Mathematical Conversations

Takeyuki Hida: Brownian Motion, White Noise >>>

Takeyuki Hida

Interview of Takeyuki Hida by Y.K. Leong

Takeyuki Hida is well-known for his pioneering work in establishing and developing a new field in probability theory - the field of white noise analysis, which has now found numerous applications outside probability in quantum dynamics and biology, and within mathematics itself in differential equations and geometry.

His doctoral thesis sowed the seeds of a new type of differential and integral calculus (now called the Hida calculus) for Gaussian processes in terms of the time derivative of Brownian motion. This was developed further in his analysis of generalized white noise functionals, first proposed in his Carleton University lectures in 1975. In the decades that followed, he initiated a program of investigations into functionals of general noise, in particular Poisson noise, and the analysis of random complex systems. He has also applied his results to provide new approaches to Feynman (Lagrangian) path integrals and the Chern-Simons action integral and to problems in mathematical biology.

Born in Okazaki in Aichi Prefecture, Hida had his undergraduate education at Nagoya University and obtained his PhD from Kyoto University under the official supervision of the distinguished probabilist Kiyosi Itô, the founder of stochastic analysis. Immediately after he obtained his B.Sc., he taught for 7 years in a teachers' college, Aichi Gakugei University. Subsequently, he taught at Kyoto University (Yoshida College) for 5 years before joining Nagoya University as Professor of Mathematics. On his official retirement in 1991, he was bestowed the title of Professor Emeritus by Nagoya University and he also took up a professorship at Meijo University. Since 2000, he holds a Special Professorship position at Meijo.

At an age when others would be content to bask in past achievements, he continues to collaborate with an active team of researchers in Nagoya University in pushing the frontiers of his discipline further afield into the scientific unknown. Hida's research output includes more than 130 research papers and 7 books. He has been invited to leading universities and major scientific meetings in the west. He has served as Dean of Science at Nagoya University and as Dean of Science and Technology at Meijo University. He was Chairman of the Committee of Conference, Stochastic Processes and Applications. For his scientific contributions, he was awarded the Chunichi Cultural Prize in 1980 and made an Aoi Citizen of the city of Okazaki. More recently, in 2007, he was awarded the Zuihou Jyuukou Shoh (瑞宝重光 章), one of the highly prestigious awards in Japan.

Hida's connections with NUS go back to 1981 when he was an invited speaker at the International Mathematical Conference organized by the Department of Mathematics, NUS. Since then he has maintained close ties with mathematicians in NUS through personal visits to Singapore and official invitations. He was invited by the Institute and the Department of Mathematics to give colloquium lectures and seminars on white noise analysis in April 2007. It was during this occasion that *Imprints* had the opportunity to interview him on 2 April 2007.

The following is an edited and enhanced version of the transcript of the interview in which he traces the emergence from research isolation in a teachers' college to international prominence in the world of probability and leads us through the excitement of a newly emerging field that is as profoundly abstract as it is diversely applicable.

Imprints: You had your undergraduate education at Nagoya University shortly after the war at a comparatively late age. How did it affect your studies?

Takeyuki Hida: Not quite at a late age because the educational system in Japan at that time was different from the present. Usually you graduate at 23, starting from elementary school to junior high school, high school and university. In my case, 24 - so not much difference. However, compared to other people, I was delayed for almost two years.

I: You taught at Aichi Gakugei University for 7 years before going to Kyoto University where you got PhD. Was there any

Continued on page 11

Continued from page 10

particular reason for this comparatively long time gap?

H: Of course, I had wished to go to graduate school sooner. But when I finished my undergraduate studies, I immediately took up a job. I had to work because I came from a farmer's family and I had brothers and sisters. It was financially difficult for me to continue with my graduate studies. I had a responsibility to my family. Aichi Gakugei University is actually a teachers' college. I only had to teach and I had heavy teaching duties. I didn't have much time to do research. I corresponded with Professor Itô, of course. At that time, he had already left for Kyoto University but I was able to visit him sometimes. In addition, I communicated with Professor Paul Lévy who was in France. The reason was that as an undergraduate I was taught by Professor Itô, and the textbook used was Lévy's book [*Processus stochastiques et movement brownien*], starting from Chapter 6 on Brownian motion. So I felt very close to Professor Lévy, and also I wanted to hear from him - his comments and suggestions, etc. I was then teaching at Aichi Gakugei University.

I: Did Professor Itô suggest a topic for your research?

H: Not in particular. In my last undergraduate year, he suggested that I read about Brownian motion from the book by Paul Lévy.

I: Your PhD topic was on Gaussian processes. Why did you choose to work on this topic? Did Kiyoshi Itô have any influence on your choice?

