SOLUTIONS OF THE PROBLEMS AND RIDERS

PROPOSED

IN THE SENATE-HOUSE EXAMINATION

FOR 1854.

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FOR 1854.

BY

THE MODERATORS AND EXAMINERS.

Cambridge (Eng.) - University.

WITH AN APPENDIX

CONTAINING THE EXAMINATION PAPERS IN FULL.

"It is good to vary and intermingle asking of questions with telling of opinions."
BACON.

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PREFACE.

THE Moderators and Examiners have been induced to publish the present volume, mainly on the following account.

The value of a problem frequently depends in great measure upon its illustrating clearly some general principle or exemplifying some analytical process; and thus a solution, which is as it were forced out, and which misses the method designed, is worth little in point of the instruction it affords.

It is hardly possible for any but the framers of the questions to produce a complete series of solutions, shewing the method which they wished the student to pursue.

In the present instance the writers have availed themselves of their opportunities of inspecting the answers returned by the candidates for honours, and have appended to their own solutions some of the more striking of those which were submitted to them.

Cambridge, Oct. 1854.

SOLUTIONS OF SENATE-HOUSE PROBLEMS AND RIDERS

FOR THE YEAR EIGHTEEN HUNDRED AND FIFTY-FOUR.

THURSDAY, *Jan.* 5, 1854. 1 *to* 4.

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1. *ABD*, *ACE* are two straight lines touching a circle in *B* and *C*, and, if *DE* be joined, *DE* is equal to *BD* and *CE* together; shew that *DE* touches the circle.

If *DE*, fig. (1), be not a tangent, from *D* draw *DFG* to touch the circle. Then, since (*Euc.* I. 47) *BD* is equal to *DF*, and *CG* to *GF*; therefore, *BD* and *CG* are together equal to *DG*. But *BD* and *CE* are together equal to *DE*. Therefore the difference between *DG* and *DE* is equal to the difference between *CG* and *CE*, which is *EG*: that is two sides of the triangle *DEG* are equal to the third, which is impossible; therefore no line except *DE* can be drawn from *D* to touch the circle; therefore *DE* touches it.

Direct Proof.—Let *O*, fig. (2), be the centre of the circle. Make *DF* equal to *DE*. Join *OB*, *OC*, *OD*, *OE*, *OF*, and draw *OG* perpendicular to *DE*.

Since DF is equal to DE, therefore DF is equal to DB and EC together; therefore BF is equal to EC; and $OB = OC$ and $\angle OBF = OCE$, therefore $OF = OE$; therefore, in the triangles DOF, DOE, DO, OF, DF are equal to DO, OE, DE respectively, therefore $\angle ODF = ODE$; again, in the triangles ODB, ODG, OD is common, and the angles ODB, OBD, are equal to ODG, OGD, respectively, therefore $OB = OG$; therefore the circle passes through G, and, since OG is perpendicular to DE, DE touches it.

2. O, A, B, C, are four points arranged in order in a straight line, so that OA, OB, OC, form an harmonic progression. Prove that, A and C being stationary, if O move towards A, B will also move towards A.

$$\frac{2}{OB} = \frac{1}{OA} + \frac{1}{OC},$$

$$\frac{1}{OB} - \frac{1}{OC} = \frac{1}{OA} - \frac{1}{OB},$$

$$\frac{BC}{OC} = \frac{AB}{OA}, \text{ see fig. (3)},$$

$$\frac{AC - AB}{OA + AC} = \frac{AB}{AO},$$

$$(AC - AB) \,.\, OA = (OA + AC) \,.\, AB,$$

$$\frac{1}{AB} - \frac{1}{AC} = \frac{1}{AC} + \frac{1}{OA},$$

$$\frac{1}{AB} = \frac{1}{OA} + \frac{2}{AC}.$$

If then OA decreases, AB also decreases.

3. If a, b, c be positive integers, and $a^{\frac{2}{b}}$, $b^{\frac{1}{ac}}$, $c^{\frac{2}{b}}$ be in geometrical progression, shew that $a^{\frac{2}{b^n}}$, $b^{\frac{1}{a^nc^n}}$, $c^{\frac{2}{b^n}}$ are also in geometrical progression.

Since $a^{\frac{2}{b}}$, $b^{\frac{1}{ac}}$, $c^{\frac{2}{b}}$ are in geometrical progression,

$$a^{\frac{2}{b}}c^{\frac{2}{b}} = \left(b^{\frac{1}{ac}}\right)^2,$$

$$(ac)^{\frac{1}{b}} = b^{\frac{1}{ac}},$$

$$(ac)^{ac} = b^b,$$

$$ac = b,$$

$$(ac)^{\frac{2}{b^n}} = (b)^{\frac{2}{a^nc^n}},$$

therefore, $a^{\frac{2}{b^n}}$, $b^{\frac{1}{a^nc^n}}$, $c^{\frac{2}{b^n}}$, are in geometrical progression.

4. If either of the two quantities $1 + 3^m$, $1 + 3^{m+4r}$, is a multiple of 10, prove that the other is also a multiple of 10, m and r being positive integers.

Assuming that $\dfrac{1 + 3^{m+4r}}{10}$ is integral, it is evident that the following quantities also are integral:

$$\frac{3^{m+4r} - 3^2 + 10}{10},$$

$$\frac{3^2\,(3^{m+4r-2} - 1)}{10},$$

$$\frac{(10-1)\,(3^{m+4r-2} - 1)}{10},$$

$$\frac{3^{m+4r-2} - 1}{10},$$

$$\frac{3^{m+4r-2} + 3^2 - 10}{10},$$

$$\frac{3^2\,(3^{m+4r-4} + 1)}{10},$$

$$\frac{(10-1)\,(3^{m+4r-4} + 1)}{10},$$

$$\frac{3^{m+4r-4} + 1}{10}.$$

Proceeding in the same way, we see that $\dfrac{3^m+1}{10}$ is integral.

The reasoning here given, taken backwards, shews that, if $\dfrac{3^m+1}{10}$ is integral, $\dfrac{3^{m+4r}+1}{10}$ also is integral.

The following is a somewhat different solution of the same problem.

Suppose 3^m+1 divisible by 10; then 3^m must have 9 for its last digit. Now $3^4=81$: hence 3^{4r} has 1 for its last digit. Hence $3^m\times 3^{4r}$ has 9 for its last digit, and therefore $1+3^m\times 3^{4r}$ is divisible by 10.

Suppose $3^{m+4r}+1$ divisible by 10: then 3^{m+4r} has 9 for its last digit: therefore 3^m must have 9 for its last digit; for otherwise $3^m\times 3^{4r}$ would not have 9 for its last digit.

Hence 3^m+1 is divisible by 10.

5. Find the value of $\tan\alpha$ or $\tan\beta$ from the equations

$$\tan(\alpha+\beta)=\tan\alpha\cot\beta+\cot\alpha\tan\beta,$$
$$\tan(\alpha-\beta)=\tan\alpha\cot\beta-\cot\alpha\tan\beta.$$

Adding together the two equations, we get

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=2\tan\alpha\cot\beta,$$

$$2\tan\alpha\cdot\frac{1+\tan^2\beta}{1-\tan^2\alpha\tan^2\beta}=2\tan\alpha\cot\beta,$$

or $$\tan\beta(1+\tan^2\beta)=1-\tan^2\alpha\tan^2\beta\ldots\ldots\ldots\ldots(1).$$

By symmetry, $$\tan\alpha(1+\tan^2\alpha)=1-\tan^2\beta\tan^2\alpha\ldots\ldots\ldots\ldots(2).$$

From (1) and (2), $$\tan\alpha-\tan\beta+\tan^3\alpha-\tan^3\beta=0,$$

$$(\tan\alpha-\tan\beta)\,.\,\{1+\tan^2\alpha+\tan\alpha\tan\beta+\tan^2\beta\}=0,$$

$$(\tan\alpha-\tan\beta)\,.\,\{(\tan\alpha+\tfrac{1}{2}\tan\beta)^2+1+\tfrac{3}{4}\tan^2\beta\}=0,$$

and therefore, since the second factor cannot be zero,

$$\tan\alpha=\tan\beta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (2) and (3),

$$\tan\alpha\,(1+\tan^2\alpha) = 1 - \tan^4\alpha,$$

$$\tan^2\alpha + \tan\alpha + \tfrac{1}{4} = \tfrac{5}{4},$$

$$\tan\alpha = \frac{\pm\sqrt{5}-1}{2} = \tan\beta.$$

6. If $A + B + C = 90°$, shew that the least value of $\tan^2 A + \tan^2 B + \tan^2 C$ is 1.

$$0 = \cot(A+B+C) = \frac{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}{\tan A + \tan B + \tan C - \tan A\tan B\tan C},$$

therefore $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1.$

But since $\tan^2 A + \tan^2 B = (\tan A - \tan B)^2 + 2\tan A\tan B,$

$$\tan^2 B + \tan^2 C = (\tan B - \tan C)^2 + 2\tan B\tan C,$$

$$\tan^2 C + \tan^2 A = (\tan C - \tan A)^2 + 2\tan C\tan A\,;$$

therefore $\tan^2 A + \tan^2 B + \tan^2 C = 1 + \tfrac{1}{2}\{(\tan A - \tan B)^2 + (\tan B - \tan C)^2 + (\tan C - \tan A)^2\}$;

therefore $\tan^2 A + \tan^2 B + \tan^2 C$ is not < 1.

7. Lines, drawn through Y, Z, at right angles to the major axis of an ellipse, cut the circles, of which SP, HP, are diameters, in I, J, respectively. Prove that IS, JH, BC, produced indefinitely, intersect each other in a single point.

Let IY, JZ, fig. (4), produced if necessary, intersect the major axis in Y', Z', respectively: then

$$\angle\, SIY' = \text{supplement of } SIY,$$

$$= \angle\, SPY = \angle\, HPZ$$

$$= \angle\, HJZ';$$

$$\angle\, SY'I = \text{right angle} = \angle\, HZJ,$$

and therefore $\angle\, ISY' = \angle\, JHZ'$;

whence the triangle formed by producing IS, HJ, is isosceles, and therefore, CS, CH, being equal, the vertex of the triangle must lie in BC produced.

Since the angles SIY', HJZ', are equal respectively to the angles SPY, HPZ, they can never be zero, and therefore SI, HJ, can never be perpendicular to the major axis. Thus the point of intersection of IS, JH, BC, can never move off to an infinite distance from C.

8. From any point T, (fig. 5), two tangents are drawn to a given ellipse, the points of contact being Q, Q': CQ, CQ', QQ', CT, are joined; V is the intersection of QQ', CT. Prove that the area of the rectilinear triangle QCQ' varies inversely as

$$\left(\frac{CV}{TV}\right)^{\frac{1}{2}} + \left(\frac{TV}{CV}\right)^{\frac{1}{2}}.$$

Draw CK at right angles to QQ'. Then

$$\begin{aligned}(\text{area } QCQ')^2 &= QV^2 . CK^2 \\ &= \frac{CD^2}{CP^2} . (CP^2 - CV^2) . CK^2 \\ &= \frac{AC^2 . BC^2}{CP^2 . PE^2} . (CP^2 - CV^2) . CK^2.\end{aligned}$$

But $\quad CP^2 - CV^2 = CT.CV - CV^2 = CV . TV.$

Also $\dfrac{CK^2}{CP^2 . PE^2} = \dfrac{1}{CP^2} \cdot \dfrac{CV^2}{CP^2} = \dfrac{CV^2}{CT^2 . CV^2} = \dfrac{1}{CT^2} = \dfrac{1}{(CV+TV)^2}.$

Hence $(\text{area of } QCQ')^2 = \dfrac{AC^2 . BC^2 . CV . TV}{(CV+TV)^2}$,

and therefore area of $QCQ' = \dfrac{AC . BC}{\left(\frac{CV}{TV}\right)^{\frac{1}{2}} + \left(\frac{TV}{CV}\right)^{\frac{1}{2}}}$,

or area of QCQ' varies inversely as

$$\left(\frac{CV}{TV}\right)^{\frac{1}{2}} + \left(\frac{TV}{CV}\right)^{\frac{1}{2}}.$$

9. A piece of uniform wire is bent into three sides of a square $ABCD$, of which the side AD is wanting; shew that, if it be hung up by the two points A and B successively, the angle between the two positions of BC is $\tan^{-1} 18$.

Let EF, fig. (6), be drawn parallel to BA, through E the middle point of BC. Then, if G be the centre of gravity of the piece of wire, EG equals two-thirds of BE.

Draw HG parallel to BC, and join AG, BG.

When the wire is hung up by A, AG will be vertical, and when hung up by B, BG will be vertical; therefore the inclinations of BC to the vertical will be equal to the angles which BC makes with AG and BG. Therefore the angle between the two positions of BC, (supposing it to be kept in the same plane,) will be the angle between AG and BG.

Now $\tan AGB = \tan(AGH + HGB)$

$$= \frac{\frac{4}{3} + \frac{2}{3}}{1 - \frac{4}{3} \cdot \frac{2}{3}} = 18;$$

therefore the angle between the two positions of BC is $\tan^{-1} 18$.

10. A weight of given magnitude moves along the circumference of a circle, in which are fixed also two other weights: prove that the locus of the centre of gravity of the three weights is a circle. If the immoveable weights be varied in magnitude, their sum being constant, prove that the corresponding circular loci intercept equal portions of the chord joining the two immoveable weights.

Let R, fig. (7), be the moveable weight, P and Q the stationary ones. Let G be the centre of gravity of P and Q, H that of P, Q, R.

Then
$$GH = \frac{R \cdot GR}{P + Q + R}$$
$$\propto GR.$$

But the locus of R is a circle; hence that of H is a circle, G being a similar point in the two circles, and GR, GH, similar lines. Hence, if HP', HQ', be drawn parallel to RP, RQ, P' and Q' will be points in the locus of H.

Also $$P'Q' : PQ :: GH : GR,$$

and therefore, $GH : GR$ being constant, and PQ being constant, $P'Q'$ is constant.

11. A ball of elasticity e is projected from a point in an inclined plane, and, after once impinging upon the inclined plane, rebounds to its point of projection: prove that, α being the inclination of the inclined plane to the horizon, and β that of the direction of projection to the inclined plane,

$$\cot\alpha \,.\, \cot\beta = 1 + e.$$

Let V be the velocity of projection.

This is equivalent to $V\sin\beta$ and $V\cos\beta$ respectively perpendicular and parallel to the plane.

Also the force of gravity is equivalent to $g\cos\alpha$ and $g\sin\alpha$, perpendicular and parallel to the plane.

Consider the motion perpendicular to the plane. The time of flight = twice the time in which the velocity $V\sin\beta$ can be generated by the force $g\cos\alpha$

$$= 2\,\frac{V\sin\beta}{g\cos\alpha};$$

after rebounding, the velocity perpendicular to the plane is $eV\sin\beta$,

therefore time of returning to the point of projection

$$= 2\,\frac{eV\sin\beta}{g\cos\alpha};$$

therefore whole time of flight

$$= 2\,\frac{V\sin\beta}{g\cos\alpha}(1+e).$$

Again, the motion parallel to the plane is not affected by the impact, therefore whole time = twice the time in which the velocity $V \cos\beta$ can be generated by the force $g \sin\alpha$

$$= 2\,\frac{V\cos\beta}{g\sin\alpha}.$$

Therefore, equating these expressions,

$$2\,\frac{V\cos\beta}{g\sin\alpha} = 2\,\frac{V\sin\beta}{g\cos\alpha}\,.\,(1+e),$$

therefore $$\cot\alpha\,\cot\beta = 1 + e.$$

[The student may gain instruction by endeavouring to draw a correct figure.]

12. Two heavy bodies are projected from the same point at the same instant in the same direction, with different velocities; find the direction of the line joining them at any subsequent time.

By the second law of motion, the positions of the bodies at any time after their projection will be the same as if they moved for that time unaffected by gravity, and then fell from rest, from the positions they had reached, for the same time.

Now after the first part of the motion, each will be in the common line of projection; and after the second part of the motion, since they fall through equal and parallel spaces, the line joining them will be parallel to the line joining them before they fell, that is, to the line of projection. Therefore, in the actual motion, the line joining them will be always parallel to the line of projection.

