

1-D Shock Front

Kunal Marwaha

March 5, 2019

This work is licensed under a Creative Commons “Attribution 4.0 International” license.



1 Introduction

Consider a 1-dimensional shock in a γ -law (adiabatic, $\gamma > 1$) ideal gas. In the frame of the shock, the flow is steady before and after the shock front. Over long distances, any terms that go $\frac{\partial}{\partial x}$ will be small, such as τ (viscosity) and F_{cond} (conductive flux). So, the continuity, momentum, and energy equations are as follows:

$$\frac{\partial}{\partial x}(\rho u) = 0 \quad (1)$$

$$\frac{\partial}{\partial x}(\rho u^2 + P) = 0 \quad (2)$$

$$\frac{\partial}{\partial x}(\rho u(\frac{1}{2}u^2 + \epsilon) + uP) = 0 \quad (3)$$

In the above, P is pressure, ρ is density, u is bulk velocity, and ϵ is internal energy. These are functions of space, but not of time, since the flow is steady in this frame.

2 Internal energy

In general, the following equation holds:

$$md\epsilon = -PdV + TdS \quad (4)$$

For a given mass, we know $\frac{dV}{m} = d(\frac{1}{\rho}) = -\frac{d\rho}{\rho^2}$, and for an adiabatic gas, $dS = 0$. Thus,

$$d\epsilon = \frac{P}{\rho^2}d\rho \quad (5)$$

For an adiabatic gas, $P = K\rho^\gamma$ for some K , so we can take the full integral:

$$\epsilon = \int d\epsilon = \int \frac{K\rho^\gamma}{\rho^2}d\rho = \frac{1}{\gamma-1}K\rho^{\gamma-1} = \frac{1}{\gamma-1}\frac{P}{\rho} \quad (6)$$

For equation 6, we ignore the integration constant, assuming $\epsilon = 0$ at $\rho = 0$.

3 Combining equations

Given equation 6, the energy equation (equation 3) becomes the following:

$$\frac{\partial}{\partial x}(\rho u(\frac{1}{2}u^2) + \frac{\gamma}{\gamma-1}uP) = 0 \quad (7)$$

Equations 1, 2, and 7 can be rewritten as invariants, where A , B , and C are conserved quantities (constants) based on initial conditions:

$$\rho u = A \quad (8)$$

$$\rho u^2 + P = B \quad (9)$$

$$\rho u(\frac{1}{2}u^2) + \frac{\gamma}{\gamma-1}uP = C \quad (10)$$

We can substitute equation 8 in the others to remove dependence on ρ :

$$Au + P = B \quad (11)$$

$$\frac{Au^2}{2} + \frac{\gamma}{\gamma-1}uP = C \quad (12)$$

Removing dependence on P , we have an equation quadratic in u :

$$C = \frac{Au^2}{2} + \frac{\gamma}{\gamma-1}u(B - Au) = -\frac{A(\gamma+1)}{2(\gamma-1)}u^2 + \frac{B\gamma}{\gamma-1}u \quad (13)$$

What's surprising here is that there can be at most 2 (real) values for u , given initial conditions (which include u before the shock!). Hopefully the other value of u exists, and matches with our physical intuitions.

4 Ratio of quadratic equation solutions

The solutions for a quadratic equation $ax^2 + bx + c = 0$ are of the form $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If we are interested in the ratio of the two solutions (call them x_1 and x_2 , it doesn't matter which is larger):

$$\frac{x_2}{x_1} = \frac{x_2 + x_1}{x_1} - 1 = \left(\frac{-b}{a}\right)\frac{1}{x_1} - 1 \quad (14)$$

So, given equation 13, we can find:

$$\frac{u_2}{u_1} = \frac{2B\gamma}{A(\gamma+1)}\frac{1}{u_1} - 1 = \frac{2\gamma}{\gamma+1}\left(\frac{B}{Au_1}\right) - 1 \quad (15)$$

This is a strange form, but we will exploit some nice properties of $\frac{B}{Au}$.

$$\frac{B}{Au} = 1 + \frac{P}{Au} = 1 + \frac{P}{\rho u^2} = 1 + \frac{1}{\gamma} \frac{c_s^2}{u^2} = 1 + \frac{1}{\gamma M^2} \quad (16)$$

In equation 16, c_s is the sound speed $\sqrt{\frac{\partial P}{\partial \rho}}$ (which is $\sqrt{\gamma \frac{P}{\rho}}$ for a γ -law ideal gas), and M is the (dimensionless) Mach number $\frac{u}{c_s}$. Note that neither are conserved quantities, i.e. they vary across space.

