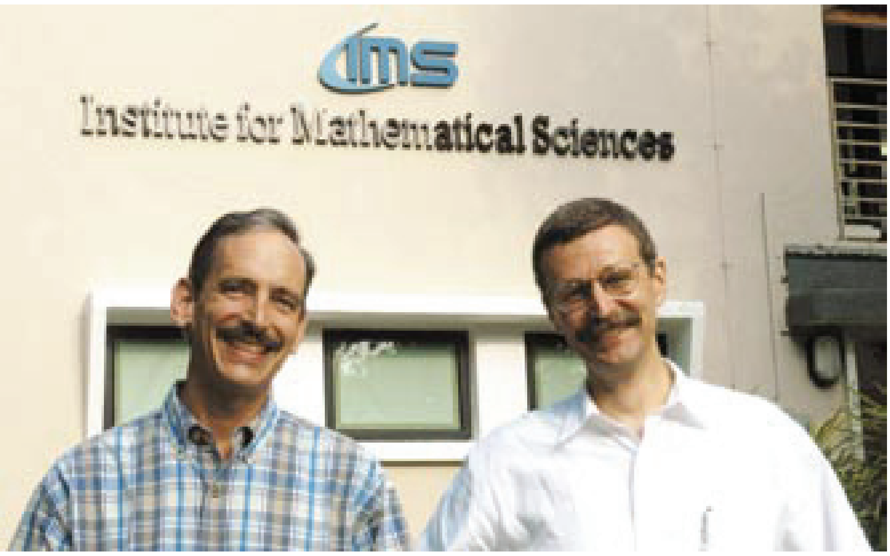
Theodore Slaman and Hugh Woodin: Logic and Mathematics >>>



From right to left: Theodore Slaman and Hugh Woodin

Interview of Theodore Slaman and Hugh Woodin by Y.K. Leong

Theodore Slaman and Hugh Woodin have recently made important contributions to logic, especially to recursion theory and set theory respectively.

Slaman did a bachelor's degree in physics at Pennsylvania State University before going to Cambridge, Massachusetts to do his doctorate in mathematics at Harvard University. He taught at University of Chicago from 1983 to 1996 and subsequently at University of California at Berkeley, where he is a professor of mathematics and is currently chairman of the mathematics department. He has received the Presidential Young Investigator Award and the Alexander von Humboldt Research Award. He has been invited to give lectures at major mathematical meetings such as the International Congress of Mathematicians, meetings of the American Mathematical Society, British

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Mathematics Colloquium and the Association of Symbolic Logic's Logic Colloquia. He has given the Gödel Lecture and has been invited to give lectures and engage in research collaboration in many parts of the world, in particular Japan, United Kingdom, Germany, France, Canada, China, and Singapore. Slaman is best known for his contributions to Recursion Theory.

Woodin obtained his bachelor's degree in mathematics from California Institute of Technology and doctorate from University of California at Berkeley. He taught at CalTech and then at University of California, Berkeley, where he is professor of mathematics. He has received many awards and grants for his research work, among them the NSF Principal Investigator, SERC Senior Visiting Fellowship Research Grant, Presidential Young Investigator Award, Sloan Research Fellow, Carol Karp Prize, Humboldt Research Award, Miller Research Professorship. He is a member of the American Academy of Arts and Sciences. He has been invited to give lectures at the International Congress of Mathematicians and in many parts of the world, in particular, UK, France, Netherlands, Italy, Spain, Canada, New Zealand, Germany and China. He is well-known for fundamental contributions to modern set theory and large cardinals.

The Editor (Y.K. Leong) interviewed Slaman and Woodin at the Department of Mathematics on 12 June 2004 when they were invited to participate in the Institute's program on the Computational Prospects of Infinity. The following is an enhanced version of the edited transcript of the interview in which they share their passion for and fascination with the highest form of abstraction in one of the possibly most remote area of knowledge at the boundaries of mathematics, logic and philosophy.

Imprints: Why did you choose logic and set theory in your graduate studies?

Slaman: When I was an undergraduate, I majored in physics and I took courses in physics and pure mathematics. But nothing fit my way of thinking as well as mathematical logic. I cannot say how logic felt about me, but I was attracted to it from the beginning. It was an intuitive rather than rational decision.

Woodin: When I was an undergraduate, I began in analysis and the problems in analysis that I was interested in led naturally to set theory. So I studied set theory.

I: Mathematical logic and set theory seem to be linked together like inseparable Siamese twins. Why is that so?

W: Methods from logic form an essential part of the study of set theory because the study of definable sets is such an important part of the subject. In that sense, logic is very much a part of set theory.

S: There are many natural questions in set theory that cannot be settled within a naive mathematical setting. The best example is probably the continuum hypothesis. Gödel and Cohen made a big breakthrough by showing that the continuum hypothesis is neither provable nor refutable within the axioms of Zermelo-Frankel set theory. It is a mathematical question in set theory but you have to use logic to understand it.