13. Three equal and perfectly elastic balls A, B, C move with equal velocities towards the same point, in directions equally inclined to each other; suppose first, that they impinge upon each other at the same instant; secondly, that B and C impinge on each other, and immediately afterwards simultaneously on A; and thirdly, that B and C impinge simultaneously on A just before touching each other; and let $V_1 V_2 V_3$ be the velocities of A after impact on these suppositions respectively: shew that

$$V_2 = \tfrac{1}{5}V_1, \text{ and that } V_3 = \tfrac{7}{5}V_1.$$

Let V be the velocity of each ball before impact. Let A, B, C, fig. (8), be the centres of the balls at the instant when they impinge, O the point towards which they are all moving. Join AO and produce it to D; AOD is perpendicular to BC.

In the first case, by symmetry, the three balls are reduced to rest at the same instant, and since the forces of restitution are equal to those of compression, and act in the same directions, the velocities generated are equal to those destroyed; therefore each ball has the same velocity after impact as it had before; therefore

$$V_1 = V.$$

In the second case, B and C impinge first on each other; their velocity parallel to DA is therefore unchanged, while their velocities perpendicular to DA are reversed. Now, before striking C, B was moving in the direction BO; therefore, after striking C, B moves with the velocity V in a direction perpendicular to AB; it has therefore no velocity in the direction of the normal AB.

Let R be the whole impulsive force between A and B, measured as an accelerating impulsive force, R' the force of compression, [it is convenient to measure them as accelerating forces, the balls being equal]: then the velocity of A at the time of greatest compression is

$$V - 2R' \cos 30° = V - R'\sqrt{3};$$

therefore the velocity of A, resolved along the normal AB, is

$$(V - R'\sqrt{3}) \cos 30° = \frac{\sqrt{3}}{2} V - \tfrac{3}{2} R';$$

and the velocity of B, resolved along the same line AB, is R'; but, at the time of greatest compression, the normal velocities are equal, therefore

$$\frac{\sqrt{3}}{2} V - \tfrac{3}{2} R' = R',$$

therefore $$R' = \frac{\sqrt{3}}{5} V; \text{ and } R = 2R' = \frac{2\sqrt{3}}{5} V;$$

therefore the velocity of A, after the impact is completed, is

$$V - 2R\cos 30^\circ = V - R\sqrt{3} = V - \tfrac{6}{5}V = -\tfrac{1}{5}V;$$

therefore $$V_2 = \tfrac{1}{5}V_1.$$

In the third case, B and C impinge on A just before striking each other. Let then R be the whole accelerating impulsive force between A and B, R' the force of compression; then, as before, the normal velocity of A at the time of greatest compression

$$= \frac{\sqrt{3}}{2}V - \tfrac{3}{2}R'.$$

Also the normal velocity of B before impact $= V\cos 30^\circ = \dfrac{\sqrt{3}}{2}V$,

therefore the normal velocity of B at the time of greatest compression

$$= R' - \frac{\sqrt{3}}{2}V;$$

therefore, equating these normal velocities,

$$\frac{\sqrt{3}}{2}V - \tfrac{3}{2}R' = R' - \frac{\sqrt{3}}{2}V;$$

therefore $$R' = \frac{2\sqrt{3}}{5}V; \text{ and } R = 2R' = \frac{4\sqrt{3}}{5}V;$$

therefore the velocity of A after impact $= V - 2R\cos 30^\circ$

$$= V - R\sqrt{3} = V - \tfrac{12}{5}V = -\tfrac{7}{5}V;$$

therefore $$V_3 = \tfrac{7}{5}V_1.$$

14. CP, CD, are two conjugate semidiameters of an ellipse described by a body about a centre of force in the focus S: PP', DD', chords of the ellipse parallel to the major axis. Prove that, α, α', β, β', being the angular velocities of the body about S at P, P', D, D', respectively,

$$\frac{1}{(\alpha\alpha')^{\frac{1}{4}}} + \frac{1}{(\beta\beta')^{\frac{1}{4}}} = \text{a constant quantity.}$$

We know that $$SP \,.\, SP' = CD^2.$$

But $$\alpha = \frac{h}{SP^2}, \quad \alpha' = \frac{h}{SP'^2};$$

hence $$\frac{h}{(\alpha\alpha')^{\frac{1}{2}}} = CD^2.$$

Similarly $$\frac{h}{(\beta\beta')^{\frac{1}{2}}} = CP^2.$$

Hence $$\frac{1}{(\alpha\alpha')^{\frac{1}{2}}} + \frac{1}{(\beta\beta')^{\frac{1}{2}}} = \frac{CP^2 + CD^2}{h}$$

$$= \frac{CA^2 + CB^2}{h}$$

$$= \text{a constant quantity.}$$

15. Supposing the velocity of a body in a given elliptic orbit to be the same at a certain point, whether it describe the orbit in a time t about one focus, or in a time t' about the other, prove that, $2a$ being the major axis, the focal distances of the point are equal to

$$\frac{2at'}{t+t'}, \quad \frac{2at}{t+t'}.$$

Let S, H be the foci, $\frac{1}{2}h$, $\frac{1}{2}h'$ the areas described in a unit of time when t, t' are the periodic times respectively. Let SP, HP be the focal distances of the point where the velocity is V in each case; SY, HZ, perpendiculars on the tangent at P.

Then, since the whole area described in the times t, t' is the same,

$$h : h' :: t' : t,$$

$$SY.V : HZ.V :: t' : t,$$

$$SY : HZ :: t' : t,$$

$$SP : HP :: t' : t,$$

$$SP : 2AC :: t' : t + t',$$

$$SP = \frac{t'}{t+t'} . 2AC,$$

$$HP = \frac{t}{t+t'} . 2AC.$$

16. Three candles are placed in a room, and the two shorter being lighted throw shadows of the third upon the ceiling; if the directions of these shadows be produced, where will they meet?

The shadow of any straight line, caused by a luminous point, is in the plane passing through the luminous point and the line. Therefore the two shadows on the ceiling are the intersections of the ceiling by the two planes passing through the longer candle and the two flames respectively; the shadows if produced will meet in the line in which these two planes meet, that is, in the point when the direction of the longer candle meets the ceiling, that is, the point directly over the longer candle.

17. Within a reflecting circle on the same side of the centre are two parallel rays, one dividing the circumference into arcs which are as 3 to 1, the other dividing it into arcs which are as 8 to 1; find the least value of n such that, after each ray has suffered n reflections, they may be again parallel.

Let AB (fig. 9) be the original direction of the first ray, BC its direction after one reflection; the deviation of the ray

$$= \pi - \angle ABC = \angle AOB.$$

Now the arc ADB = three times the arc AEB, therefore AEB is one fourth of the whole circumference; therefore the angle $AOB = \frac{2\pi}{4}$.

And since the deviation at each successive reflection is always the same, the deviation after n reflections $= n.\frac{1}{2}\pi$.

Similarly, for the other ray, the deviation at each reflection

$$= \frac{2\pi}{8+1} = \frac{2\pi}{9};$$

therefore the deviation after n reflections $= \frac{2n\pi}{9}$.

Now after n reflections the rays are parallel to each other; therefore the deviation of one must exceed the deviation of the other by some multiple of two right angles; therefore

$$\frac{n}{2}\pi - \frac{2n}{9}\pi = p\pi,$$

$$5n = 18p,$$

and, since p is an integer, the least value of n is 18.

18. One asymptote of an hyperbola lies in the surface of a fluid: find the depth of the centre of pressure of the area included between the immersed asymptote, the curve, and two given horizontal lines in the plane of the hyperbola.

Let $BB'C'C$ (fig. 10) be the included area. Draw $P_{\prime}M_{\prime}$, horizontally, equidistantly from BB', CC'. Take any two strips PM, $P'M'$, of equal breadths, and equidistant from $P_{\prime}M_{\prime}$. Then, τ denoting the breadth,

$$\begin{aligned} \text{Pressure on } PM &= \tau . PM . OM . \sin\alpha \\ &= \tfrac{1}{4}\tau\,(a^2 + b^2)\sin\alpha \\ &= \text{pressure on } P'M'. \end{aligned}$$

Hence PM, $P'M'$, balance about $P_{\prime}M_{\prime}$. Similarly for all like pairs of strips. Hence the centre of pressure of $BB'C'C$ lies in the line $P_{\prime}M_{\prime}$.

19. A cone is totally immersed in a fluid, the depth of the centre of its base being given. Prove that, P, P', P'', being the resultant pressures on its convex surface, when the sines of the inclination of its axis to the horizon are s, s', s'', respectively,

$$P^2(s' - s'') + P'^2(s'' - s) + P''^2(s - s') = 0.$$

Let R = the resultant pressure on the whole surface of the cone, the base included; P = the resultant pressure on the convex surface, when the axis is inclined at an angle α to the horizon; B = the pressure on the base; h = the altitude of the cone; k = the depth of the centre of its base; r = the radius of its base; σ = the density of the fluid.

Then $\qquad P^2 = R^2 - 2B.R.\sin\alpha + B^2.$

Now $\qquad R = \frac{1}{3}\sigma\pi r^2 h, \quad \text{and } B = \sigma\pi r^2 k;$

hence $\qquad P^2 = \frac{1}{9}\sigma^2\pi^2 r^4 (h^2 - 6hks + 9k^2).$

Similarly, $\qquad P'^2 = \frac{1}{9}\sigma^2\pi^2 r^4 (h^2 - 6hks' + 9k^2),$

$$P''^2 = \frac{1}{9}\sigma^2\pi^2 r^4 (h^2 - 6hks'' + 9k^2).$$

Multiplying these three equations in order by $s' - s''$, $s'' - s$, $s - s'$, and adding, we have

$$P^2 (s' - s'') + P'^2 (s'' - s) + P''^2 (s - s') = 0.$$

20. Light emanating from a luminous circular disk, placed horizontally on the ceiling of a room, passes through a rectangular aperture in the floor: ascertain the form and area of the luminous patch on the floor of the room below.

Shew that neither the shape nor the area of the patch will be affected by any movement of the disk along the ceiling.

Let O (fig. 11) be the centre of the disk, M any point in its circumference. Through P, any point in a side of the aperture $ABCD$, draw OPO' to meet the floor of the lower room in O'. Draw MP and produce it to M', a point in the floor. With O' as centre and radius $O'M'$ describe a circle on the floor. This circle will be the area illuminated by the rays which pass through the point P.

Again, lines drawn from O through A, B, C, D, to meet the floor will form a rectangle $A'B'C'D'$ on the floor.

The form of the patch is therefore such as represented in (fig. 12).

Let $AB = a$, $BC = b$, r = radius of disk,

$$A'B' = a', \quad B'C' = b', \quad r' = O'M',$$

and let h, h', denote the heights of the higher and lower rooms.

Then $r' = r.\dfrac{h'}{h}, \quad a' = a.\dfrac{h+h'}{h}, \quad b' = b.\dfrac{h+h'}{h}.$

Then area of patch $= \pi r'^2 + a'b' + 2(a'+b')r'$

$$= \frac{1}{h^2}\{\pi r^2 h'^2 + ab(h+h')^2 + 2rh'(h+h')(a+b)\}.$$

This result shews that the form and area of the patch are independent of the position of the disk on the ceiling of the upper room.

21. If c_1, c_2, c_3, be the lengths of the meridian shadows of three equal vertical gnomons, on the same day, at three different places on the same meridian, prove that the latitudes $\lambda_1, \lambda_2, \lambda_3$, of the places are connected together by the equation

$$c_1 \cdot \frac{(c_2 - c_3)^2}{\tan(\lambda_2 - \lambda_3)} + c_2 \cdot \frac{(c_3 - c_1)^2}{\tan(\lambda_3 - \lambda_1)} + c_3 \cdot \frac{(c_1 - c_2)^2}{\tan(\lambda_1 - \lambda_2)} = 0.$$

Let x = the altitude of the gnomon, δ = the sun's declination when on the meridian.

Then $\dfrac{c_2}{x}$ = tan (sun's zenith distance when on the meridian),

or $$\frac{c_2}{x} = \tan(\lambda_2 - \delta).$$

Similarly, $\dfrac{c_3}{x} = \tan(\lambda_3 - \delta)$.

Hence $$\lambda_3 - \lambda_2 = \tan^{-1}\frac{c_3}{x} - \tan^{-1}\frac{c_2}{x} = \tan^{-1}\left\{\frac{c_3 - c_2}{\dfrac{c_2 c_3}{x} + x}\right\},$$

and therefore $$\frac{c_2 c_3}{x} + x = (c_3 - c_2)\cot(\lambda_3 - \lambda_2) \;\ldots\ldots\ldots\ldots (1).$$

Similarly $$\frac{c_3 c_1}{x} + x = (c_1 - c_3)\cot(\lambda_1 - \lambda_3) \;\ldots\ldots\ldots\ldots (2),$$

$$\frac{c_1 c_2}{x} + x = (c_2 - c_1)\cot(\lambda_2 - \lambda_1) \;\ldots\ldots\ldots\ldots (3).$$

Multiplying (1), (2), (3), in order by $c_1(c_3 - c_2)$, $c_2(c_1 - c_3)$, $c_3(c_2 - c_1)$, adding, and observing that

$$(c_3 - c_2) + (c_1 - c_3) + (c_2 - c_1) = 0,$$

and $$c_1(c_3 - c_2) + c_2(c_1 - c_3) + c_3(c_2 - c_1) = 0,$$

we have

$$0 = c_1(c_3 - c_2)^2\cot(\lambda_3 - \lambda_2) + c_2(c_1 - c_3)^2\cot(\lambda_1 - \lambda_3) + c_3(c_2 - c_1)^2\cot(\lambda_2 - \lambda_1),$$

or $$c_1 \cdot \frac{(c_2 - c_3)^2}{\tan(\lambda_2 - \lambda_3)} + c_2 \cdot \frac{(c_3 - c_1)^2}{\tan(\lambda_3 - \lambda_1)} + c_3 \cdot \frac{(c_1 - c_2)^2}{\tan(\lambda_1 - \lambda_2)} = 0.$$

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1. IF $\overset{m}{\underset{s}{C}}$ denote generally the number of combinations of m things s together, and $\overset{m}{\underset{0}{C}}$ be taken to denote unity for all values of m; prove that, if

$$\underset{r}{S} = 1^r.\overset{n}{\underset{r}{C}} + 2^{r-1}.\overset{n-1}{\underset{r-1}{C}} + 3^{r-2}.\overset{n-2}{\underset{r-2}{C}} + 4^{r-3}.\overset{n-3}{\underset{r-3}{C}} + \ldots\ldots + r^1.\overset{n-r+1}{\underset{1}{C}} + \overset{n-r}{\underset{0}{C}},$$

then

$$\underset{0}{S} + \underset{1}{S} + \underset{2}{S} + \underset{3}{S} + \ldots + \underset{n}{S} = 1^{n+1} + 2^n + 3^{n-1} + \ldots + (n-1)^3 + n^2 + (n+1)^1.$$

From the expression for $\underset{r}{S}$ we see that

$$\underset{0}{S} = 1,$$

$$\underset{1}{S} = \overset{n}{\underset{1}{C}} + 1,$$

$$\underset{2}{S} = \overset{n}{\underset{2}{C}} + 2^1.\overset{n-1}{\underset{1}{C}} + 1,$$

$$\underset{3}{S} = \overset{n}{\underset{3}{C}} + 2^2.\overset{n-1}{\underset{2}{C}} + 3\,.\overset{n-2}{\underset{1}{C}} + 1,$$

....

....

$$\underset{n}{S} = \overset{n}{\underset{n}{C}} + 2^{n-1}.\overset{n-1}{\underset{n-1}{C}} + 3^{n-2}.\overset{n-2}{\underset{n-2}{C}} + 4^{n-3}.\overset{n-3}{\underset{n-3}{C}} + \ldots\ldots + n^1.\overset{1}{\underset{1}{C}} + 1.$$

But $$1 + \overset{n}{\underset{1}{C}} + \overset{n}{\underset{2}{C}} + \overset{n}{\underset{3}{C}} + \ldots\ldots + \overset{n}{\underset{n}{C}} = (1+1)^n = 2^n,$$

$$1 + 2^1.\overset{n-1}{\underset{1}{C}} + 2^2.\overset{n-1}{\underset{2}{C}} + \ldots\ldots + 2^{n-1}.\overset{n-1}{\underset{n-1}{C}} = (1+2)^{n-1} = 3^{n-1},$$

....

