1-D Shock Front

Kunal Marwaha

March 5, 2019

1 Introduction

Consider a 1-dimensional shock in a γ-law (adiabatic, $\gamma > 1$) ideal gas. In the frame of the shock, the flow is steady before and after the shock front. Over long distances, any terms that go $\frac{\partial}{\partial x}$ will be small, such as τ (viscosity) and F_{cond} (conductive flux). So, the continuity, momentum, and energy equations are as follows:

$$\frac{\partial}{\partial x}(\rho u) = 0 \tag{1}$$

$$\frac{\partial}{\partial x}(\rho u^2 + P) = 0 \tag{2}$$

$$\frac{\partial}{\partial x}(\rho u(\frac{1}{2}u^2 + \epsilon) + uP) = 0 \tag{3}$$

In the above, P is pressure, ρ is density, u is bulk velocity, and ϵ is internal energy. These are functions of space, but not of time, since the flow is steady in this frame.

2 Internal energy

In general, the following equation holds:

$$m\mathrm{d}\epsilon = -P\mathrm{d}V + T\mathrm{d}S \tag{4}$$

For a given mass, we know $\frac{\mathrm{d}V}{m} = \mathrm{d}(\frac{1}{\rho}) = -\frac{\mathrm{d}\rho}{\rho^2}$, and for an adiabatic gas, $\mathrm{d}S = 0$. Thus,

$$\mathrm{d}\epsilon = \frac{P}{\rho^2}\mathrm{d}\rho \tag{5}$$

For an adiabatic gas, $P = K\rho^\gamma$ for some K, so we can take the full integral:

$$\epsilon = \int \mathrm{d}\epsilon = \int \frac{K\rho^\gamma}{\rho^2}\mathrm{d}\rho = \frac{1}{\gamma - 1}K\rho^{\gamma-1} = \frac{1}{\gamma - 1}\frac{P}{\rho} \tag{6}$$

For equation 6, we ignore the integration constant, assuming $\epsilon = 0$ at $\rho = 0$.

3 Combining equations

Given equation 6, the energy equation (equation 3) becomes the following:

$$\frac{\partial}{\partial x}(\rho u(\frac{1}{2}u^2) + \frac{\gamma}{\gamma - 1}uP) = 0 \tag{7}$$

Equations 1, 2, and 7 can be rewritten as invariants, where A, B, and C are conserved quantities (constants) based on initial conditions:

$$\rho u = A \tag{8}$$

$$\rho u^2 + P = B \tag{9}$$

$$\rho u(\frac{1}{2}u^2) + \frac{\gamma}{\gamma - 1}uP = C \tag{10}$$

We can substitute equation 8 in the others to remove dependence on ρ:

$$Au + P = B \tag{11}$$

$$\frac{Au^2}{2} + \frac{\gamma}{\gamma - 1}uP = C \tag{12}$$

Removing dependence on P, we have an equation quadratic in u:

$$C = \frac{Au^2}{2} + \frac{\gamma}{\gamma - 1}u(B - Au) = -\frac{A(\gamma + 1)}{2(\gamma - 1)}u^2 + \frac{B\gamma}{\gamma - 1}u \tag{13}$$

What's surprising here is that there can be at most 2 (real) values for u, given initial conditions (which include u before the shock!). Hopefully the other value of u exists, and matches with our physical intuitions.

4 Ratio of quadratic equation solutions

The solutions for a quadratic equation $ax^2 + bx + c = 0$ are of the form $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If we are interested in the ratio of the two solutions (call them x_1 and x_2, it doesn't matter which is larger):

$$\frac{x_2}{x_1} = \frac{x_2 + x_1}{x_1} - 1 = (\frac{-b}{a})\frac{1}{x_1} - 1 \tag{14}$$

So, given equation 13, we can find:

$$\frac{u_2}{u_1} = \frac{2B\gamma}{A(\gamma + 1)}\frac{1}{u_1} - 1 = \frac{2\gamma}{\gamma + 1}(\frac{B}{Au_1}) - 1 \tag{15}$$

This is a strange form, but we will exploit some nice properties of $\frac{B}{Au}$.

$$\frac{B}{Au} = 1 + \frac{P}{Au} = 1 + \frac{P}{\rho u^2} = 1 + \frac{1}{\gamma}\frac{{c_s}^2}{u^2} = 1 + \frac{1}{\gamma M^2} \tag{16}$$

In equation 16, c_s is the sound speed $\sqrt{\frac{\partial}{\partial\rho}P}$ (which is $\sqrt{\gamma\frac{P}{\rho}}$ for a γ-law ideal gas), and M is the (dimensionless) Mach number $\frac{u}{c_s}$. Note that neither are conserved quantities, i.e. they vary across space.