H: He did, and so did Professor Lévy. The main reason is that I started studying Brownian motion from Lévy's book. Brownian motion is very important and more basic than the Gaussian process. Professor Itô was my supervisor in my last year as undergraduate. At the same time that I graduated as an undergraduate, he moved to Kyoto. There was no direct communication afterwards, but sometimes I did go to Kyoto, but not often. We usually communicated by mail. With Professor Itô in Kyoto, how could I continue my studies? I had wished to be directed by him. Once I asked him what action I should take. Then he replied that I should follow the pioneers' work - pioneers, he said, like Kolmogorov, Feller, Lévy. He didn't include himself but I add: I understood that a pioneer's work was difficult and not easy to understand, and I found that his [Itô's] paper was interesting to follow. However, he suggested that I should investigate the pioneer's work and find out what the pioneering idea was and that I should do it by myself. I think this is a very nice suggestion.

I: Professor Itô himself also followed the pioneers.

H: I cannot say. I followed his work.

I: Did your PhD work have any influence on your later research?

H: Sure. That was very important. Actually, in 1955, there was a famous paper by Paul Lévy at the Berkeley symposium on the Gaussian process and its close connection with Brownian motion. More precisely, we hope to express any general Gaussian process as a linear function of the Brownian motion - linear meaning some kernel function and some Brownian motion integrated with respect to it. However, Paul Lévy said that the representation is not unique. If there is a way to determine uniqueness, then the combination of kernel function and Brownian motion preserves everything about the Gaussian process. I was very impressed by this idea. I tried to prove the general theory. Unfortunately [it does] not always exist, not even for the single Brownian motion. We may need many Brownian motions that are independent. However, the important part is that the basic process is Brownian motion and the general Gaussian process is expressed as a linear function of Brownian motion. I was thinking about the uniqueness of the representation. In order to characterize the Gaussian process, if there is unique representation, one can say everything about the kernel function and the Brownian motion. For many other representations, it is impossible to say so. There is a very simple example with different kind of representation. If the kernel function is taken to be polynomial, the Gaussian process has more than two different expressions. I was thinking, and the answer was eventually very simple, but the meaning is very deep. That was the idea of the canonical representation. It took me a year to come to a better understanding of this canonical representation. I tried to find out the meaning, I computed an example and finally got the definition of the canonical representation and the uniqueness theorem and characterization. So the basic part of Gaussian theory is solved.

Given a Gaussian process satisfying some conditions, we can construct a Brownian motion which leads to the canonical representation. The Gaussian process and Brownian motion have the same information - otherwise there is no meaning at all. Given a stochastic process, it would be fine, if we can find some nice basic processes (independent increment process like additive process) such that any process can be expressed as a function of those independent increment processes. Suppose we find the required properties of the process, then we can combine a (non-linear) kernel function and the basic processes just obtained. The Gaussian case is a very particular case and is, in some sense, elementary. We should generalize. We now come to the general idea. Given a general random complex system, we can find independent (increment) systems and non-linear functions

Continued on page 12

Continued from page 11

to express the random complex phenomenon in question. We can think of the general theory of the analysis of those random systems. This is a very big problem. In some sense, it is vague. But still, I wish to extend from the Gaussian process to the general random complex system, including stochastic processes and random fields.

I: Is this a kind of program you are proposing?

H: Yes. This is the original idea of white noise analysis. This is, of course, a very big program. We can do it step by step. First, the basic process is an independent increment process, or I would say, white noise, time-derivative of the Brownian motion. For that purpose, we need some detailed analysis where the functional is non-linear. I was not dreaming. Actual program was there. That is what I wrote in the Carleton Lecture Notes Series in 1975. Two years ago, we celebrated its 30th anniversary. Many people have referred to the notes. It is included at the beginning of the publication of my selected papers.

I: Has this got to do with your calculus (the Hida calculus)?

H: Yes. That's the origin of it. At the beginning of 1975, at the request of Don Dawson of Carleton University, I gave a series of lectures in the summer, mainly for researchers and some graduate students.

I: You taught at Kyoto University for only 5 years.

H: There is an unusual story behind this. Because I followed the suggestion of Itô in following a pioneer (Paul Lévy's direction), I communicated with Itô and Lévy when I was in Aichi Gakugei University. Prof Itô suggested to Prof Akizuki, who was then director of the department of mathematics of Kyoto University, to invite me to Kyoto - that was very unusual because Kyoto University is a very prestigious university and I was only an instructor in a teachers' college which was of not so high a standard. I was extremely happy and honored about it, and because of this, my studies very much accelerated.

I: You were in Nagoya University for a long time until you retired. Were you very strongly attached to your prefecture?