....

$$1 + (n-1)^1.\overset{2}{\underset{1}{C}} + (n-1)^2.\overset{2}{\underset{2}{C}} = \{1 + (n-1)\}^2 = n^2,$$

$$1 + n^1.\overset{1}{\underset{1}{C}} = (n+1)^1,$$

$$1 = 1^{n+1}.$$

Hence

$$\underset{0}{S} + \underset{1}{S} + \underset{2}{S} + \underset{3}{S} + \ldots + \underset{n}{S} = 1^{n+1} + 2^n + 3^{n-1} + \ldots + (n-1)^3 + n^2 + (n+1)^1.$$

2. Straight lines $A\alpha$, $B\beta$, $C\gamma$, (fig. 13), are drawn from the angular points A, B, C, of a triangle to bisect the opposite sides in α, β, γ, O being the point of intersection of the three lines. If the radii of the circles inscribed in the triangles $BO\alpha$, $CO\alpha$; $CO\beta$, $AO\beta$; $AO\gamma$, $BO\gamma$; be represented by a_β, a_γ; b_γ, b_α; c_α, c_β; respectively; prove that

$$\frac{1}{a_\beta} - \frac{1}{a_\gamma} + \frac{1}{b_\gamma} - \frac{1}{b_\alpha} + \frac{1}{c_\alpha} - \frac{1}{c_\beta} = 0.$$

We see that, u denoting the area of the triangle ABC, and h, k, l, the distances AO, BO, CO,

$$\tfrac{1}{2}a_\beta (B\alpha + O\alpha + BO) = \text{area } BO\alpha = \tfrac{1}{6}\text{ area } BAC,$$

$$a_\beta (B\alpha + \tfrac{1}{2}h + k) = \tfrac{1}{3}u:$$

similarly, $$a_\gamma (C\alpha + \tfrac{1}{2}h + l) = \tfrac{1}{3}u.$$

Hence $$k - l = \frac{u}{3}\cdot\left(\frac{1}{a_\beta} - \frac{1}{a_\gamma}\right).$$

By similarity, $$l - h = \frac{u}{3}\cdot\left(\frac{1}{b_\gamma} - \frac{1}{b_\alpha}\right),$$

and $$h - k = \frac{u}{3} \cdot \left(\frac{1}{c_\alpha} - \frac{1}{c_\beta}\right).$$

Hence $$\frac{1}{a_\beta} - \frac{1}{a_\gamma} + \frac{1}{b_\gamma} - \frac{1}{b_\alpha} + \frac{1}{c_\alpha} - \frac{1}{c_\beta} = 0.$$

3. P is a point in a branch of an hyperbola, P' a point in a branch of its conjugate, CP, CP', being conjugate semi-diameters. If S, S', be the interior foci of the two branches, prove that
$$S'P' - SP = AC - BC.$$

Draw PN, $P'N'$, (fig. 14), to meet CA, CB, produced, at right angles. Let $CA = a$, $CB = b$, $CN = x$, $CN' = x'$.

$$SP = ex - a.$$
$$\begin{aligned} S'P' &= e'x' - b \\ &= \frac{(a^2+b^2)^{\frac{1}{2}}}{b} \cdot x' - b \\ &= \frac{(a^2+b^2)^{\frac{1}{2}}}{b} \cdot b \cdot \frac{x}{a} - b \\ &= \frac{(a^2+b^2)^{\frac{1}{2}}}{a} \cdot x - b \\ &= ex - b. \end{aligned}$$

Hence $$S'P' - SP = a - b.$$

✓ 4. On any chord of a parabola as diameter is described a circle cutting the parabola again in two points; if these points be joined, shew that the portion of the axis of the parabola included between the two chords is equal to its latus rectum.

Let $y^2 = mx$ be the equation to the parabola;

$(x_1 y_1)$, $(x_2 y_2)$ coordinates of the ends of the given chord,
$(x_3 y_3)$, $(x_4 y_4)$ .... other ....

Since y_1, y_2 are the roots of the equation
$$y^2 - (y_1 + y_2)\,y + y_1 y_2 = 0,$$

and since x_1y_1, and also x_2y_2 satisfy the equation

$$y^2 = mx,$$

therefore they satisfy the equation

$$mx - (y_1 + y_2)\,y + y_1y_2 = 0:$$

this is therefore the equation to the given chord.

Similarly, the equation to the other chord is

$$mx - (y_3 + y_4)\,y + y_3y_4 = 0.$$

Putting 0 for y in each of these equations, and subtracting the resulting values of x, we find the portion of the axis intercepted between the chords equal to $\dfrac{y_3y_4 - y_1y_2}{m}$.

Now the equation to the circle, of which the given chord is a diameter, is

$$\left(x - \frac{x_1 + x_2}{2}\right)^2 + \left(y - \frac{y_1 + y_2}{2}\right)^2 = \left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2,$$

or $\quad x^2 - (x_1 + x_2)\,x + y^2 - (y_1 + y_2)\,y + x_1x_2 + y_1y_2 = 0.$

Combining this with the equation to the parabola, by eliminating x, we have

$$\frac{y^4}{m^2} - (x_1 + x_2)\,\frac{y^2}{m} + y^2 - (y_1 + y_2)\,y + \frac{y_1^2y_2^2}{m^2} + y_1y_2 = 0,$$

an equation whose roots are y_1, y_2, y_3, y_4;

therefore $\qquad y_1y_2y_3y_4 = y_1^2y_2^2 + m^2y_1y_2;$

therefore $\qquad y_3y_4 = y_1y_2 + m^2;$

therefore the portion of axis intercepted is equal to

$$\frac{y_1y_2 + m^2 - y_1y_2}{m} = m.$$

Otherwise. Let a, b be the coordinates of the middle point of the given chord; then the equation to the circle may be written

$$x^2 + y^2 - 2ax - 2by + f = 0,$$

and the equations to the chords will be

$$x - a - m(y-b) = 0, \qquad x + m'y - c = 0.$$

Now the equation to any conic section possessing these same chords must be of the form

$$k(x^2 + y^2 - 2ax - 2by + f) = \{x - a - m(y-b)\}.\{x + m'y - c\};$$

let this conic section coincide with the parabola $y^2 = lx$; then by comparison of the coefficients of like terms, we have

$$k = 1, \quad m = m', \quad -2bk = mc + m'(mb - a),$$
$$(k + mm')\,l = 2ak + mb - a - c;$$

therefore $$c + mb - a = -\frac{2b}{m} \quad\text{.................. (1)},$$

and $$(1 + m^2)\,l = mb + a - c$$
$$= 2mb + \frac{2b}{m} \quad \text{by (1)},$$
$$= (1 + m^2)\,\frac{2b}{m};$$

therefore $$l = \frac{2b}{m} = a - mb - c;$$

therefore the latus rectum is equal to the portion of the axis included between the chords.

5. If $r = f(\theta)$ and $y = f\left(\frac{x}{a}\right)$ be the equations to two curves, $f(\theta)$ being a function which vanishes for the values θ_1, θ_2, and is positive for all values between these limits, and if A be the area of the former between the limits

$$\theta = \theta_1, \quad \theta = \theta_2,$$

and M be the arithmetic mean of all transverse sections of the solid generated by the revolution, about the axis of x, of the portion of the latter curve between the limits $x = a\theta_1$, $x = a\theta_2$; shew that

$$M = \frac{2\pi}{\theta_2 \sim \theta_1} A.$$

To find an expression for M, conceive the portion of the axis $a\theta_2 - a\theta_1$, (taking θ_2 to represent the greater of the two angles θ_1, θ_2) to be divided into n equal parts, each equal to δx, so that

$$a\theta_2 - a\theta_1 = n\delta x;$$

and through each point of division draw a plane making a transverse section of the solid: then

$$\text{the arithmetic mean of these sections} = \frac{\Sigma\pi y^2}{n}$$

$$= \frac{\Sigma\pi y^2\delta x}{a\theta_2 - a\theta_1}:$$

therefore, when the number of sections is increased indefinitely,

$$\text{the arithmetic mean of all transverse sections} = \frac{\pi\int_{a\theta_1}^{a\theta_2} y^2 dx}{a(\theta_2 - \theta_1)}$$

$$= \frac{\pi\int_{a\theta_1}^{a\theta_2}\left\{f\left(\frac{x}{a}\right)\right\}^2 dx}{a(\theta_2 - \theta_1)}$$

$$= \frac{\pi\int_{\theta_1}^{\theta_2}\{f(\theta)\}^2 d\theta}{\theta_2 - \theta_1}, \qquad \text{writing } \theta \text{ for } \frac{x}{a};$$

and
$$A = \int_{\theta_1}^{\theta_2}\frac{r^2}{2}d\theta = \tfrac{1}{2}\int_{\theta_1}^{\theta_2}\{f(\theta)\}^2 d\theta;$$

therefore
$$\frac{M}{A} = \frac{2\pi}{\theta_2 - \theta_1}.$$

6. A brick is divided by a plane, passing through one corner, and making an angle of 45° with the length of the brick; find the position of this plane in which the two parts are the most nearly equal.

Let O (fig. 15) be the given corner, OC the direction of the length of the brick, $OA = a$, $OB = b$. The brick is supposed to be transparent, so that O, which is the furthest corner, may be seen.

Let $OFGE$ be the cutting plane;

α, β the angles AOE, BOF respectively.

Describe a spherical surface with O as centre, meeting the planes AC, BC, FE in the arcs hl, lk, kh respectively: then $hl=\alpha$, $lk=\beta$, $hlk=\frac{1}{2}\pi$; and by the given condition the arc drawn from l perpendicular to $hk=\frac{1}{4}\pi$; therefore

$$1=\tan^2\tfrac{1}{4}\pi\cot^2\alpha+\tan^2\tfrac{1}{4}\pi\cot^2\beta=\cot^2\alpha+\cot^2\beta=p^2+q^2\ldots(1),$$

writing p and q for $\cot\alpha$ and $\cot\beta$.

Again, the equation to the cutting plane referred to OA, OB, OC as axes, is

$$z=x\cot\alpha+y\cot\beta=px+qy;$$

therefore the volume of the part between $OBDA$ and $OFGE$

$$=\int_0^a\int_0^b z\,dy\,dx=\int_0^a\int_0^b(px+qy)\,dy\,dx$$

$$=p\frac{a^2b}{2}+q\frac{ab^2}{2}$$

$$=\frac{ab}{2}(ap+bq).$$

And since this is always the smaller part, the two will be most nearly equal when this part is the greatest possible;

therefore $\qquad 0=adp+bdq.$

And from (1) $\qquad 0=pdp+qdq;$

therefore $\qquad \dfrac{p}{a}=\dfrac{q}{b}=\dfrac{1}{\sqrt{(a^2+b^2)}}=\dfrac{ap+bq}{a^2+b^2};$

therefore the volume $AOBG=\dfrac{ab}{2}\sqrt{(a^2+b^2)}$,

and $\qquad DG=ap+bq=\sqrt{(a^2+b^2)}=OD;$

therefore the angle $DOG=\frac{1}{4}\pi$;

therefore $COG=\frac{1}{4}\pi=$ inclination of OC to $OFGE$;

therefore the plane COG is perpendicular to the plane FE, which defines the position of the cutting plane.

It may be shewn further that

$$AE = \frac{a^2}{\sqrt{(a^2+b^2)}}, \quad BF = \frac{b^2}{\sqrt{(a^2+b^2)}}.$$

7. If r, r', be the radii of curvature of an involute and evolute at corresponding points (x, y), (x', y'); prove that

$$rdx' \pm r'dy = 0, \quad rdy' \mp r'dx = 0;$$

and shew that, the involute being an ellipse of which the semi-axes are a, b, the greatest value of $\frac{r'}{r}$ is equal to

$$\frac{3}{2}\left(\frac{a}{b} - \frac{b}{a}\right).$$

We know that $$\frac{dx'}{dy'} = -\frac{dy}{dx} \quad\text{......................} (1),$$

and therefore, by differentiation,

$$\frac{d^2x'.dy' - d^2y'.dx'}{dy'^2} = -\frac{dx\,d^2y - dy\,d^2x}{dx^2}.$$

Hence $$\frac{1}{r'^2} = \frac{(dx'd^2y' - dy'd^2x')^2}{(dx'^2+dy'^2)^3} = \frac{(dx\,d^2y - dy\,d^2x)^2}{(dx'^2+dy'^2)^3} \cdot \frac{dy'^4}{dx^4}.$$

But $$\frac{1}{r^2} = \frac{(dx\,d^2y - dy\,d^2x)^2}{(dx^2+dy^2)^3}.$$

Hence $$\frac{r^2}{r'^2} = \left(\frac{dx^2+dy^2}{dx'^2+dy'^2}\right)^3 \cdot \frac{dy'^4}{dx^4} = \frac{dx^2}{dy'^2}, \quad \text{by (1)},$$

and therefore $$r\,dy' \mp r'\,dx = 0;$$

whence also $$r\,dx' \pm r'\,dy = 0.$$

These relations follow also directly from the formulæ

$$r = \frac{ds}{d\psi}, \quad r' = \frac{ds'}{d\psi},$$

$$\frac{dx'}{ds'} = \cos\psi = \pm\frac{dy}{ds},$$

$$\frac{dy'}{ds'} = \sin\psi = \mp\frac{dx}{ds},$$

where ψ represents the inclination of the tangent of the evolute to the axis of x.

In the case of the ellipse, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, we know that $y'=-\dfrac{a^2-b^2}{b^4}y^3$; hence

$$\frac{dy'}{dx}=-3\frac{a^2-b^2}{b^4}.y^2.\frac{dy}{dx}=3\frac{a^2-b^2}{b^4}.y^2.\frac{b^2x}{a^2y}=3\frac{a^2-b^2}{a^2b^2}.x.y.$$

Hence $$\frac{r'}{r}=3\frac{a^2-b^2}{a^2b^2}xy.$$

Now xy is a maximum, under the condition $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, when $x=\dfrac{a}{\sqrt{2}}$, $y=\dfrac{b}{\sqrt{2}}$: hence $\dfrac{r'}{r}$ has a maximum value equal to

$$\frac{3}{2}\left(\frac{a}{b}-\frac{b}{a}\right).$$

8. Trace the curve whose equation is

$$y^2=\frac{x^4-c^4}{x(x-a)},$$

first supposing a to be less than c, then equal, then greater; and shew how the three forms of the curve pass into each other, when the value of a is supposed to increase gradually through the value c.

To find the asymptotes.

$$x=0 \text{ and } x=a \text{ each make } y=\infty;$$

therefore the lines $x=0$ and $x-a=0$ are asymptotes.

Also, $$y^2=x^2\left(1-\frac{c^4}{x^4}\right)\left(1-\frac{a}{x}\right)^{-1},$$

therefore $$y=\pm x\left(1-\frac{c^4}{x^4}\right)^{\frac{1}{2}}\left(1-\frac{a}{x}\right)^{-\frac{1}{2}}$$

$$=\pm x\left(1-\frac{1}{2}\frac{c^4}{x^4}+\&c.\right)\left(1+\frac{1}{2}\frac{a}{x}+\frac{1}{8}\frac{a^2}{x^2}+\&c.\right)$$

$$=\pm\left(x+\frac{a}{2}+\frac{a^2}{8x}+\&c.\right);$$

therefore $y = \pm\left(x + \frac{a}{2}\right)$ is a pair of oblique asymptotes, and if we consider points far enough from the origin, the asymptotes lie between the curve and the axis of x.

First take $a < c$.

When $x = \infty$,	$y = \pm x$, possible.*
$x = c$,	$y = 0$, impossible.
$x = a$,	$y = \infty$, possible.
$x = 0$,	$y = \infty$, impossible.
$x = -c$,	$y = 0$, possible.
$x = -\infty$,	$y = \pm x$;

therefore the form of the curve is that given in fig. (16), where $OA = a$, $OC = c$, $OD = c$, $OB = \frac{1}{2}a$.

Secondly, let $a = c$.

We may say that the equation

$$y^2 = \frac{x^4 - c^4}{x(x - c)}$$

degenerates into the form

$$y^2 = \frac{(x^2 + c^2)(x + c)}{x},$$

or rather we should say that when $x = c$, y may have any value. Thus the line $x - c = 0$ is part of the locus.