I: Logic seems to be a closed book to many people, even to many mathematicians. What do you think is the reason for this?

S: In fact, much of mathematical logic percolated into the general mathematical consciousness without the general mathematician's being aware of it. For example, logic is instrumental in the analysis of solvability problems for Diophantine equations. Given a polynomial with integer coefficients, can you tell whether it has integer solutions? A lot of people contributed and then Matiyasevich provided the final step in the 1970's to show that the answer is "no": there is no such algorithm. Of course, before Matiyasevich's theorem can be proven, one has to analyze what it means to have an algorithm. That preliminary work was done by Gödel, Church, Turing, and others in the 1930's. The very recent result that primality can be decided in polynomial time is something any contemporary mathematician should be able to understand. Again, one is building upon: what it means to be computable, what it means to have an algorithm, the model of computation using Turing machines. All of that comes from logic. These days, there's a concrete feet-on-the-ground understanding of logical issues.

I: Probably for the new generation of mathematicians, but for the older generation of mathematicians, they are quite happy that they never had a course in logic.

S: I recall taking a variety of courses in analysis, topology, algebra, and geometry, and I am very happy about that. Yes, there are mathematicians who are happy that they don't know some branch of mathematics. But with a different attitude, they would have a richer mathematical experience. People should be happy for all the courses that they have taken.

W: I think the Gödel incompleteness theorem is one of the greatest theorems of the 20th Century. How could one be happy about being ignorant of it?

I: They may know it but only at the superficial level. Even now, a first course in logic is not part of the essential mathematics curriculum, even in the second year.

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S: It's true that in most universities a course in logic is not required but usually it is available.

I: Do you think that logic itself could be subject to immutable logical laws? How do we know that the logical processes that we use in mathematics follow the genuine laws and not some approximation of them?

W: I think that if one comes up with a proof that is a formal proof, there is no question that you have proved what you have set out to prove. In that sense it is a sufficient condition. Of course, no one exhibits a formal proof. On the other hand, I do happen to believe that there are truths of number theory or set theory which are not formal consequences of the axioms. But it's not that we are missing a logic to discover those truths. We discover those truths by understanding formal consequences of the axioms and by following our intuition. But, I don't think there is a super logic that transcends classical logic waiting to be discovered and things you can prove in super logic from the axioms give you more truth than the formal truth.

I: If I remember correctly, when a person defines "implication", it is not a "real" definition, it's just a symbol and somehow it assumes that the person already knows what it means. It is one of those things that you don't define. It seems to me that the logical process is taken for granted right from the beginning.

S: Sure, a logical process is taken for granted at the beginning. It's similar to taking the integers for granted when introducing the axioms for a ring. The point is that we analyze logic with the same mathematical precision and the same success that we achieve with the integers.

W: The issue is if you want to show that a problem is unsolvable, to make that precise you have to formalize reason. Otherwise how do you get mathematical content into the statement like "The continuum hypothesis cannot be solved from the axioms of set theory"? You have to set up some formal system of logic and there are many different ways to do it.

I: Is there a unique system of logic?

W: No, but I think whatever approach you take, you are going to end up with the same collection of unsolvable propositions.

I: Has logic or set theory been able to solve any long-standing problems in areas like analysis or algebra that are not essentially of a logical nature?

S: One way to give a logician's solution to a problem would be to show, to say, that the problem does not have a solution within the axioms of set theory - that the basic principles used by people in the field who originally formulated the problem are insufficient to settle the problem - in the same way that the axioms of a group are insufficient to settle whether the operation commutes. There are lots of examples like that.

Another way that logic could be used to settle a problem would be to actually give an answer, in the usual sense, which rests upon perspectives or techniques which are intrinsically logical, having something to do with language, definability, and so on. In the past few years, Hrushovski has brilliantly applied model theory to algebra and number theory. Of course, ideas flow in both directions; Sela recently solved a long-standing problem in logic by applying ideas from topology.

W: I give you another example. When I was an undergraduate, I became interested in set theory and was given a summer research project on a well-known problem of Kaplansky in Banach algebras to think about. I did an analysis which led to the theorem of Solovay that it was unsolvable. It was a non-logical problem and the answer is impossible within set theory.

I: What exactly is that problem?

W: The simplest formulation is this question. Consider the set of continuous functions on the unit interval. This is naturally a Banach algebra with the norm being the sup norm. Suppose you have an algebra homomorphism of that Banach algebra into another Banach algebra. Must it be continuous? You can recast the problem as the following. C[0,1] is a Banach algebra under the sup norm. Suppose you put another norm on it that makes it a normed Banach algebra. Must the norm topology be the same? In other words, are all algebraic norms on C[0,1] topologically equivalent? This problem is independent. If the continuum hypothesis is true, then there exist mutually inequivalent algebraic norms on C[0,1]. However it is possible to build a universe of sets so that there is only one norm up to topological equivalence.