We can also check how many solutions exist with the sign of $b^2 - 4ac$. Remember that $\gamma > 1$.

$$\text{sgn}(b^2 - 4ac) = \text{sgn}\left(\frac{B^2\gamma^2}{(\gamma-1)^2} - \frac{2AC(\gamma+1)}{\gamma-1}\right) = \text{sgn}(B^2\gamma^2 - 2AC(\gamma^2 - 1)) \quad (17)$$

Substituting in $B^2 = A^2u^2 + 2PAu + P^2$, and $2AC = A^2u^2 + \frac{\gamma}{\gamma-1}(2PAu)$ from equations 11 and 12:

$$\text{sgn}(b^2 - 4ac) = \text{sgn}\left(\gamma^2\left(P^2 + \frac{2PAu}{\gamma}\right) + 2AC\right) = \text{sgn}(\gamma^2P^2 + 2\gamma PAu + 2AC) \quad (18)$$

We typically consider P and ρ as positive quantities (so $Au = \rho u^2 > 0$), thus AC is positive. So, $b^2 - 4ac > 0$: there are always two real solutions.

5 Putting it all together

Plugging equation 16 into equation 15, we get the following:

$$\frac{u_2}{u_1} = \frac{2\gamma}{\gamma+1}\left(1 + \frac{1}{\gamma M_1^2}\right) - 1 = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2} \quad (19)$$

The other ratios $\frac{\rho_2}{\rho_1}$ and $\frac{P_2}{P_1}$ now fall out easily:

$$\frac{\rho_2}{\rho_1} = \frac{A}{u_2} \frac{u_1}{A} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2} \quad (20)$$

$$\frac{P_2}{P_1} = \frac{B - Au_2}{B - Au_1} = \frac{\left(1 + \frac{1}{\gamma M_1^2}\right) - \frac{u_2}{u_1}}{\left(1 + \frac{1}{\gamma M_1^2}\right) - 1} = \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right) + 1 = \frac{2\gamma}{\gamma+1} M_1^2 - \frac{\gamma-1}{\gamma+1} \quad (21)$$

We can check that the two solutions make physical sense, by considering M_2 in terms of M_1 . Since $M^2 = \frac{u^2}{c_s^2} = \frac{u^2 \rho}{\gamma P}$:

$$\frac{M_2^2}{M_1^2} = \frac{u_2^2 \rho_2}{u_1^2 \rho_1} \frac{P_1}{P_2} = \frac{u_2}{u_1} \frac{P_1}{P_2} = \frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2} \cdot \frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)} \quad (22)$$

Some terms cancel, and we can write M_2 in terms of M_1 :

$$M_2^2 = \frac{2 + (\gamma-1)M_1^2}{2\gamma M_1^2 - (\gamma-1)} = \frac{\gamma-1}{2\gamma} + \frac{4\gamma + (\gamma-1)^2}{4\gamma^2 M_1^2 - 2\gamma(\gamma-1)} = \frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma} \cdot \frac{\gamma+1}{2\gamma M_1^2 - \gamma+1} \quad (23)$$

When $M_1 = M_1^2 = 1$, $M_2^2 = \frac{2+(\gamma-1)}{2\gamma-(\gamma-1)} = \frac{1+\gamma}{1+\gamma} = 1$ also. From there, increasing M_1 (which increases M_1^2) will decrease M_2^2 (and the positive root M_2). So whenever M_1 is supersonic (as it should be before the shock), M_2 (describing flow after the shock) is subsonic. Nice! This matches physical intuition.

For hard shocks (as in supernovae), $M_1 \gg 1$, so $M_2^2 \rightarrow \frac{\gamma-1}{2\gamma}$. This surprises me: why would the post-shock flow approach a particular proportion of the (post-shock) sound speed?

6 Conclusion

Given the fundamental equations of fluid dynamics and some simplifying assumptions, we can describe the rough depiction of flow before and after a shock front. We solve by systematically removing variables and simplifying the quotient $\frac{u_2}{u_1}$ just as a function of initial Mach number. We then guarantee the solutions of P , ρ , and u exist for our purposes and make physical sense.

If something in this document is unclear, please let me know at marwahaha@berkeley.edu .