To find the general form of the rest of the locus,

* The notation in the text is used for stating concisely whether the value of y is possible or impossible between particular values.

$$
\begin{array}{lll}
x = \infty & \text{makes} & y = \pm x, \\
 & & \text{possible.} \\
x = c & & y = \pm 2a, \\
 & & \text{possible.} \\
x = 0 & & y = \infty, \\
 & & \text{impossible.} \\
x = -c & & y = 0, \\
 & & \text{possible.} \\
x = -\infty & & y = \pm x.
\end{array}
$$

The form of the curve is given in fig. (17), where $OA = c$, $OD = c$, $OB = \frac{1}{2}c$.

Thirdly, let $a > c$.

$$
\begin{array}{lll}
x = \infty & \text{makes} & y = \pm x, \\
 & & \text{possible.} \\
x = a & & y = \infty, \\
 & & \text{impossible.} \\
x = c & & y = 0, \\
 & & \text{possible.} \\
x = 0 & & y = \infty, \\
 & & \text{impossible.} \\
x = -c & & y = 0, \\
 & & \text{possible.} \\
x = -\infty & & y = \pm x.
\end{array}
$$

The form of the curve is given in fig. (18), where $OA = a$, $OB = \frac{1}{2}a$, $OC = OD = c$.

[It will be easier to draw the curve if we find the points where it cuts its oblique asymptotes. The abscissæ of these points are given by the equation

$$x = -\frac{a}{b} \pm \sqrt{\left(\frac{4c^4}{3a^2} + \frac{a^2}{3b}\right)};$$

and it may be shewn that the negative value is always less than $-c$, except when $a = 2c$, in which case the value of x is $-c$: also that the positive value is greater or less than c

according as c is greater or less than a. Hence there are always, except when $a = 2c$, four real points at which the curve crosses its oblique asymptotes.]

To explain how the form of the curve changes gradually from fig. (16) to fig. (18) as a passes through the value c, we must observe that, when $a = c$, the straight line AE is part of the locus; and that the curved branch cuts AE in F, AF being equal to $2a$: also that when $a = c$, A and C coincide. It appears then, that as A approaches C, the arc CG becomes less curved, and approximates to the straight line AF. Similarly the branch HK becomes less and less curved, and at last coincides with FE. Also as H and G approach F, the two branches LH and GM ultimately unite and the curve assumes the form of fig. (17). It is clear that the curvature at G and at H must increase indefinitely as the curve fig. (16) approaches its limiting form. The above explanation holds for the change from fig. (18) to fig. (17).

9. $SPHQ$ is a quadrilateral, P and Q being points in an ellipse of which S and H are the foci; if Q be fixed while P moves, find the locus of the centre of gravity of the perimeter of the quadrilateral.

Let G_1, fig. (19), be the centre of gravity of SP and PH,
G_2 .... SQ and QH;

then G the centre of gravity of the whole perimeter is the middle point of G_1G_2, and since G_2 is fixed the locus of G will be similar to the locus of G_1, and of half the linear magnitude: also when PCQ is a straight line, G will be at C.

To find the locus of G_1. Join UV the middle points of SP, PH: UV evidently passes through G_1. Again, a perpendicular CP' from C upon the tangent at P also passes through G_1; for if SP be produced to H', so that $PH' = PH$, CP', which is parallel to HH' and bisects SH, will also bisect SH', (*Euc.* VI. 2); therefore a line SH' would balance on CP'; and since PH and PH' are equally inclined to CP', SP and

PH will balance on CP'. Draw G_1N; then if (xy), (x_1y_1) be coordinates of P and G_1,

$$y_1 = \frac{y}{2},\ x_1 = y_1 \tan CG_1N = y_1 \tan (\text{inclination of tangent at } P \text{ to axis of } x)$$

$$= \frac{y}{2}\frac{b^2x}{a^2y} = \frac{b^2}{a^2}\frac{x}{2};$$

$$\therefore\ 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \left\{\frac{x_1}{\frac{b^2}{2a}}\right\}^2 + \left\{\frac{y_1}{\frac{b}{2}}\right\}^2:$$

the equation to an ellipse whose semi-axes are half the semi-latus-rectum, and half the minor axis of the given ellipse; therefore the locus of G is an ellipse whose semi-axes are one-fourth of the semi-axis minor, and one-fourth of the semi-latus-rectum of the given ellipse, passing through C, having its centre on the perpendicular from C upon the tangent at Q, and its major axis perpendicular to the major axis of the given ellipse.

Otherwise. Let $SP = r$, $HP = r'$;

and let (xy), $(x'y')$, $(\bar{x}\bar{y})$ be coordinates of P, Q, G respectively.

$$4a\,\bar{x} = r\,\frac{x+ae}{2} + r'\,\frac{x-ae}{2} + \text{like terms for } Q$$

$$= ax + \tfrac{1}{2}ae(r - r') + \text{like terms}$$

$$= ax(1 - e^2) + \text{like terms};$$

therefore $$4\bar{x} = (1-e^2)(x + x').$$

Again, $$4a\bar{y} = r\frac{y}{2} + r'\frac{y}{2} + \text{like terms}$$

$$= a(y + y');$$

therefore $$\frac{1}{a^2}\left(\frac{4\bar{x}}{1-e^2} - x'\right)^2 + \frac{1}{b^2}(4\bar{y} - y')^2 = 1,$$

the equation to an ellipse.

10. From an external point P two tangents are drawn to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Supposing the locus of the centre of

gravity of the triangle, included between the two tangents and the chord of contact, to be an ellipse $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$, find the equation to the locus of P.

What must be the relation between a, b, a_1, b_1, in order that the locus of P may be an ellipse?

Let h, k, be the coordinates of P, and (x_1, y_1), (x_2, y_2), be the two points of contact. The equation to the chord of contact is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1.$$

When it intersects the ellipse, viz. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have, eliminating y,

$$\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right) x^2 - 2hx + a^2 \left(1 - \frac{k^2}{b^2}\right) = 0;$$

hence
$$x_1 + x_2 = \frac{2h}{\frac{h^2}{a^2} + \frac{k^2}{b^2}};$$

and, similarly,
$$y_1 + y_2 = \frac{2k}{\frac{h^2}{a^2} + \frac{k^2}{b^2}}.$$

Now, x, y, denoting the coordinates of the centre of gravity of the triangle,

$$x = \tfrac{1}{3}(h + x_1 + x_2), \qquad y = \tfrac{1}{3}(k + y_1 + y_2);$$

whence
$$\frac{3x}{h} = 1 + \frac{2}{\frac{h^2}{a^2} + \frac{k^2}{b^2}} = \frac{3y}{k}.$$

Hence $\frac{x}{a'} = \frac{h}{3a'}\left\{1 + \frac{2}{\frac{h^2}{a^2} + \frac{k^2}{b^2}}\right\}$, $\frac{y}{b'} = \frac{k}{3b'}\left\{1 + \frac{2}{\frac{h^2}{a^2} + \frac{k^2}{b^2}}\right\}$.

But, by the hypothesis,

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1;$$

hence the equation to the locus of P is

$$\left(\frac{h^2}{a'^2}+\frac{k^2}{b'^2}\right)\left\{1+\frac{2}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}\right\}^2=9.$$

This equation cannot be reduced to one of the second order unless

$$a' : b' :: a : b;$$

under this condition it will plainly represent an ellipse, its equation being of the form

$$\frac{h^2}{a^2}+\frac{k^2}{b^2}=n^2.$$

COR. Let $a'=a$, $b'=b$: then the equation to the locus of P becomes

$$\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)\left\{1+\frac{2}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}\right\}^2=9.$$

Solving this quadratic equation, we get

$$\frac{h^2}{a^2}+\frac{k^2}{b^2}=4 \text{ or } 1.$$

The former result shews that the locus of P is an ellipse with axes $4a$, $4b$. The latter belongs to the case when the triangle is constantly zero.

11. The radii vectores of any series of points in the path of a particle, moving about a centre of force, being in arithmetical progression, the times of arriving at these points, reckoned from a given epoch, form another arithmetical progression. Find the equation to the path.

By the condition of the problem, it is plain that dt is constant when dr is constant; but, t being some function of r which we may denote by $f(r)$,

$$dt=f'(r)\,dr;$$

hence $f'(r)$ is constant. But

$$r^2\frac{d\theta}{dt}=h.$$

Hence, β denoting some constant,

$$r^2 d\theta = \beta dr,$$

and therefore, α being a constant,

$$r = \frac{\beta}{\alpha - \theta},$$

which is the equation to the curve.

12. In any machine in which two weights P and W are suspended by strings and balance each other in all positions, let P be replaced by a weight Q equal to pP; if in the ensuing motion W and Q move vertically, find the tensions of these strings, neglecting the friction of the machine and the masses of its several parts.

Let $W = mP$, then, by the principle of virtual velocities, P describes a space m times as great as W in the same time; and after P is replaced by Q, Q must describe m times the space described by W in the same time; therefore the whole accelerating force on Q must be m times as great as that on W. Let T, T' be the tensions of the strings to which Q and W are attached, then

$$\frac{Q - T}{\frac{Q}{g}} = m\,\frac{T' - W}{\frac{W}{g}},$$

or
$$1 - \frac{T}{Q} = m\left(\frac{T'}{W} - 1\right);$$

and since the machine has no inertia, the forces which act on it must have the same relation as if it were at rest, (otherwise a finite velocity would be *instantaneously* generated,) therefore

$$\frac{T'}{W} = \frac{T}{P};$$

therefore
$$m + 1 = \frac{T}{P}\left(m + \frac{1}{p}\right),$$

therefore $$\frac{W+P}{W+\frac{1}{p}P}=\frac{T}{P}=\frac{T''}{W}.$$

13. There are generally two directions in which a projectile may be projected with given velocity from a point A, so as to pass through another point B; shew that one of these directions is inclined to the vertical at the same angle that the other is inclined to the line AB. Hence shew that the locus of points, for which a given sight must be used in firing with a given charge of powder, is the surface generated by the revolution, about the vertical, of the path of the bullet obtained by aiming at the zenith with the given sight, and with the given charge of powder.

The former part of this problem is solved in Phear's *Dynamics*, Sect. III. Art. 30, and the latter part follows at once.

Or the latter part may be worked independently as follows:

To find the locus of points, for which the same sight must be used.

Let α be the inclination of the line of the sights to the axis of the barrel; r, θ the polar coordinates of a point for which this sight is adjusted; then, substituting $r\cos\theta$ and $r\sin\theta$ for x and y in the ordinary equation to the path of a projectile, $\theta+\alpha$ being the angle of projection,

$$r\sin\theta = r\cos\theta\,.\tan(\theta+\alpha) - \frac{g}{2V^2}\cdot\frac{(r\cos\theta)^2}{\cos^2(\theta+\alpha)},$$

$$r\{\sin\theta\cos(\theta+\alpha)-\cos\theta.\sin(\theta+\alpha)\}\cos(\theta+\alpha) = -\frac{g}{2V^2}(r\cos\theta)^2$$

$$r\sin\alpha.(\cos\theta\cos\alpha-\sin\theta\sin\alpha)=\frac{g}{2V^2}(r\cos\theta)^2$$

$$x\cot\alpha - y = \frac{g}{2V^2\sin^2\alpha}x^2$$

or $$y = x\cot\alpha - \frac{g}{2V^2\sin^2\alpha}x^2,$$

the equation to the path of a particle projected with velocity V at an angle α to the vertical: that is, if a man, facing the south for instance, aim, with a given sight, at the zenith, the ball, which falls behind him, will pass through all those points to the north of the man, for which the given sight is adapted.

14. A prism whose base is a given regular polygon is surmounted by a regular pyramid whose base coincides with the head of the prism; find the inclination of the faces of the pyramid to its axis in order that the whole solid may contain a given volume with the least possible surface.

Let a be the perpendicular distance of one of the sides of the polygon from its centre; θ the inclination of a face of the pyramid to the axis; x the height of the prism; A the area of polygon; P the perimeter. Then

$$A.(x + \tfrac{1}{3}a \cot\theta) = \text{const.}$$

$$P.(x + \tfrac{1}{2}a \operatorname{cosec}\theta) = \text{min.};$$

$$\therefore\ \tfrac{1}{2}a \operatorname{cosec}\theta + C - \tfrac{1}{3}a \cot\theta = \text{min.}$$

$$\tfrac{1}{2}\cot\theta \operatorname{cosec}\theta - \tfrac{1}{3}\operatorname{cosec}^2\theta = 0;$$

$$\therefore\ \cos\theta = \tfrac{2}{3},$$

which gives the required inclination.

15. An ellipsoid is intersected in the same curve by a variable sphere, and a variable cylinder: the cylinder is always parallel to the least axis of the ellipsoid, and the centre of the sphere is always at one focus of a principal section containing this axis. Prove that the axis of the cylinder is invariable in position, and that the area of its transverse section varies as the surface of the sphere.

Let e, ε, be the eccentricities of the two principal sections through c.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{...........ellipsoid,}$$

$$(x - ae)^2 + y^2 + z^2 = \delta^2 \text{...........sphere;}$$

at their intersection

$$\frac{\delta^2}{c^2} = \frac{(x-ae)^2}{c^2} + \frac{y^2}{c^2} + 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$= \left(\frac{1}{c^2} - \frac{1}{a^2}\right) x^2 + \left(\frac{1}{c^2} - \frac{1}{b^2}\right) y^2 - \frac{2aex}{c^2} + 1 + \frac{a^2e^2}{c^2}$$

$$= \left(\frac{1}{c^2} - \frac{1}{a^2}\right) x^2 + \left(\frac{1}{c^2} - \frac{1}{b^2}\right) y^2 - \frac{2aex}{c^2} + \frac{a^2}{c^2}$$

$$= \frac{e^2}{c^2} x^2 + \frac{\epsilon^2}{c^2} y^2 - \frac{2aex}{c^2} + \frac{a^2}{c^2},$$

$$(ex-a)^2 + \epsilon^2 y^2 = \delta^2,$$

the equation to that cylinder which intersects the ellipsoid in the same curve as the cylinder.

Therefore the axes of all the cylinders coincide with a directrix of the principal section, and the area of a transverse section varies as δ^2 and therefore as the surface of the sphere.

16. An elastic tube of circular bore is placed within a rigid tube of square bore which it exactly fits in its unstretched state, the tubes being of indefinite length; if there be no air between the tubes, and air of any pressure be forced into the elastic tube, shew that this pressure is proportional to the ratio of the part of the elastic tube that is in contact with the rigid tube, to the part that is curved.

Let $ABCD$ (fig. 20) be a section of the rigid tube, $EGHF$ part of the section of the elastic tube: it is clear from symmetry that if E and F be the middle points of the sides AB, AD, the part $EGHF$ is one-fourth of the perimeter of the elastic tube. Also the free portion GH is circular: for the pressure and tension being the same at every point, the radius also must be the same, by the formula $T = pr$. Also, since the pressure is finite the curvature must be finite throughout, so that the sides of the rigid tube, with which the elastic tube coincides for a certain space, must be tangents to the free portions of the elastic tube: the circular arc GH is therefore a quadrant.

Join OG, and draw the radii GK, HKQ.

Let $AB = 2a$, $EOG = \theta$, $p =$ the pressure of the air within the tube.

Consider an annulus of the elastic tube whose breadth is the unit of length; and let T be the tension of this portion, E being the tension required to stretch this annulus to twice its natural length. Then

$$1 + \frac{T}{E} = \frac{\text{stretched length of annulus}}{\text{unstretched length}},$$

$$= \frac{8EG + 4GK\,\frac{1}{2}\pi}{2\pi a} = \frac{\tan\theta + \frac{1}{4}\pi(1 - \tan\theta)}{\frac{1}{4}\pi};$$

$$\therefore\ \frac{T}{E} = \frac{1 - \frac{1}{4}\pi}{\frac{1}{4}\pi}\tan\theta.$$

Again, $$p = \frac{T}{GK} = E\,\frac{1 - \frac{1}{4}\pi}{\frac{1}{4}\pi}\,\frac{\tan\theta}{a(1 - \tan\theta)},$$

$$= \frac{E}{a}\left(1 - \tfrac{1}{4}\pi\right)\frac{8a\tan\theta}{2\pi a(1 - \tan\theta)},$$

$$= \frac{E}{a}\left(1 - \tfrac{1}{4}\pi\right)\frac{\text{length in contact}}{\text{length that is curved}};$$

therefore the pressure of the air in the tube is proportional to the ratio of the part that is in contact to the part that is curved.