H: Nagoya University knew and appreciated what I was doing in Kyoto, and was kind in inviting me as a professor. I consulted with Prof Itô and he agreed that I should move to Nagoya. That was why I moved to Nagoya University. I worked there for 27 years. My hometown is actually Okazaki, not Nagoya.

I: Nagoya University has two parts, one is the School of General Education and the other School of Science, isn't it?

H: Formally, I spent 2 years in the Mathematics Department of School of Education. Two years later, I formally moved to the School of Science. Nagoya University is very famous in science. There are seven national universities which are very prestigious in Japan - Tokyo, Kyoto, and so on. Nagoya is one of them, and in fact, the youngest.

I: Was there much work done on white noise analysis before 1960 (the beginning of your second research period on white noise analysis)?

H: 1960 was the year of my thesis dealing with Gaussian processes. It is difficult to say [it is] the origin of white noise analysis. White noise theory is, in a sense, a generalization of the study of Gaussian processes. For 5 years I studied very hard to understand the meaning of canonical representation using the methods of stochastic analysis. My thesis came from the work of those 5 years.

I: Could you give us a brief idea of white noise?

H: Take the time derivative of Brownian motion, $\dot{B}(t)$. It is an independent basic system, and the structure is linear. I wish to come to the nonlinear case. That is the first idea. The second step is Paul Lévy's proposal in 1937 in his famous book *Addition des variables aléatoires* (*Addition of random variables*) to have integration. There he took a discrete time random process X_n. Suppose we know all the information of X_n until time n, then at the next step X_{n+1} is a function of the known value plus independent variable. That was the innovation theory for a discrete time series. Then in 1953, Lévy wrote a booklet published by the University of California, Berkeley, in which he proposed to think of the innovation for a continuous time parameter stochastic process $X(t)$ - he didn't say differential equations - but variation of $X(t)$,

$$\delta X(t) = \phi(X(\tau), \tau \leq t, Y = \dot{B}, t, dt).$$

To obtain δX, of course, depends on the parameters t and dt - that are nonrandom and not of much interest. The important part is that within time dt, the stochastic process brings new information, and this new information is independent of the past. He gives a formal expression for the stochastic process, and it serves, in a sense, as a generalization of the classical stochastic differential equation. He asked for a general expression theoretically. I was very much impressed by this idea. So we should find an independent system that will serve to express a given complex random system. The basic

Continued on page 13

Continued from page 12

variables are independent, the function is a nonrandom function. So we can combine two mathematical theories. One is a system of independent random variables, and the other is functional analysis. We can then establish a general theory for random systems.

One more motivation comes from Norbert Wiener's approach. In 1940, he wrote about time series and proposed prediction theory. Linear prediction is rather easy but nonlinear prediction is difficult and more important. Sometime later, in 1958, Wiener wrote a famous book on nonlinear problems in random theory. There he discussed many applications of the nonlinear function of the white noise $\dot{B}$. I was inspired, I would say, by this approach. I thought I should study nonlinear functions of $\dot{B}$, not the Brownian motion B, and even more we should establish calculus. So we have to introduce partial derivatives and integration with respect to the variable $\dot{B}$.

In 1967/8, I was invited to the mathematics department of Princeton University by William Feller. He had listened to my Berkeley lecture and agreed very much with what I was thinking and with my plan of research. At Princeton I was able to give a graduate course and undergraduate course too. Feller appreciated my white noise approach. He had published the third edition of a famous book on probability theory (Volume 1) and at the time I was there. Volume 2 was published a little bit earlier. In the preface of his book, he wrote that he wished to make probability theory a part of pure mathematics. Many people think that probability theory is about gambling and related topics. Feller was not willing to do it in that way. He said that probability theory should be one of the branches of pure mathematics with good connections with analysis. In the weekends, I often met him and he was fascinated by the picture of the chromosome. He was very much interested in applications of probability theory to biology. Two viewpoints are important: one is that it is part of pure mathematics and the other is that we should have good contact with applications to discover good problems in mathematics. The important thing is that though we are applying the theory to applications, it is not quite applied mathematics. We should investigate concrete problems and if we are lucky, we can discover mathematics in the applications. Even though you are studying biology, you are not a biologist - you are a mathematician. We should try to find mathematical theory in biology. That was the way of Feller's research so far as applied mathematics is concerned. I was very much impressed by his idea.

I: Were you surprised by all those connections with conformal groups and geometry?

H: Yes. And it is quite natural. I have not yet obtained good results so far in group theory or good connections with Lie groups. Only for the simple case like $SO(\infty)$. I observed those groups through white noise. The basic part is conformal invariance. That was in Princeton, 1967-68.