We can also check how many solutions exist with the sign of $b^2 - 4ac$. Remember that $\gamma > 1$.

$$sgn(b^2 - 4ac) = sgn(\frac{B^2\gamma^2}{(\gamma-1)^2} - \frac{2AC(\gamma+1)}{\gamma-1}) = sgn(B^2\gamma^2 - 2AC(\gamma^2-1)) \tag{17}$$

Substituting in $B^2 = A^2u^2 + 2PAu + P^2$, and $2AC = A^2u^2 + \frac{\gamma}{\gamma-1}(2PAu)$ from equations 11 and 12:

$$sgn(b^2 - 4ac) = sgn(\gamma^2(P^2 + \frac{2PAu}{\gamma}) + 2AC) = sgn(\gamma^2P^2 + 2\gamma PAu + 2AC) \tag{18}$$

We typically consider P and ρ as positive quantities (so $Au = \rho u^2 > 0$), thus AC is positive. So, $b^2 - 4ac > 0$: there are always two real solutions.

5 Putting it all together

Plugging equation 16 into equation 15, we arrive at the Rankine-Hugoniot jump conditions:

$$\frac{u_2}{u_1} = \frac{2\gamma}{\gamma+1}(1 + \frac{1}{\gamma {M_1}^2}) - 1 = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1){M_1}^2} \tag{19}$$

The other ratios $\frac{\rho_2}{\rho_1}$ and $\frac{P_2}{P_1}$ now fall out easily:

$$\frac{\rho_2}{\rho_1} = \frac{A}{u_2}\frac{u_1}{A} = \frac{u_1}{u_2} = \frac{(\gamma+1){M_1}^2}{2 + (\gamma-1){M_1}^2} \tag{20}$$

$$\frac{P_2}{P_1} = \frac{B - Au_2}{B - Au_1} = \frac{(1 + \frac{1}{\gamma {M_1}^2}) - \frac{u_2}{u_1}}{(1 + \frac{1}{\gamma {M_1}^2}) - 1} = \gamma {M_1}^2(1 - \frac{u_2}{u_1}) + 1 = \frac{2\gamma}{\gamma+1}{M_1}^2 - \frac{\gamma-1}{\gamma+1} \tag{21}$$

We can check that the two solutions make physical sense, by considering M_2 in terms of M_1. Since $M^2 = \frac{u^2}{{c_s}^2} = \frac{u^2\rho}{\gamma P}$:

$$\frac{{M_2}^2}{{M_1}^2} = \frac{{u_2}^2\rho_2}{{u_1}^2\rho_1}\frac{P_1}{P_2} = \frac{u_2}{u_1}\frac{P_1}{P_2} = \frac{2 + (\gamma-1){M_1}^2}{(\gamma+1){M_1}^2} \cdot \frac{\gamma+1}{2\gamma {M_1}^2 - (\gamma-1)} \tag{22}$$

Some terms cancel, and we can write M_2 in terms of M_1:

$${M_2}^2 = \frac{2 + (\gamma-1){M_1}^2}{2\gamma {M_1}^2 - (\gamma-1)} = \frac{\gamma-1}{2\gamma} + \frac{4\gamma + (\gamma-1)^2}{4\gamma^2{M_1}^2 - 2\gamma(\gamma-1)} = \frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma} \cdot \frac{\gamma+1}{2\gamma {M_1}^2 - \gamma + 1} \tag{23}$$

When $M_1 = {M_1}^2 = 1$, ${M_2}^2 = \frac{2+(\gamma-1)}{2\gamma-(\gamma-1)} = \frac{1+\gamma}{1+\gamma} = 1$ also. From there, increasing M_1 (which increases M_1^2) will decrease M_2^2 (and the positive root M_2). So whenever M_1 is supersonic (as it should be before the shock), M_2 (describing flow after the shock) is subsonic. Nice! This matches physical intuition.

For hard shocks (as in supernovae), $M_1 >> 1$, so ${M_2}^2 \to \frac{\gamma-1}{2\gamma}$. This surprises me: why would the post-shock flow approach a particular proportion of the (post-shock) sound speed?

6 Conclusion

Given the fundamental equations of fluid dynamics and some simplifying assumptions, we can describe the rough depiction of flow before and after a shock front. We solve by systematically removing variables and simplifying the quotient $\frac{u_2}{u_1}$ just as a function of initial Mach number. We then guarantee the solutions of P, ρ, and u exist for our purposes and make physical sense.

If something in this document is unclear, please let me know at marwahaha@berkeley.edu .