17. OA, OB, are any equal arcs of two given great circles of a sphere, intersecting in O. A and B are joined by an arc of a great circle, and also by an arc of a small one described about O. Find the area of the lune included between the two joining arcs.

If $OA = \lambda$ and $\angle AOB = 2\omega$, prove that the lune is greatest when

$$\cos^2\lambda = \frac{\tan\omega - \omega}{\omega\tan^2\omega}.$$

ACB (fig. 21) is the arc of the small circle,

$AC'B$ is the arc of the great circle:

$$\text{area of } ACBO = 2\omega(1 - \cos\lambda),$$

$$\text{area of } AC'BO = 2\omega + 2\psi - \pi,$$

$$u = \text{area of lune} = \pi - 2\omega\cos\lambda - 2\psi.$$

By one of Napier's Rules,

$$\cos\lambda = \cot\omega \,.\, \cot\psi \text{.....................} (1).$$

Hence $\qquad u = \pi - 2\omega \,.\, \cot\omega \,.\, \cot\psi - 2\psi.$

Putting $\dfrac{du}{d\psi} = 0$, we have

$$\frac{du}{d\psi} = 2\omega \,.\, \cot\omega \,.\, \operatorname{cosec}^2\psi - 2 = 0 \text{...........} (2),$$

$$\frac{d^2u}{d\psi^2} = -4\omega \,.\, \cot\omega \,.\, \cot\psi \,.\, \operatorname{cosec}^2\psi.$$

From (2), $\qquad \sin^2\psi = \omega \,.\, \cot\omega\,;$

hence, from (1), $\qquad \cos^2\lambda = \dfrac{\tan\omega - \omega}{\omega \,.\, \tan^2\omega}\,.$

Since $\dfrac{d^2u}{d\psi^2}$ is negative, this result corresponds to a maximum value of u.

18. The ridges of two roofs are at right angles to each other, and the inclination of each roof to the horizon is θ; the shadow of a chimney falling upon them makes angles α and β with their ridges; shew that

$$\cos^2\theta = \cot\alpha \cot\beta.$$

Let $ACDB$ (fig. 22) be one side of the shadow on one roof; through C draw the vertical CE, and through D draw a horizontal plane cutting CE in E, and meeting the roof in DF, which is parallel to the ridge; draw EF perpendicular to DF, and join CF, DE.

Now CFE = inclination of roof to horizon = θ,

CDF = inclination of shadow to ridge = α;

and since CDE is the vertical plane passing through the sun,

EDF is equal to the sun's azimuth measured from the direction of the ridge of the roof, $= \phi$ suppose. Then

$$\cot\alpha = \frac{DF}{CF} = \frac{DF}{FE} \cdot \frac{FE}{CF} = \cot\phi \,.\, \cos\theta.$$

Similarly, $$\cot\beta = \tan\phi \,.\, \cos\theta,$$

$$\therefore\ \cos^2\theta = \cot\alpha \,.\, \cot\beta.$$

19. The hour angles of two stars being ε, ε', and the azimuths α, α', when $\alpha \sim \alpha'$ has for a moment a stationary value; prove that the latitude λ of the place of observation is given by the formula

$$\sin\lambda = \frac{\sin 2\alpha \,.\, \cot\varepsilon - \sin 2\alpha' \,.\, \cot\varepsilon'}{\cos 2\alpha - \cos 2\alpha'}.$$

By spherical trigonometry we have

$$\cot\alpha \,.\, \sin\varepsilon = \sin\lambda \,.\, \cos\varepsilon - \cos\lambda \,.\, \tan\delta,$$

and $$\cot\alpha' \,.\, \sin\varepsilon' = \sin\lambda \,.\, \cos\varepsilon' - \cos\lambda \,.\, \tan\delta'.$$

Differentiating these two equations, α, ε, α', ε', being variables, we get

$$(\cot\alpha \,.\, \cos\varepsilon + \sin\lambda \,.\, \sin\varepsilon)\ d\varepsilon = \operatorname{cosec}^2\alpha \,.\, \sin\varepsilon \,.\, d\alpha,$$

$$(\cot\alpha' \,.\, \cos\varepsilon' + \sin\lambda \,.\, \sin\varepsilon')\ d\varepsilon' = \operatorname{cosec}^2\alpha' \,.\, \sin\varepsilon' \,.\, d\alpha'.$$

But $d\varepsilon = d\varepsilon'$, and, $\alpha \sim \alpha'$ being for an instant stationary, $d\alpha = d\alpha'$: hence

$$\operatorname{cosec}^2\alpha \,.\, \sin\varepsilon \,.\, (\cot\alpha' \cos\varepsilon' + \sin\lambda \sin\varepsilon')$$
$$= \operatorname{cosec}^2\alpha' \,.\, \sin\varepsilon' \,.\, (\cot\alpha \cos\varepsilon + \sin\lambda \sin\varepsilon),$$

$$\sin\varepsilon \,.\, \sin\varepsilon' \,.\, \sin\lambda\, (\cot^2\alpha - \cot^2\alpha')$$
$$= \sin\varepsilon' \,.\, \cos\varepsilon \,.\, \cot\alpha \,.\, \operatorname{cosec}^2\alpha' - \sin\varepsilon \,.\, \cos\varepsilon' \,.\, \cot\alpha' \,.\, \operatorname{cosec}^2\alpha,$$

$$\sin\varepsilon \,.\, \sin\varepsilon' \,.\, \sin\lambda\, (\sin^2\alpha' \cos^2\alpha - \sin^2\alpha \cos^2\alpha')$$
$$= \tfrac{1}{2} \sin\varepsilon' \,.\, \cos\varepsilon \,.\, \sin 2\alpha - \tfrac{1}{2} \sin\varepsilon \,.\, \cos\varepsilon' \,.\, \sin 2\alpha',$$

$$2\sin\lambda \,.\, \sin(\alpha' - \alpha) \,.\, \sin(\alpha' + \alpha) = \sin 2\alpha \,.\, \cot\varepsilon - \sin 2\alpha' \,.\, \cot\varepsilon',$$

$$\sin\lambda = \frac{\sin 2\alpha \,.\, \cot\varepsilon - \sin 2\alpha' \,.\, \cot\varepsilon'}{\cos 2\alpha - \cos 2\alpha'}.$$

20. A thin hollow ring, of which the plane is vertical, and which contains a bead, is placed upon a smooth horizontal plane: prove that the bead, having been placed near the lowest point of the ring, will oscillate isochronously with a perfect pendulum the length of which is equal to

$$\frac{\mu a}{m+\mu},$$

a being the radius of the ring, μ its mass, and m the mass of the bead.

Let C (fig. 23) be the centre of the ring, A its point of contact with the horizontal plane, Ox the rectilinear locus of A, O being a fixed point. From P, the place of the bead, draw PM at right angles to Ox.

Let $OA = x$, $OM = x'$, $PM = y'$, $\angle ACP = \theta$, $R =$ the mutual action between the ring and the bead.

The equations of motion are

$$\mu \frac{d^2x}{dt^2} = R \sin\theta,$$

$$m \frac{d^2x'}{dt^2} = - R \sin\theta,$$

$$m \frac{d^2y'}{dt^2} = R \cos\theta - mg.$$

From the geometry,

$$x' = x + a \sin\theta, \quad y' = a - a \cos\theta.$$

As far as the first order of small quantities,

$$\mu \frac{d^2x}{dt^2} = R\theta, \quad m\left(\frac{d^2x}{dt^2} + a\frac{d^2\theta}{dt^2}\right) = - R\theta, \quad 0 = R - mg.$$

Hence $$\frac{d^2\theta}{dt^2} + \frac{m+\mu}{\mu a} \cdot g\theta = 0.$$

Hence the vibration of P is isochronous with a perfect pendulum of length equal to

$$\frac{\mu a}{m+\mu}.$$

21. A uniform rod, not acted on by any forces, is in motion, its ends being constrained to slide along two fixed rods at right angles to each other in one plane. Prove that, during the whole motion, the wrenching force at any point of the moving rod varies as the product of the distances of the point from the two fixed rods.

Let AB (fig. 24) be the moving rod, O being the intersection of the two fixed rods. Let C be any point in AB. Draw CH, CK, at right angles to OA, OB. Let $AB = 2a$, $AC = 2u$, $BC = 2v$, $\angle BAO = \theta$, $m =$ the mass of AB, $\omega =$ its angular velocity, which will be invariable. The actions and reactions and the wrenching force are indicated in the figure.

Since $\dfrac{d\theta}{dt}$ is equal to a constant quantity ω,

$$\frac{d\cos\theta}{dt} = -\omega\sin\theta, \quad \frac{d^2\cos\theta}{dt^2} = -\omega^2\cos\theta,$$

$$\frac{d\sin\theta}{dt} = \omega\cos\theta, \quad \frac{d^2\sin\theta}{dt^2} = -\omega^2\sin\theta.$$

For the motion of AC, we have

$$\frac{mu}{a}(2a-u)\frac{d^2\cos\theta}{dt^2} = -X,$$

$$\frac{mu}{a}(2a-u)\,\omega^2\cos\theta = X \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{mu}{a}\,u\,\frac{d^2\sin\theta}{dt^2} = S - Y,$$

$$\frac{mu^2}{a}\,\omega^2\sin\theta = Y - S \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

$$\mu = (S+Y)\,u\cos\theta + Xu\sin\theta \;\ldots\ldots\ldots\ldots(3).$$

For the motion of BC,

$$\frac{mv}{a}(2a-v)\frac{d^2\sin\theta}{dt^2} = Y,$$

$$-\frac{m}{a}(a^2-u^2)\,\omega^2\sin\theta = Y \;\ldots\ldots\ldots\ldots\ldots(4).$$

From (2) and (3),

$$\begin{aligned}
\mu &= Xu\sin\theta + 2\,Yu\cos\theta - \frac{mu^3}{a}\,\omega^2\sin\theta\cos\theta \\
&= \frac{m}{a}\,\omega^2 u\sin\theta\cos\theta\,\{u\,(2a-u) - 2\,(a^2-u^2) - u^2\} \\
&= \frac{m}{a}\,\omega^2 u\sin\theta\cos\theta\,.\,2a\,(u-a) \\
&= -\,2m\omega^2.\,u\sin\theta\,.\,(a-u)\cos\theta \\
&= -\,\tfrac{1}{2}m\omega^2.\,2u\sin\theta\,.\,(2a-2u)\cos\theta \\
&= -\,\tfrac{1}{2}m\omega^2.CH.CK \\
&\propto CH.CK.
\end{aligned}$$

WEDNESDAY, *Jan.* 18, 1854. 9½...12½.

1. THERE are n points in space, of which p are in one plane, and there is no other plane which contains more than three of them; how many planes are there, each of which contains three of the points?

Conceive n points such that no plane contains more than three of them; the number of planes, each of which contains three points, being equal to the number of combinations of n things taken three at a time, is equal to

$$\frac{n(n-1)(n-2)}{1.2.3};$$

now if p of these n points be selected, the number of planes, each of which contains three of these points, is

$$\frac{p(p-1)(p-2)}{1.2.3};$$

hence, if these p points move so as to lie in one plane, this one will replace the $\frac{p(p-1)(p-2)}{1.2.3}$ planes last mentioned; the number required is therefore

$$\frac{n(n-1)(n-2)}{1.2.3} - \frac{p(p-1)(p-2)}{1.2.3} + 1.$$

2. A bag contains nine coins, five are sovereigns, the other four are equal to each other in value; find what this value must be, in order that the expectation of receiving two coins at random out of the bag may be worth twenty-four shillings.

Let the value of each of the four coins be x shillings.

The chance of drawing two of the sovereigns

$$=\frac{5.4}{1.2}\div\frac{9.8}{1.2}=\frac{10}{36};$$

the chance of drawing one sovereign and one other coin

$$=5.4\div\frac{9.8}{1.2}=\frac{20}{36};$$

the chance of drawing two of the other coins

$$=\frac{4.3}{1.2}\div\frac{9.8}{1.2}=\frac{6}{36}:$$

therefore the value of the expectation in shillings $=\frac{10}{36}$ of 40 shillings $+\ \frac{20}{36}$ of $(20+x)$ shillings $+\ \frac{6}{36}$ of $2x$ shillings;

therefore $\quad 36\times 24=400+20\,(20+x)+6\times 2x,$

$$864-800=32x,$$

$$x=2;$$

therefore the coin is worth two shillings.

Otherwise. Let x shillings be the value of one of the four coins; then the contents of the bag are worth $(100+4x)$ shillings; therefore one coin at random is worth $\dfrac{100+4x}{9}$ shillings, two coins at random are worth $\frac{2}{9}(100+4x)$ shillings, and so on;

therefore $\quad \frac{2}{9}(100+4x)=24,$

$$25+x=27,$$

$$x=2;$$

therefore the coin is a florin.

3. Having given that u, v, and z are functions of the independent variables x and y, and that one of the equations for determining them is $\dfrac{du}{dx}=v\dfrac{dz}{dx}$; transform this equation into one in which x and z shall be the independent variables.

This is simply an example in the change of the independent variables.

In general, if u be a function of two variables x, y, and x', y' are to be made the independent variables, x, y being given by the equations

$$x = \phi(x', y'), \quad y = f(x', y') \quad \text{..............(1)},$$

we have

$$\left.\begin{aligned} \frac{du}{dx'} &= \left(\frac{du}{dx}\right)\frac{d\phi}{dx'} + \left(\frac{du}{dy}\right)\frac{df}{dx'} \\ \frac{du}{dy'} &= \left(\frac{du}{dx}\right)\frac{d\phi}{dy'} + \left(\frac{du}{dy}\right)\frac{df}{dy'} \end{aligned}\right\} \text{............ (2)},$$

where the brackets indicate differential coefficients formed on the supposition that x, y are the independent variables.

From these equations the values of $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ might be found: but before solving the equations it will be simpler to introduce the peculiar conditions of the problem.

Now x and z are to be two independent variables; therefore ϕ, which denotes the value of x in terms of the new variables, stands for x, and f denotes the function that y is of x, z: hence

$$\frac{d\phi}{dx'} = 1, \quad \frac{d\phi}{dy'} = 0, \quad \frac{df}{dx'} = \frac{dy}{dx}, \quad \frac{df}{dy'} = \frac{dy}{dz}.$$

Substituting these values, equations (2) become

$$\frac{du}{dx} = \left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right)\frac{dy}{dx}, \quad \frac{du}{dz} = \left(\frac{du}{dy}\right)\frac{dy}{dz}.$$

From these equations

$$\left(\frac{du}{dx}\right) = \frac{du}{dx} - \frac{du}{dz}\frac{\frac{dy}{dx}}{\frac{dy}{dz}}.$$

To find the value of $\frac{dz}{dx}$, we must observe that in the last equation $\frac{du}{dx}$ and $\frac{du}{dz}$ are found on the supposition that x and z

are the independent variables; so that if u be a function of z alone, $\frac{du}{dx}$ will be equal to zero. Hence, writing z for u,

$$\left(\frac{dz}{dx}\right) = -\frac{\frac{dy}{dx}}{\frac{dy}{dz}};$$

and substituting these values in the given equations,

$$\frac{du}{dx} - \frac{du}{dz}\frac{\frac{dy}{dx}}{\frac{dy}{dz}} = v\frac{-\frac{dy}{dx}}{\frac{dy}{dz}},$$

$$\frac{du}{dx}\frac{dy}{dz} - \frac{du}{dz}\frac{dy}{dx} + v\frac{dy}{dx} = 0,$$

the form required.

4. (1) Trace the curve whose equation is

$$\tan^2\frac{x}{a} + \tan^2\frac{y}{a} = 1.$$

Let x', y' be the coordinates of a point in the curve; then evidently the values $x = m\pi a + x'$, $y = n\pi a + y'$ satisfy the equation: hence the whole locus consists of portions similar to the portion obtained by taking x and y from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$.

Again, the curve is symmetrical about the axes of x and y; we may therefore ascertain the complete form by considering positive values only of x and y.

Thirdly, $x = 0$ makes $y = \frac{\pi}{4}a$, and y decreases as x increases till $x = \frac{\pi}{4}a$, when $y = 0$; and, if x lie between $\frac{\pi}{4}a$ and $\frac{\pi}{2}a$, y is impossible; therefore the general form is a closed round curve, and the entire locus consists of an infinite number of such round figures, at equal distances, on a series of equidistant lines at right angles to each other.