In fact there is an area where methods of logic and set theory have led to a solution of a classical problem in analysis and not just by showing that the problem is unsolvable. Here there are various ways to cast the problem. A nice way of doing it is to use projective sets which are the sets of real numbers which can be generated from the Borel sets by closing under continuous images and complements. These were studied extensively in the beginning of 1900. By 1925, I would say, two kinds of questions had emerged. One is a measure-theoretic question. Are the projective sets Lebesgue measurable? The second was a structural question called the uniformization question: given a projective subset of a

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plane; is there a function whose graph is contained in that projective set and whose domain is the projection of that projective set? These are the two classical questions about projective sets. It turns out that both questions are unsolvable in set theory. We can say now that we have solved the questions. They are unsolvable in set theory because the axioms of set theory are insufficient.

The study of the projective sets is really second-order number theory. Beyond the integers, the next structure you might want to look at is the set of integers together with all sets of integers. Many fundamental questions you can ask there are unsolvable, even in set theory. The measure problem for the projective sets, level by level, is a question of second-order number theory. So the question is really what are the axioms for second-order number theory? We now know what they are, and it took many years to come to that understanding. In some sense, we have found the axioms for second-order number theory that correspond to the Peano axioms for number theory. So here we have an example where classical problems have been solved by other than purely formal means.

I: It would be dramatic if one could use set theory to settle the Riemann Hypothesis. Any chance of doing that?

W: Well, the Riemann Hypothesis is formally equivalent to the consistency of a theory. So it could be equivalent to the consistency of the axioms of set theory. If that is the case, probably set theory will be used. But there is no evidence that the problem cannot be solved just on the basis of the axioms for number theory.

There are examples now of algebraic questions that have been solved using set theory. There is a very nice example dealing with free left-distributive algebras with one operation. The algebra generated by one element is left-distributive: $a*(b*c) = (a*b)*(a*c)$. There are some natural questions you can ask about this. It turns out that the natural model for this comes from set theory and that model was used to gain insights into that algebra. In particular, the first solution to the word problem came from this natural model. Subsequently it was done without it. But there are still questions about this algebra that have to do with whether the free algebra is an inverse limit of a canonical sequence of finite approximations. The only proof known still uses the natural model from set theory, but there is no evidence that it really needs it.

This is an example where a truth was discovered first by invoking very powerful axioms. And there are other examples, such as those coming from the determinacy of infinite games.

S: Determinacy talks about two players and a given set of reals. You use the set to organize a game between the players. The players alternate playing integers to form an infinite sequence. If the sequence is in the set, the first player wins. If not, the second player wins.

W: Suppose the given set is a subset of the unit interval and we are going to play integers between zero and nine. We create the decimal expansion of a real from the integers played. Player 1 wins if that real is in the set, otherwise Player 2 wins.

S: Once we have defined the games we can ask whether they are determined, i.e. if one of the two players has a winning strategy. For example, if we start with the set of all reals, then the first player is going to win. Any strategy will work. The theorem that any Borel set is determined has a remarkable history and is due to Tony Martin. The first proof used measurable cardinals (so, very strong principles from set theory). Later on, Martin proved the determinacy of all Borel games using just the axioms of set theory.

I: What do you think is the greatest contribution of logic to computer science?

S: My answer to that would be the definition of a computable function. It grew purely from logical considerations. Turing presented his model of a computing machine as part of his argument that if a function could be computed at all, then it could be computed by a machine of his sort, namely a Turing machine.

I: This was introduced before the first computers …

S: Yes, the work on the foundations of computability took place in the 1930s and predates actual computers considerably. There are other examples. Logic has to do with the analysis of language and definability, the resources of definability and algorithms. If you look at logic as having a scope that wide, then computer science is looking at a certain section of logic. So you cannot in any way think of computer science without logic. Theoretical computer scientists are analyzing different levels of complexity of computation, and that's all logic.

I: We know from Gödel's incompleteness theorem that there are true results about numbers that cannot be proved within arithmetic. Is it possible to produce concrete, non-metamathematical statements of this nature?

W: A famous example concerns Goodstein sequences. Goodstein published a paper in the forties presenting a number-theoretic fact which he thought was a good candidate for statements which cannot be proved from the axioms of number theory. In the seventies, that was shown to be the case. You start with a seed number N and generate

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sequences of integers, called Goodstein sequences. The theorem is that the sequence is eventually 1.

You start with your seed number N and expand it to base 2 as a sum of powers of 2. Then you take every exponent and expand it as a sum of powers of 2 – exponents of exponents and so on. Now change all the 2s to 3s this defines a new number expanded in base 3. Subtract 1 and re-expand to base 3. Now replace the 3s by 4s. Subtract 1 and re-expand to base 4, and so on. The theorem is that you will eventually reach 1. It may take a long time but you will always reach 1. If you use your calculator and start with 33, you will overflow your calculator.