Don Dawson agreed very much with what I did and he invited me to give lectures in Carleton University. Sometime later, I visited Carleton again, in 1975, and he asked me to give a series of lectures in summer, mainly for researchers and graduate students. In this case, I started with a function of $\dot{B}$, starting from Paul Lévy's expression. I had dreamed of a new way to study stochastic differential equations, and I thought it would be fine if we could obtain differential equations in the variable, not x, but in $\dot{B}(t)$. I was inspired by Norbert Wiener's nonlinear networks whose input is white noise, and output is a non-linear function of white noise. How do you identify the unknown nonlinear network device in between input known and output known too?

I: What are some of the most important applications of white noise analysis?

H: Physical applications - the most important one is the Feynman path integral and related topics in quantum dynamics. Feynman proposed functional integration. Starting from the Lagrangian, he wished to introduce infinite-dimensional integration to obtain the propagator in quantum dynamics. I think we can imagine the original idea in Dirac's book [*Principles of Quantum Mechanics*, 1930]. Of course, Feynman improved it very much so that we can do more. However, the general idea was proposed at that time. It is known that a constant trajectory is determined uniquely by the Lagrangian. In quantum mechanics, the trajectory fluctuates and the fluctuation is expressed as a Brownian bridge - we are suggested that description from Dirac. I have discussed with Streit (German physicist) and he agreed with me. The problem is that the velocity and kinetic energy of the particle are white noise functionals, and we have to perform integration and establish a calculus.

I: What about applications to biology? For example, could one view the "junk DNA" in the human genome as some kind of white noise at the molecular level?

H: A friend of mine called Naka who used to be in New York University Medical Center had applied white noise to identify the action of the retina of catfish, which is simple compared to that of other animals. Naka was clever enough to consider the non-linear part of the action. He did some very complicated computations, but unfortunately he passed away last year after returning to Japan. It's a very sad story. Much of his work, however, can be seen in the literatures by him and his colleagues.

Continued on page 14

Continued from page 13

I: Are there people continuing his research work?

H: I hope so. Another application is in the study of polymers.

Biologists study the mechanism underlying what happens when a polymer is cut. They conduct a lot of experiments and computer simulations. I have a good friend in Kyoto who proposed some principles and compared it with actual phenomena.

Another friend of mine (Oosawa) is studying the movement of paramecium with the help of some kind of differential equations, and there is involved a fluctuation which is some kind of white noise. There was a conference on biocomputation in Italy organized by a friend of mine, Ricciardi. He asked me to give a lecture on white noise. Biologists understood that many non-linear fluctuations can be expressed in terms of white noise.

I: Do biologists understand the mathematics?

H: Honestly speaking, I don't describe everything systematically in advanced mathematics and I don't include everything in the lecture. There are many applications in biology which are interesting. I think a systematic approach will achieve something good. I was once heavily involved in the work on polymers and I even wrote a paper on it with Okada and Kiho. But I don't have any other mathematical results.

I: What about other applications?

H: Let me summarize the applications within mathematics - not in probability, but outside of probability. One is in fractional functional analysis. Fréchet, Lévy and others discussed non-linear functions, $\mathcal{L}^2$ functions essentially, not random, however, from the viewpoint of infinite-dimensional calculus. We can see very intimate connections between white noise analysis and classical functional analysis. Second is in harmonic analysis. There is a duality between $\dot{B}$ and $\dot{P}$ which is Poisson noise. I have recommended people to discuss the infinite-dimensional rotation group O_∞, symmetric group $S(\infty)$ and their subgroups to see their roles in infinite-dimensional analysis.

I: What are the future directions of white noise analysis?

H: I would answer in the following way. Many people are more interested in Gaussian noise, but Poisson noise is also interesting. In the linear additive process, the noise can be decomposed into two parts, Gaussian and compound Poisson. They can be discussed separately. One may think that Poisson noise can be similarly treated to Gaussian, but I claim that dissimilarity is more important. There should also be a duality. To the Gaussian case, we can associate the infinite-dimensional rotation group. To the Poisson case, my colleague Si Si has associated the infinite symmetric group. The future direction is to discuss the duality between the Poisson and Gaussian cases in terms of harmonic analysis arising from groups.

Another direction is to work on the foundations of white noise analysis. There is mathematical beauty to be found regarding invariance, optimality, symmetry, duality and others, which should be investigated. There are also connections with other fields - in quantum dynamics, quantum information theory (quantum probability), molecular biology. For the last field, we are still at the stage of case-by-case study. Random fields should also be investigated, hopefully, in line with white noise theory. An application is to the Tomonaga-Schwinger equations - our group at Nagoya has kept up the interest. Once I had a conversation with David Mumford, the very famous algebraic geometer, and found that he now has an interest in probability. Random fields appear on his homepage. We should revisit the ideas of Lévy and Itô.

I have organized Lévy seminars in Nagoya; last year's was our fifth Lévy seminar, and we proposed a new program for his ideas.