(2) Trace the curve whose equation is

$$xy(y-x)^2 - ay^3 = a^4.$$

This curve may be conveniently traced as an example of one of the theorems in the fourth question of the paper dated Jan. 18, 1½–4; it will be found so treated further on.

√ 5. Find the value of $\int_0^{\frac{1}{4}\pi} \tan^{-1}\{m\sqrt{(1-\tan^2 x)}\}\,dx$; and shew, either from your result, or from the area of the former of the two curves proposed in the preceding question, that $\int_0^{\frac{1}{4}\pi} \tan^{-1}\sqrt{(1-\tan^2 x)}\,dx$ is equal to $\pi\,(\cdot 17)$ nearly.

Let $f(m) = \int_0^{\frac{1}{4}\pi} \tan^{-1}\{m\sqrt{(1-\tan^2 x)}\}\,dx$,

$$f'(m) = \int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1-\tan^2 x)}}{1+m^2(1-\tan^2 x)}\,dx$$

$$= \int_0^{\frac{1}{4}\pi} \frac{\sqrt{(\operatorname{cosec}^2 x - 2)}.\operatorname{cosec} x.\cot x \operatorname{cosec} x\,dx}{\{(m^2+1)(\operatorname{cosec}^2 x - 2)+1\}\{(\operatorname{cosec}^2 x - 2)+2\}}$$

$$= \int^{\infty} \frac{v\,(v\,dv)}{\{(m^2+1)v^2+1\}\{v^2+2\}}$$

putting v^2 for $\operatorname{cosec}^2 x - 2$,

$$= \frac{1}{2m^2+1}\left\{\int_0^{\infty}\frac{2dv}{v^2+2} - \int_0^{\infty}\frac{dv}{(m^2+1)v^2+1}\right\}$$

$$= \frac{1}{2m^2+1}\left\{\sqrt{2}\tan^{-1}\frac{v}{\sqrt{2}} - \frac{1}{\sqrt{(m^2+1)}}\tan^{-1}v\sqrt{(m^2+1)}\right\}_0^{\infty}$$

$$= \frac{\pi}{\sqrt{2}}\frac{1}{2m^2+1} - \frac{\pi}{2}\frac{1}{\sqrt{(m^2+1)}(2m^2+1)}.$$

Now $f(0) = 0$;

therefore $$f(m) = \frac{\pi}{\sqrt{2}}\int_0^m \frac{dm}{2m^2+1} - \frac{\pi}{2}\int_0^m \frac{dm}{\sqrt{(m^2+1)}(2m^2+1)}$$

$$= \frac{\pi}{2}\tan^{-1}m\sqrt{2} - \frac{\pi}{2}\int_{\sqrt{\left(\frac{1}{m^2}+1\right)}}^{\infty}\frac{dv}{v^2+1}$$

putting v^2 for $\frac{1}{m^2}+1$,

$$= \frac{\pi}{2}\tan^{-1}m\sqrt{2} - \frac{\pi}{2}\left\{\frac{\pi}{2} - \tan^{-1}\sqrt{\left(\frac{1}{m^2}+1\right)}\right\}.$$

From this we obtain

$$\int_0^{\frac{1}{4}\pi} \tan^{-1}\sqrt{(1-\tan^2 x)}\,dx = \pi \tan^{-1}\sqrt{2} - \frac{\pi^2}{4}$$

$$= \pi\left(\tan^{-1}\sqrt{2} - \frac{\pi}{4}\right)$$

$$= \pi \times \text{circular measure of } (9^\circ\ 44')$$

$$= \pi\,(\cdot 1699);$$

a result which is confirmed by considering the curve traced above: for in that curve $\int_0^{\frac{1}{4}\pi} \tan^{-1}\sqrt{(1-\tan^2 x)}\,dx$ is the expression for $\frac{1}{4}$ of the area of one of the round figures; and this area has been shewn to be less than the square on $\frac{\pi}{2}a$: and it may be shewn that the curve lies outside a circle whose radius is $\frac{\pi}{4}a$; for when $x = y$,

$$\tan\frac{x}{a} = \frac{1}{\sqrt{2}};$$

therefore $$\tan\frac{x\sqrt{2}}{a} > 1;$$

therefore $$\frac{x\sqrt{2}}{a} > \frac{\pi}{4},$$

$$x > \frac{\pi}{4}a.\sqrt{\frac{1}{2}};$$

the abscissa of the curve is therefore greater than the corresponding abscissa of the circle. Therefore the value of $\int_0^{\frac{1}{4}\pi} \sqrt{(1-\tan^2 x)}\,dx$ lies between $\frac{\pi^2}{16}$ and $\frac{\pi}{4}\frac{\pi^2}{16}$, that is between $\pi(\cdot 196)$ and $\pi(\cdot 154)$; therefore

$$\int_0^{\frac{1}{4}\pi} \sqrt{(1-\tan^2 x)}\,dx = \pi\,(\cdot 17) \text{ nearly.}$$

6. Determine the form of the function $f(\theta)$ from the equation
$$f(2\theta) = \cos\theta\, f(\theta);$$
with the condition $f(0) = m$.

Apply the result to find the centre of gravity of a circular arc.

$$f(\theta) = \cos\frac{\theta}{2}\,.f\left(\frac{\theta}{2}\right)$$

$$f\left(\frac{\theta}{2}\right) = \cos\frac{\theta}{2^2}\,.f\frac{\theta}{2^2}$$

$$\vdots$$

$$f\left(\frac{\theta}{2^{n-1}}\right) = \cos\frac{\theta}{2^n}\,.f\left(\frac{\theta}{2^n}\right)$$

$$\therefore\ f(\theta) = \cos\frac{\theta}{2}\,.\cos\frac{\theta}{2^n}\ldots\cos\frac{\theta}{2^n}\,.f\left(\frac{\theta}{2^n}\right)$$

$$2^n.\sin\left(\frac{\theta}{2^n}\right).f(\theta) = \sin\theta\,.f\left(\frac{\theta}{2^n}\right)$$

$$f(\theta) = \frac{\sin\theta}{\theta}\,.\frac{\frac{\theta}{2^n}}{\sin\frac{\theta}{2^n}}\,.f\left(\frac{\theta}{2^n}\right).$$

Now as n increases, $\dfrac{\frac{\theta}{2^n}}{\sin\frac{\theta}{2^n}}$ approximates to 1 and $f\left(\frac{\theta}{2^n}\right)$ to m,

$$\therefore\ f(\theta) = m\,.\frac{\sin\theta}{\theta}.$$

The centre of gravity of a circular arc must be in the line drawn from the centre to the middle point of the arc.

Let $f(\theta)$ express the distance of the centre of gravity from the centre of the circle, when θ is the angle subtended by the arc at the centre.

Let AOB, fig. (25), $= 2\theta$, $AO = a$.

Bisect AOB by OC, and AOC, COB, by OD, OE.

Then, since the centre of gravity of AC lies in OD at a distance from O equal to $f(\theta)$, and similarly for the centre of gravity of CB; and since the line joining the centres of gravity of AC and CB must cut OC at right angles in the centre of gravity of AB, therefore

$$f(2\theta) = \cos\theta \, . f(\theta),$$

and $m = a$, in this case; therefore

$$f(\theta) = a \, . \frac{\sin\theta}{\theta}.$$

7. (1) A rod is marked at random at two points, and then divided into three parts at those points; shew that the probability of its being possible to form a triangle with the pieces is $\frac{1}{4}$.

Let AB, fig. (26), be the rod, C its middle point, D, E, the middle points of AC, CB.

In order that it may be possible to form a triangle, each of the pieces must be less than the sum of the other two, or in other words, each must be less than half the rod.

To secure this it is clear that the two points of division P, Q, must lie on opposite sides of C; the probability of their doing so is $\frac{1}{2}$.

Let x be the probability that two points lying on opposite sides of the middle point of a line contain between them less than half the line: the required probability will be $\frac{1}{2}x$. Now there are four classes of ways in which the points may fall, all equally likely, the chance of each is therefore $\frac{1}{4}$. In the first of these classes, viz. when the points of division lie in DC, CE, success is certain; in the second, viz. when the points lie in AD, EB, success is impossible; in the third, viz. when the points lie in AD, CE, the probability of success is x, for success depending on the distance between the points being less than AC, the probability is the same as if DC were removed, and success depended on the distance between the points being less than AD, and this probability is x by supposition; lastly in the fourth class, viz. when the points lie in DC, EB, it may be shewn by similar reasoning that the probability of success is x.

Hence x is equal to the sum of the four compound probabilities $\frac{1}{4}+\frac{0}{4}+\frac{x}{4}+\frac{x}{4}$, therefore $x=\frac{1}{2}$, and the probability required is $\frac{1}{4}$.

Otherwise. Let a be the length of the rod, x, y, the distances of the two points of division from one end, x being greater than y. Then the lengths of the three pieces are y, $x-y$, $a-x$.

And the conditions of the problem give, as above shewn,

$$y<\frac{a}{2},\quad x-y<\frac{a}{2},\quad a-x<\frac{a}{2}.$$

Now let x, y, be the coordinates of a point referred to the rectangular axes Ox Oy, fig. (27).

Let $\quad OA=a,\quad AB=a,\quad OAB=\frac{1}{2}\pi.$

Then every possible way of dividing the rod may be represented by a point in the triangle OAB, and the chance of succeeding will be equal to the ratio of the area which contains points corresponding to favourable cases, to the area of the whole triangle.

Now we must have $y<\frac{1}{2}a$; therefore if CD bisect OB and AB, points in CDB are not favourable. Again, since $x-y<\frac{1}{2}a$, points in EDA, E being the middle point of OA, are excluded. And lastly, since $a-x<\frac{1}{2}a$, or $x>\frac{1}{2}a$, OCE is excluded. Hence the required chance is equal to area $ECD\div$ area $OAB=\frac{1}{4}$.

7. (2) Again: a piece is cut off the end of a rod, and the remainder is cut into two pieces at random; shew that the probability of its being possible to form a triangle with the pieces is in this case $\log_e 2-\frac{1}{2}$.

Let AB (fig. 28) be the rod, C its middle point; then, if A be the end from which the piece is cut off, it is necessary that the first point of section P should fall within AC, and also that each of the parts into which PB is then divided should be less than half the rod.

First, the probability that P, which may fall anywhere in AB, falls within AC, is equal to $\frac{1}{2}$.

Let the rod be divided into n parts, each equal to δx, so that $a = n\delta x$.

Then, if P fall on the r^{th} of these parts, there is a portion of the rod equal to $\left(1 - \frac{r}{n}\right) a$ within which the other point Q may fall.

Let this part be divided into m parts, each equal to δy, so that $\left(1 - \frac{r}{n}\right) a = m\delta y$.

Then the whole number of ways in which P and Q may fall is mn; and these ways are all equally likely.

Now to estimate in how many of these ways the formation of a triangle is possible, we observe that if $r > \frac{n}{2}$, a triangle cannot be formed; and if $r < \frac{n}{2}$, then the space within which Q must fall, so as to make a triangle possible, is CP', where $PP' = AC$; and CP' is equal to $\frac{r}{n} a = m \frac{r}{n-r} \delta y$.

Therefore the number of favourable cases is $\Sigma_{r=1}^{r=\frac{1}{2}n} m \cdot \frac{r}{n-r}$.

These results are approximately true when m and n are large; they will be strictly true in the limit when m and n are indefinitely increased: therefore the chance required is equal to

$$\begin{aligned}
&\text{limit } \frac{1}{mn} \Sigma_{r=1}^{r=\frac{1}{2}n} m \cdot \frac{r}{n-r} \\
&= \text{limit } \frac{1}{n} \Sigma_{r=1}^{r=\frac{1}{2}n} \frac{r}{n-r} \\
&= \int_{x=0}^{x=\frac{1}{2}} \frac{x}{1-x}\, dx \qquad \text{writing } x \text{ for } \frac{r}{n} \text{ and } dx \text{ for } \frac{1}{n}, \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{1-x} - 1\right) dx \\
&= \{-\log(1-x) - x\}_0^{\frac{1}{2}} \\
&= \log_\varepsilon 2 - \tfrac{1}{2}.
\end{aligned}$$

If we wish to apply to this problem the geometrical method by which the former was solved, we may take $AP = x$, $PQ = \left(1 - \frac{x}{a}\right) y$. Then every permissible way of cutting the rod will correspond to a point (xy) in a certain square: and the areas containing the points which correspond to unfavourable ways will be cut off by three lines, a straight line $\left(x = \frac{a}{2}\right)$, and two hyperbolas whose equations are $(a - x) y = \frac{a^2}{2}$, and $(a - x)(a - y) = \frac{a^2}{2}$: this method will require the evaluation of the same integral as the preceding method, of which it may be considered as a geometrical illustration.

8. One helix rolls upon another, (the inclination of the curve to the axis being the same in both,) in such a way that the osculating planes of the two curves at the point of contact coincide; find the curve traced out by a point in the rolling curve.

If a helix be traced on the surface of a cylinder, and a line be drawn through any point of the curve perpendicular to the axis, the osculating plane will pass through this line, and through the tangent line to the curve. If therefore two helixes be traced on the surfaces of cylinders, and the inclination of the curve to the axis be the same in each, if one of the cylinders be placed within the other, and roll round inside it, the one curve will roll upon the other, and the osculating planes at the point of contact will always coincide; the motion will therefore be of the kind described in the enunciation: and a point in the rolling curve will evidently trace out a hypocycloid. If the one cylinder be exterior to the other, that is if the curvatures of the two helixes be in opposite directions, a point in the rolling curve will trace out an epicycloid.

9. A, B, C, are three fixed points, and P a point which moves first half-way to A, then half-way to B, then half-way

to C, then half-way to A again, and so on for ever; shew that from whatever position P start, its path approximates to the perimeter of a certain triangle whose area is one-seventh of the area of the triangle ABC.

Let ABC (fig. 29) be the triangle formed by joining the given points.

We shall first shew that there is a triangle $A'B'C'$, such that if P start from C' it will continue to move in the perimeter of $C'A'B'$.

Find the points D, E, F, such that

$$BD = 2DC, \quad CE = 2EA, \quad AF = 2FB.$$

Join AD, BE, CF, AB', BC', CA'.

Let a, b, c, x, stand for the areas of the triangles $AA'B'$, $BB'C'$, $CC'A'$, $A'B'C'$, respectively.

Then, since $$AF = 2FB,$$

$$\triangle C'AB' = 2C'BB',$$

(for $C'AF = 2C'BF$ and $B'AF = 2B'BF$,)

or $$x + a = 2b.$$

Similarly, $$x + b = 2c,$$

$$x + c = 2a;$$

therefore, multiplying the second equation by 2, and the third by 4, and adding,

$$7x + a = 8a;$$

therefore $$x = a;$$

therefore $$C'A' = A'A.$$

Similarly, $$A'B' = B'B,$$

$$B'C' = C'C.$$

If, therefore, the point P start from C', and move according to the law stated in the enunciation, it will continue to move in the perimeter of the triangle $A'B'C'$.

Now, let P start from some other point C_0; and let the successive points where it rests be A_1, B_1, C_1; A_2, B_2, C_2; &c.

Join C_0C', A_1A', B_1B', C_1C', A_2A', &c.

Then, since C_0A is bisected in A_1,

and $C'A$ .... A',

therefore $A_1A' = \frac{1}{2}C_0C'$.

Similarly, $B_1B' = \frac{1}{2}A_1A'$,

$C_1C' = \frac{1}{2}B_1B'$;

therefore $C_1C' = \frac{1}{8}C_0C'$.

Similarly, $C_nC' = (\frac{1}{8})^n C_0C'$;

therefore the successive resting-places of P approximate to the points $A'B'C'$, and the path of P to the perimeter of $A'B'C'$.

Next to find the area of $A'B'C'$.

$$CC' = C'B';$$

therefore $\triangle ACC' = AC'B' = x + a = 2x$.

Similarly, $BAA' = 2x$,

$CBB' = 2x$;

and $\triangle ABC$ is made up of these three triangles, together with $A'B'C'$;

therefore $\triangle ABC = 7x$,

or the area of $A'B'C'$ is one-seventh of the area of ABC.