S: Whether the sequences converge is not a metamathematical question. In theory, you can sit down and write them out. Remarkably, the proof that the Goodstein sequences converge involves the transfinite ordinals.

I: But practically it is not possible to compute those numbers.

W: Well, you can do it with small numbers, like 5. Very soon it gets very complicated. This is predicted by the metamathematics.

I: Are there other examples?

W: Harvey Friedman has other examples which are quite interesting. In fact, there are lots of examples of number-theoretic statements which are not solvable within number theory, which are true and which are purely combinatorial.

I: What about the Goldbach conjecture?

W: Well, it could be equivalent to the consistency of ZF or very strong theory but there is absolutely no evidence for that.

S: Or it could be false.

I: Probability has been successfully axiomatized by measure theory, which is essentially about set theory. Yet it is not clear that this gives a true understanding of randomness. Has there been any progress on shedding some light on the intrinsic concept of randomness from the logical point of view?

S: I think there has been a lot of progress. You can trace it back to Kolmogorov. He had this nice idea. He looked at infinite sequences. A random sequence should be as "complicated" as possible. It should be "unpredictable", it has no historical pattern. It's just noise. But, what does it mean to have no pattern? What is a pattern? That way of speaking has to be made mathematically precise. To have no pattern means that there is no algorithm that will describe the sequence. It has a pattern if it behaves according to some algorithmic law, like the sequence of primes – you can write a program that will give you more and more of the primes. So the sequence of primes is not random. It does not have an obvious pattern such as "every other digit is 1," but it does have a pattern – a computational pattern. Now, if you want to say that a sequence has no pattern at all, then you can say there is no way to compute a pattern. Kolmogorov brought a logical perspective to the concept of randomness.

I: That has nothing to do with probability?

S: Probability and randomness seem closely related to me, but logic has had more to say about randomness. It is not hard to describe. A finite sequence is said to be "simple" if there is a program that will compute the sequence digit by digit and the length (the number of symbols) of the program is less than the length of the sequence. A random infinite sequence has the property that after some finite clustering, none of its initial segments are simple. That's the notion of Kolmogorov complexity; a sequence can be descriptively random. That definition of "random", with some technical adjustments, is equivalent to one asserting that the sequence does not belong to easily described sets of measure 0.

On the one hand, you have the measure-theoretic definition of "random". On the other hand, you have the property about an individual sequence that says that it has this bit-by-bit indescribability. The equivalence of the two is very pretty.

I: What are the greatest advances in logic and set theory in the last century?

S: First, a caveat. The caveat is that the further in the past you look, the easier it is to tell the contributions that had the most impact, changed the way people think about mathematics. Those would be the greatest. I would say the identification of what it means to be provable and the Gödel completeness and incompletes theorems are great in that way. Gödel's incompleteness theorem states that the method of proof which he showed captures logical implication is insufficient to axiomatize the basic facts about number theory. That's a real advance and it changed the way people thought about the mathematical enterprise. A 19th Century mathematician might think, "We know what the proofs are and we should be able to find the correct set of axioms on which to base all of mathematics." This mindset was completely gone by the middle, certainly by the end, of the 20th Century.

Giving a clear definition of "computable function" was another great achievement. With that available, we can prove theorems about algorithmic solvability and

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unsolvability within a mathematical setting.

W: In set theory, I think Gödel's isolation or identification of the constructible universe is certainly one of the greatest achievements in set theory in the 20th Century. Cohen's discovery of forcing was also one of the great achievements in set theory of the last century. I think the validation and understanding of projective determinacy is also an important advance, but even among set theorists it is debated whether the axiom should be regarded as true.

S: Professor Woodin would have to be modest about this since he was involved in that work. I do think logicians looking back 50 years from now would see it as something great.

I: There were many scientists and mathematicians who started off as logicians, like Norbert Wiener, John von Neumann.

S: And Saunders MacLane.

I: Any advice for graduate students in mathematics?

S: There are two things you have to do when you are a graduate student. First, read papers (a lot of them), work out the details, learn the body of mathematics, and learn as much as possible of applied mathematics, physics and biology. Then, identify a field of interest and learn it in depth. It's very important to choose the questions to study. You should have the big questions in mind. The smaller questions you work on should have a bearing on the big ones. Secondly, and maybe this should have been primarily, you have to choose the right advisor.

I: What are your hobbies when you are not doing mathematics?

S: The richest part of my life is the company of my family. Mathematics is second to that. There is no candidate for third place. Third place is always going up for rent: running marathons, fixing my house, programming my computer, doing administrative work at the university.

W: Sleeping.