse, so $K \cup L \cup M$ is exactly the set of self-inverse matrices in $GL_2(\mathbb{R})$.

7 Figure

Figure 1 summarizes many results in this document. It was created with an online diagram maker tool.

8 Questions to explore

8.1 What is the structure of a matrix in N? Or generally in $D_{\mathbb{R}} - D_{\mathbb{C}}$?

Matrices $A \in N$ have non-real eigenvalues, but are not defective over $\mathbb{C}$. I think the best approach is to expand the real Jordan form for complex eigenvalues, and inspect the real and imaginary parts.

8.2 Is $K \cup M \cup N$ a group?

I think there are matrices in N that are not rotations (as there are matrices in G that are not reflections). I'm not sure $K \cup M \cup N$ is closed over matrix multiplication. If Y is a group, this is the subgroup of determinant-1 matrices in Y.

8.3 Is J a group?

Does the product of two matrices with unit-magnitude eigenvalues have unit-magnitude eigenvalues? I wonder if it's possible to bound the eigenvalues of a matrix product AB given $A, B \in J$. The determinant of AB is ± 1, but many matrices, such as $A = \begin{pmatrix} 2 & 0 \\ 0 & \pm 1/2 \end{pmatrix}$, satisfy $det(A) = \pm 1$ without unit-magnitude eigenvalues.

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