Otherwise: Let the plane which passes through ABC be taken as the plane of xy; and let

${}_xa\ {}_ya$ be the coordinates of A,

${}_xb\ {}_yb$ .... B,

${}_xc\ {}_yc$ .... C,

$x'y'z'$ the original coordinates of P,

${}_xa_1\,{}_ya_1\,{}_za_1$ the coordinates of P after moving half-way towards A,

${}_xb_1\,{}_yb_1\,{}_zb_1$ .... B,

${}_xc_1\,{}_yc_1\,{}_zc_1$ .... C,

${}_xa_2\,{}_ya_2\,{}_za_2$ .... A,

&c.

Then $${}_xa_1 = \frac{1}{2}{}_xa + \frac{1}{2}x,$$

$${}_xb_1 = \frac{1}{2}{}_xb + \frac{1}{2}{}_xa_1,$$

$$= \frac{1}{2}{}_xb + \frac{1}{4}{}_xa + \frac{1}{4}x',$$

$${}_xc_1 = \frac{1}{2}{}_xc + \frac{1}{4}{}_xb + \frac{1}{8}{}_xa + \frac{1}{8}x'.$$

Let c_x be such that $\frac{1}{2}{}_xc + \frac{1}{4}{}_xb + \frac{1}{8}{}_xa = \frac{7}{8}c_x$;

$$\therefore\ {}_xc_1 = \frac{7}{8}c_x + \frac{1}{8}x';$$

$$\therefore\ ({}_xc_1 - c_x) = \frac{1}{8}(x' - c_x).$$

Similarly, $$({}_xc_2 - c_x) = \frac{1}{8}({}_xc_1 - c_x)$$

$$= \frac{1}{8^2}(x' - c_x),$$

and so on; therefore $({}_xc_n - c_x) = \frac{1}{8^n}(x' - c_x)$;

therefore, when n is increased indefinitely,

$$\text{limit } ({}_xc_n - c_x) = 0,$$

$$\text{limit } {}_xc_n = c_x = \frac{4}{7}{}_xc + \frac{2}{7}{}_xb + \frac{1}{7}{}_xa.$$

Similarly, $$\text{limit } {}_yc_n = \frac{4}{7}{}_yc + \frac{2}{7}{}_yb + \frac{1}{7}{}_ya,$$

and $$\text{limit } {}_zc_n = 0.$$

If, therefore, we select every third resting-place of P, these will approximate towards a certain point in the plane of ABC, whose coordinates are given above.

Call this point C', and let $A'B'$ be the similar points; then the coordinates of A' will be

$$\frac{4}{7}{}_xa + \frac{2}{7}{}_xc + \frac{1}{7}{}_xb,$$

$$\frac{4}{7}{}_ya + \frac{2}{7}{}_yc + \frac{1}{7}{}_yb,$$

and of B',

$$\frac{4}{7}{}_xb + \frac{2}{7}{}_xa + \frac{1}{7}{}_xc,$$

$$\frac{4}{7}{}_yb + \frac{2}{7}{}_ya + \frac{1}{7}{}_yc.$$

So far we have proved that there is a triangle $A'B'C'$, to whose perimeter the path of P approximates; we proceed to find the area of this triangle.

The area of ABC is equal to

$$\pm \tfrac{1}{2}\left({}_xa\,{}_yb + {}_xb\,{}_yc + {}_xc\,{}_ya - {}_xb\,{}_ya - {}_xc\,{}_yb - {}_xa\,{}_yc\right);$$

and a similar expression might be written down for the area of $A'B'C'$, involving the values of the coordinates of $A'B'C'$ given above.

In this latter expression the coefficient of ${}_xa\,{}_yb$ would be

$$\frac{1}{2}\left(\frac{16}{49} + \frac{4}{49} + \frac{1}{49} - \frac{2}{49} - \frac{4}{49} - \frac{8}{49}\right)$$

$$= \frac{1}{2}\left(\frac{1}{7}\right);$$

and by symmetry the expression for the area of $A'B'C'$ will be similar to that for ABC, every coefficient in the one being one-seventh of the corresponding coefficient in the other: therefore the area of $A'B'C'$ is one-seventh of the area of ABC.

10. A string has a heavy particle at one end, and a small smooth ring at the other; a loop, formed by passing the particle through the ring, surrounds a fixed rough horizontal cylinder, the string being in one plane perpendicular to the axis: find the limiting positions of equilibrium; and shew that

in every position of equilibrium the three angles at the ring will be all obtuse unless the coefficient of friction exceed $\frac{2\log_e 2}{7\pi}$.

Let P (fig. 30) be the heavy particle, A the ring, $PACBA$ the string, PAC passing through the ring; O the axis of the cylinder.

Let P be the weight of the particle,

T the tension of AB,

θ the angle BOC,

μ the coefficient of friction.

We must first find in what positions the system will rest, and then limit μ so as to exclude the possibility of acute angles at A.

If the system be at rest, the conditions of equilibrium of the ring will be satisfied, and also the relation between P and T will be such that the string may not slip round in either direction. To secure this latter we must have

$$T > P\epsilon^{-\mu(2\pi-\theta)}, \text{ and } T < P\epsilon^{\mu(2\pi-\theta)} \quad \text{.........(1)},$$

and the ring will rest if

$$\text{angle } BAC = \text{angle } BAP \quad \text{............... (2)},$$

and
$$T = 2P\cos\frac{PAC}{2},$$

or
$$T = 2P\cos\theta \quad \text{.......................... (3)},$$

(for $BAC = \pi - \theta$, and $BAC = BAP$; therefore $\frac{PAC}{2} = \theta$).

If (1), (2), (3) be satisfied the system will rest.

Again, since $PAC < \pi$, therefore BAC, BAP must each of them be greater than $\frac{\pi}{2}$. If, therefore, there be an acute angle at A, it must be PAC. It is required therefore so to limit μ that no value of θ less than $\frac{\pi}{4}$ shall satisfy the conditions (1).

Substitute in (1) the value of T given by (3): then

$$2\cos\theta > \epsilon^{-\mu(2\pi-\theta)} \dots\dots\dots\dots (\alpha),$$

$$2\cos\theta < \epsilon^{\mu(2\pi-\theta)} \dots\dots\dots\dots (\beta).$$

First, consider the condition (α): as θ increases, $\cos\theta$ decreases, and $\epsilon^{-\mu(2\pi-\theta)}$ increases, so that the smaller the value of θ the more likely is the condition (α) to be satisfied: it is clear then that this condition cannot exclude small values of θ.

Consider then condition (β), which may be thrown into the form

$$\epsilon^{\mu\theta}\cos\theta < \tfrac{1}{2}\epsilon^{2\mu\pi} \dots\dots\dots\dots (\gamma).$$

Now $\epsilon^{\mu\theta}\cos\theta$ is a maximum relative to θ for the value $\theta = \tan^{-1}\mu$. *First*, suppose $\mu < 1$; then, since the value of $\epsilon^{\mu\theta}\cos\theta$ increases as θ changes from 0 to $\tan^{-1}\mu$, and decreases as θ changes from $\tan^{-1}\mu$ to $\frac{\pi}{4}$, it is clear that (γ) will be inconsistent with all the values of θ from 0 to $\frac{\pi}{4}$, provided it be inconsistent with these limiting values themselves, (one or the other of these being the value most likely to satisfy (γ)). It is required therefore so to limit μ that (γ) may be inconsistent with $\theta = 0$ and also with $\theta = \frac{\pi}{4}$; that is, in order to exclude acute angles, we must have

$$1 > \tfrac{1}{2}\epsilon^{2\mu\pi}, \qquad \text{or} \quad \mu < \frac{\log_\epsilon 2}{2\pi},$$

and $$\sqrt{2} > \epsilon^{\mu(2\pi-\frac{1}{4}\pi)}, \quad \text{or} \quad \mu < \frac{\frac{1}{2}\log_\epsilon 2}{\frac{7}{4}\pi},$$

which are both satisfied by

$$\mu < \frac{2\log_\epsilon 2}{7\pi}.$$

Secondly. Suppose $\mu > 1$. Now, since $\tan^{-1}\mu$, which is the value of θ which makes $\epsilon^{\mu\theta}\cos\theta$ a maximum, is greater than $\frac{\pi}{4}$, therefore $\epsilon^{\mu\theta}\cos\theta$ increases as θ changes from 0 to $\frac{\pi}{4}$;

therefore (γ) will be inconsistent with every value of θ between 0 and $\frac{\pi}{4}$, provided it be inconsistent with $\theta = 0$; that is, in order to exclude acute angles, we must have

$$\mu < \frac{\log_e 2}{2\pi}:$$

but this is impossible, for by supposition $\mu > 1$; therefore, if $\mu > 1$, we cannot exclude acute angles.

On the whole, therefore, there will be no acute angle at the ring if

$$\mu < \frac{2 \log_e 2}{7\pi}.$$

11. Two parallel vertical walls are one smooth and the other rough, and between them is supported a hemisphere with its curved surface in contact with the smooth wall, and a point in its rim in contact with the rough wall: find the pressures on the walls, and the least coefficient of friction consistent with equilibrium.

The hemisphere is at rest under the action of three forces; their directions must therefore lie in one plane, and pass through one point; this plane must be vertical, and perpendicular to the two walls.

In this plane let O (fig. 31) be the centre of the hemisphere ACB, G its centre of gravity, COE a horizontal line through O, GD vertical through G: join OG, DB.

Let $a = AO$, the radius of the hemisphere;
$b = CE$, the distance between the walls;
$\alpha = AOC$, the inclination of the plane face to the horizon;
$\theta = BDE$.

When the system rests the whole action at B must act along BD, since the directions of the two other forces pass through D: and the only condition to be fulfilled is, that the wall EB be sufficiently rough to exert a force in a direction making an angle θ with the normal; that is, we must have

$$\mu \nless \tan\theta.$$

Now $$b = a(1 + \cos\alpha),$$

and
$$\tan\theta = \frac{BE}{DE}$$
$$= \frac{BO \sin\alpha}{BO\cos\alpha + OG\sin\alpha}$$
$$= \frac{1}{\cot\alpha + \frac{3}{8}}, \text{ since } OG = \tfrac{3}{8}a,$$
$$= \frac{1}{\frac{b-a}{\sqrt{(2ab-b^2)}} + \frac{3}{8}};$$

therefore the least coefficient of friction consistent with equilibrium is

$$\frac{1}{\frac{b-a}{\sqrt{(2ab-b^2)}} + \frac{3}{8}}.$$

12. A body moves under the action of a force whose direction always touches a given plane curve, shew that, so long as the curvature is continuous, the areas, which it sweeps out about the moving point of contact, are not proportional to the times.

This may be proved in the way in which Newton proves his first proposition.

Suppose that a body in motion is deflected by a succession of impulses, at equal intervals of time, but that the directions of the impulses do not pass through one point.

Let AB (fig. 32) be the space described by the body in the first interval;

BC' the space it would describe in the next;

$B\beta$ the direction of the impulse at B.

BC the direction in which the body moves after the impulse;

then if $C'C$ be drawn parallel to $B\beta$, the body will at the end of the second interval be found at C; and, as in Newton,

$$\text{area } \beta AB = \beta BC.$$

Similarly, if $C\gamma$ be the direction of the impulse at C, and CD the space actually described in the third interval, it may be proved that

$$\text{area } \beta BC = \beta CD;$$

therefore $$\gamma CD > \beta BC.$$

Similarly, $$\delta DE > \gamma CD,$$
&c.

Therefore, in any number of intervals the body describes an area which is not proportional to the time, but (as the figure is drawn) describes an area which increases more rapidly than in proportion to the time: therefore, in the limit, when the number of intervals is increased and the magnitude of each diminished, (in which case the series of impulses approximates to a continuous force, and the locus of $\beta\gamma\delta\ldots$ to a curve touching the line in which the impulse acts,) the areas described by the moving body are not proportional to the times of describing them.

Note. If the line $D\gamma$ had been drawn to cut $C\beta$ instead of cutting $C\beta$ produced, and so on, then the area would have increased less rapidly than in proportion to the time.

13. A body describes a cycloid under the action of a force, which in every position of the body is directed towards the centre of the corresponding generating circle; find the law of the force and of the motion of the centre of force.

Let the equations to the cycloid be

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta);$$

therefore $$\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2} - a\sin\theta\left(\frac{d\theta}{dt}\right)^2 + a\cos\theta\frac{d^2\theta}{dt^2},$$

$$\frac{d^2y}{dt^2} = a\cos\theta\left(\frac{d\theta}{dt}\right)^2 + a\sin\theta\frac{d^2\theta}{dt^2}.$$

In any position of the generating circle, the radius drawn through the tracing point is inclined at an angle θ to the axis of y. Therefore, representing the force by F, and remembering that there is no force perpendicular to this, we have

$$F = \frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta$$

$$= a\left(\frac{d\theta}{dt}\right)^2 - a\sin\theta\,\frac{d^2\theta}{dt^2};$$

$$0 = \frac{d^2y}{dt^2}\sin\theta + \frac{d^2x}{dt^2}\cos\theta$$

$$= a(1+\cos\theta)\,\frac{d^2\theta}{dt^2};$$

therefore $$\frac{d^2\theta}{dt^2} = 0,\quad \frac{d\theta}{dt} = \text{const.};$$

therefore the centre of the generating circle moves uniformly and the force is constant.

This result might have been expected: for if a circle be fixed, a body, if projected with proper velocity, will move uniformly in its circumference under a constant force towards the centre; and if an initial velocity of translation be communicated both to the body and to the centre of force, the relative motion will not be disturbed: but if this velocity of translation be equal to that of the body in the circle, the absolute motion of the body will be in a cycloid, and will be preserved under the action of a constant force directed towards the centre of the corresponding generating circle, that circle moving uniformly in a straight line.

14. A surface of the second order circumscribes a tetrahedron, and each face of the tetrahedron is parallel to the tangent plane at the opposite angular point; shew that the centre of the surface coincides with the centre of gravity of the tetrahedron.

Let $ABCD$ be the angular points of the tetrahedron.

If the plane of the face BCD be produced, it will cut the surface in a curve of the second order: we shall shew first that the centre of this curve coincides with the centre of gravity of the triangle BCD.

Since the tangent plane at B is parallel to the plane ACD, therefore the intersections of these planes with BCD are parallel; that is, CD is parallel to the tangent to the curve BCD at B. Therefore the line drawn through B to bisect CD will pass through the centre of the curve BCD. But this line evidently passes through the centre of gravity of the triangle BCD: therefore the centre of the curve BCD, and the centre of gravity of the triangle BCD, lie on the same straight line through B. Similarly it may be shewn that a line through C or D passes through the aforesaid centres; therefore they coincide.

Let this point be called G, and join AG.

Then, since the tangent plane at A is parallel to the plane BCD, and a line is drawn from A to the centre of the curve BCD, therefore this line passes through the centre of the surface. And since this line is drawn from the vertex of a pyramid to the centre of gravity of the base, therefore it passes through the centre of gravity of the pyramid. Hence a line can be drawn through A, passing through the centre of the surface, and through the centre of gravity of the tetrahedron. And the same may be proved of any other angular point. Therefore the two centres, the centre of the surface and the centre of gravity of the tetrahedron, coincide.

15. A horizontal cylinder revolves with uniform velocity about its axis, and an endless chain, passing round it, revolves with it in such a manner that the form of the chain in space is always the same; shew that the form of the curve is independent of the velocity.

Let V be the velocity of any point of the chain,

k the mass of a unit of length of the chain,

T the tension at any point of the chain not in contact with the cylinder,

ρ the radius of curvature,

θ the inclination of the tangent to the horizon at this point.

V and k are absolute constants: T and ρ may be considered as functions of θ, since the position of any point is determinate when a value is assigned to θ.

Since the velocity is constant, the accelerating effective force at any point is $\frac{V^2}{\rho}$ along the normal; and by D'Alembert's principle the chain would hang in its existing shape under the action of gravity, and the reversed effective force acting at every point. Therefore, resolving the forces on any element along the normal and tangent,

$$g\cos\theta + \frac{V^2}{\rho} = \frac{T}{k\rho} \quad\text{.....................(1)},$$

$$g\sin\theta = \frac{1}{k\rho}\frac{dT}{d\theta} \quad\text{.....................(2)},$$

therefore, eliminating ρ,

$$\frac{\frac{1}{k}\frac{dT}{d\theta}}{\frac{T}{k} - V^2} = \tan\theta;$$

therefore, integrating,

$$\log\left(\frac{T}{k} - V^2\right) = \log(C\sec\theta);$$

therefore $$\frac{T}{k} - V^2 = C\sec\theta;$$

therefore, by (1), $$\frac{1}{\rho} = \frac{g}{C}\cos^2\theta;$$

therefore $$\frac{ds}{d\theta} = \frac{C}{g}\sec^2\theta,$$

s being the length of the chain measured from some point fixed in space, as the lowest point;

therefore $$s = \frac{C}{g}\tan\theta,$$

the equation to the common catenary.

We have thus proved that the *species* of curve which the moving chain may assume is a catenary; it remains to shew that the magnitude and position of the catenary are independent of the velocity.

For this purpose, let the student draw a circle to represent a transverse section of the cylinder, and a catenary with its axis vertical to represent the free part of the chain; the circle and catenary will touch each other in two points, on opposite sides of the axis; if he now draw another catenary with its axis vertical, and (like the former) touching the circle, he will find that he is obliged to draw it either entirely within or entirely without the former, according as the parameter of the second catenary is greater or less than that of the former; and this would require the length of the chain to be less or greater than before: but the length of the chain is given, therefore the magnitude and position of the catenary are determinate, and this without reference to the velocity of the chain.

Otherwise. (The following ingenious proof was sent up in the Senate-House by one of the candidates: it is given with a few additions, which have been placed between brackets.)

If we suppose there to be perfect friction between the chain and the cylinder, [or if we suppose the cylinder to be smooth,] since the string is always stretched, the principle of virtual velocities will hold for the effective forces reversed, and the impressed force of gravity.

But the effective forces are by themselves in equilibrium, because the velocity and direction [of motion] of each point, [fixed in the chain] are the same each time it reaches the same point [in space; and therefore the resultant of all the effective forces on it during a revolution is nothing, since the motion is the same at the end as at the beginning of that time;] and to each point at different times correspond all the points at the same time, [that is, the effective forces which act on an element of the chain in its revolution, are the same as the effective forces acting on the several elements of the chain at any particular epoch: therefore, as above asserted, the effective forces on the whole chain at any epoch are in equilibrium by themselves].

Therefore the principle of virtual velocities holds for gravity only, and therefore the form of the [moving] curve [satisfying the same conditions as if it were hanging subject to gravity only,] is the common catenary.

Note. This problem is a particular case of the following: If a uniform endless chain rest in any form, subject to the action of forces depending only on the position of the particle acted on, and to the reactions of smooth surfaces, it would continue to move in the same form if put in motion in such a manner that every point of the chain begins to move in the direction of the tangent at that point.

This proposition may be easily proved by referring to the general equations for a flexible string. The equations of equilibrium are three,

$$\left.\begin{array}{c}\dfrac{d}{ds}\left(T\dfrac{dx}{ds}\right)+X+X_1=0\\ \text{\&c.}\end{array}\right\}\dots\dots\dots\dots\dots (1),$$

the &c. standing for two other equations related to the axes of y and z in the same manner that the above equation is related to the axis of x; and X, X_1 standing for the resolved parts of the forces which depend on the position of the particle, and of the reactions of the fixed surfaces, if the point (x, y, z) be in contact with such surface.

If T were eliminated from equations (1), the two resulting equations would be satisfied by the coordinates of every point in the chain.

Now suppose the chain to be moving in the form in which it would rest, in such a manner that every point is moving in the direction of the tangent, and therefore that every point has the same velocity; let V be this velocity, T' the tension at x, y, z, and suppose X', Y', Z', to be the forces required to continue such motion, while the reactions continue the same. Then we shall have the three following equations of motion:

$$\left.\begin{array}{c}\dfrac{d}{ds}\left(T'\dfrac{dx}{ds}\right)+X'+X_1-k\dfrac{d^2x}{dt^2}=0\\ \text{\&c.}\end{array}\right\}\dots\dots\dots (2).$$

Now, $$\frac{dx}{dt} = V\frac{dx}{ds}, \quad \frac{d^2x}{dt^2} = V^2\frac{d^2x}{ds^2},$$

therefore equations (2) become

$$\left.\begin{array}{c}\dfrac{d}{ds}\left\{(T' - kV^2)\dfrac{dx}{ds}\right\} + X' + X_1 = 0\\ \&c.\end{array}\right\} \text{.........} \; (3);$$

and, as before, if T' were eliminated from equations (3), the two resulting equations would be satisfied by the coordinates of every point of the moving chain.

In order that the curve, whose equations result from eliminating T from (1), may be the same as the curve whose equations result from eliminating $(T' - kV^2)$ from (3), X', Y', Z', must be certain functions of xyz; whatever forms of X', Y', Z', proper to effect this, may exist, it is clear that if X', Y', Z', be equal to X, Y, Z, respectively, the curves will be the same. If therefore the chain when in motion be subject to the same system of forces as when at rest, it will continue to move in the same form, the reactions of the fixed surfaces will be the same as before, and the tension at every point will be greater than before, by kV.

16. An inclined plane is fixed on a table, and from the foot of it a body is projected upwards along the plane with the velocity due to the height c; after passing over the top of the plane the body strikes the table at a distance z from the foot of the plane; shew that, if the length of the plane be l, and α its inclination to the horizon be less than $\frac{\pi}{4}$, the greatest value of z for given values of c and α is $\frac{c}{\sin\alpha\cos\alpha}$, and corresponds to the value $l = 2c\frac{\cot 2\alpha}{\cos\alpha}$.

Let V be the original velocity, so that $V^2 = 2gc$;

h the height of the inclined plane, so that $h = l\sin\alpha$;

v the velocity on reaching the top of the plane, so that $v^2 = 2g(c-h)$;

x the horizontal distance of flight, so that $z = x + h\cot\alpha$.

From the equation to the path of the projectile, (the top of the plane being origin,)

$$-h = x\tan\alpha - \frac{g}{2}\,\frac{x^2}{v^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{g}{2}\,\frac{x^2}{2g(c-h)\cos^2\alpha};$$

therefore $$(c-h)(h+x\tan\alpha) = \frac{x^2}{4\cos^2\alpha};$$

therefore, substituting $z - h\cot\alpha$ for x,

$$(c-h)\,z\tan\alpha = \frac{(z-h\cot\alpha)^2}{4\cos^2\alpha};$$

therefore

$$z^2 - 2(h\cot\alpha\cos2\alpha + c\sin2\alpha)\,z + h^2\cot^2\alpha = 0 \;\ldots\; (1):$$

and, differentiating with respect to h, in order to find the maximum value of z,

$$\{z-(h\cot\alpha\cos2\alpha+c\sin2\alpha)\}\frac{dz}{dh} - \cot\alpha\cos2\alpha.z + h\cot^2\alpha = 0 \;\ldots\ldots\; (2),$$

therefore $$\frac{dz}{dh} = 0, \quad \text{if } z = h\frac{\cot\alpha}{\cos2\alpha};$$

therefore, substituting this value of z in (1), and reducing,

$$h = c\frac{\cos2\alpha}{\cos^2\alpha}:$$

also substituting in the coefficient of $\frac{dz}{dh}$ in (2),

$$h\frac{\cot\alpha}{\cos2\alpha} - h\cot\alpha\cos2\alpha - c\sin2\alpha = c\sin2\alpha;$$

therefore, as h increases through the value $c\frac{\cos2\alpha}{\cos^2\alpha}$, the coefficient of $\frac{dz}{dh}$ continues nearly equal to $c\sin2\alpha$, and therefore continues positive, while the remainder of equation (2), viz. $-\cot\alpha\cos2\alpha.z + h\cot^2\alpha$, increases from $-$ to $+$, for z is stationary and h is increasing;

therefore $\frac{dz}{dh}$ changes from $+$ to $-$;

therefore z is a maximum when

$$l = \frac{h}{\sin\alpha} = 2c\,\frac{\cot 2\alpha}{\cos\alpha},$$

and

$$z = h\,\frac{\cot\alpha}{\cos 2\alpha} = \frac{c}{\sin\alpha\cos\alpha}.$$

17. A slender ring, moveable in a vertical plane, has a fixed rough cylinder passing through it, the axis of the cylinder being perpendicular to the plane of the ring; the ring whirls round in its own plane so as always to be in contact with the cylinder, and to roll on it without sliding: if $V_1 V_2$ be the velocities of the centre of the ring when in its highest and lowest positions respectively, and if P be the point of contact, O the centre of the ring, when the tendency to slide is greatest, and OA a vertical drawn downwards through O, shew that

$$\cos POA = 2\,\frac{V_2^2 - V_1^2}{V_2^2 + V_1^2}.$$

Explain the result when $V_2^2 > 3V_1^2$.

Let a, b, be the radii of the ring and of the cylinder,
ϕ the angle POA (fig. 33),
θ the angle which a particular radius fixed in the ring makes with a fixed line in the plane of the ring,
F, R, the friction, and normal action at P, estimated as accelerating forces.

The tendency to slide will be greatest when $\frac{F}{R}$ is a maximum, provided it never become infinite; we must therefore find an expression for $\frac{F}{R}$ and make it a maximum.

Applying D'Alembert's principle, and resolving forces parallel and perpendicular to OP, and taking moments about O, we obtain

$$(a-b)\left(\frac{d\phi}{dt}\right)^2 = g\cos\phi + R \quad\text{..............(1)},$$

$$(a-b)\,\frac{d^2\phi}{dt^2} = g\sin\phi - F \quad\text{..............(2)},$$

$$a^2\,\frac{d^2\theta}{dt^2} = Fa \quad\text{..........................(3)}.$$

Also, since there is no sliding at P,

$$(a-b)\frac{d\phi}{dt} - a\frac{d\theta}{dt} = 0,$$

by which (3) becomes

$$(a-b)\frac{d^2\phi}{dt^2} = F \qquad\qquad (4).$$

To express $\left(\frac{d\phi}{dt}\right)^2$ in terms of ϕ;

from (2) and (4), $\qquad 2(a-b)\frac{d^2\phi}{dt^2} = g\sin\phi;$

$$\therefore\ (a-b)\left(\frac{d\phi}{dt}\right)^2 = C - g\cos\phi \qquad\qquad (5).$$

From (1) and (5), $\qquad R = C - 2g\cos\phi \qquad\qquad (6),$

and from (2) and (4), $\qquad F = \frac{1}{2}g\sin\phi;$

$$\therefore\ \frac{F}{R} = \frac{1}{4}\frac{\sin\phi}{\frac{C}{2g} - \cos\phi},$$

and this is a maximum when

$$\cos\phi = \frac{2g}{C}.$$

It remains to introduce V_1 and V_2 instead of C;

from (5),
$$\left.\begin{aligned} V_1^2 &= (C-g)(a-b) \\ V_2^2 &= (C+g)(a-b) \end{aligned}\right\} \qquad\qquad (7);$$

$$\therefore\ \frac{g}{C} = \frac{V_2^2 - V_1^2}{V_2^2 + V_1^2}.$$

Hence the tendency to slide is greatest when

$$\cos\phi = 2\frac{V_2^2 - V_1^2}{V_2^2 + V_1^2}.$$

Since $2\frac{V_2^2 - V_1^2}{V_2^2 + V_1^2} = 1 + \frac{V_2^2 - 3V_1^2}{V_2^2 + V_1^2}$, it follows that the expression for $\cos\phi$ is less than 1 only when $V_2^2 < 3V_1^2$; hence arises the question, What happens when V_2^2 is greater than

$3V_1^2$? Now, by substituting for V_2^2, after eliminating C from equations (7), the expression for $\cos\phi$ becomes

$$2\frac{g(a-b)}{g(a-b)+V_1^2};$$

therefore, as V_1^2 decreases from ∞ to $g(a-b)$, ϕ decreases from $\frac{\pi}{2}$ to 0; and if V_1^2 be less than $g(a-b)$, ϕ becomes impossible, and at the same time V_2^2 becomes greater than $3V_1^2$.

Hence, if the ring revolve very rapidly, the risk of sliding is greatest when the centre of the ring is only just above a horizontal line through the centre of the cylinder; if the speed be diminished, the position in which sliding is more probable than in any other becomes nearer to the highest position of the ring; if the speed be such that $V_1^2 = g(a-b)$, the risk of sliding is greater when the ring is highest, than at any other part of the revolution. In this case (*i.e.* when $V_1^2 = g(a-b)$,)

$$C = 2g \quad \text{by (7)},$$

and therefore when the ring is highest,

$$R = 0 \quad \text{by (6)};$$

and though $F = 0$, yet $\frac{F}{R} = \infty$.

A smaller value of V_1^2, besides making $V_2^2 > 3V_1^2$, would make the expression for R, given by (6), negative; hence we see that the contact between the ring and the cylinder would be broken before the ring completed a revolution, and that the risk of sliding would never be a maximum, in the proper sense of the word, but would increase without limit as the ring approached the critical position at which it would fall.

18. A cylindrical vessel is moveable about a horizontal axis passing through its centre of gravity, and is placed so as to have its axis vertical; if water be poured in, shew that the equilibrium is at first unstable; and find the condition which must be satisfied, in order that it may be possible to make the equilibrium stable by pouring in enough water.

Let CFD (fig. 34) be the base of the vessel,
G its centre of gravity,
AEB the surface of the water,
H its centre of gravity,
$CF = a$, $FE = h$, $FG = c$.

The equilibrium will be stable if, on the vessel being turned round through a very small angle, the resultant of the fluid pressures tends to bring the vessel back to its former position, the weight of the vessel producing no effect, because the centre of gravity lies on the axis of motion.

Now the line of action of the resultant pressure is the same as if a solid cylinder, that would just fit into the given cylinder, were floating on a fluid, in such a manner that the volume of fluid displaced were equal to the volume contained in the given cylindrical vessel; for the pressures would be the same in the two cases, except that in the one they would act downwards and outwards, in the other they would act upwards and inwards; therefore in the existing case the downward resultant acts through M, the metacentre of the space AD; and by the usual formula we have

$$HM = \frac{\frac{\pi a^4}{4}}{\pi a^2 h} = \frac{a^2}{4h};$$

hence the equilibrium will be unstable if

$$\frac{h}{2} + \frac{a^2}{4h} > c,$$

or if $h^2 - 2hc + \frac{a^2}{2}$ be positive.

Now by the Theory of Equations this is always positive unless h be between

$$c \pm \sqrt{\left(c^2 - \frac{a^2}{2}\right)};$$

Now if it be possible to make the equilibrium stable, these two quantities must be real; hence the required condition is that c^2 be greater than $\frac{a^2}{2}$.

19. Given the directions of three plane mirrors in space, construct a straight line, such that, if light from it be reflected by the three mirrors in succession, the third image shall be parallel to the straight line.

From the centre O of any sphere draw radii perpendicular to the mirrors, and let A, B, C, (fig. 35) be the extremities of these radii: then, if we can construct a spherical triangle $A'B'C'$, such that A, B, C, shall be the middle points of its sides, the radii OA', OB', OC', will severally satisfy the required condition. For, supposing the images to be formed by the mirrors corresponding to A, B, C, in succession since OB', OC', are equally inclined to the normal OA, the image, formed by A, of OB', will be parallel to OC'; similarly, the image, formed by B, of a line parallel to OC', will be parallel to OA'; and the image, formed by C, of a line parallel to OA', will be parallel to OB'; so that if there be a luminous object parallel to OB', the first and second images of it will be parallel to OC' and OB', and the third image will be parallel to the object. The problem is thus reduced to the determination of a spherical triangle, the middle points of whose sides shall coincide with three given points on the surface of a sphere.

Join BC by an arc of a great circle, and produce it both ways to meet $B'C'$ produced in D, E; DAE is a semicircle, and A is its middle point. (Hymers' *Spherical Trigonometry*, Prob. 7.)

Hence the triangle $A'B'C'$ is to be found by the following construction: Join BC, and produce it both ways to meet the great circle, of which A is the pole, in the points D, E: join DAE; part of this circle will be the side $B'C'$ of the required triangle; similarly, the sides $A'B'$, $A'C'$, may be constructed; and hence the lines OA', OB', OC', may be drawn.

The above construction for the required line may be put into the following form, which is independent of spherical trigonometry.

Let α, β, γ, be the names of the mirrors in the order in which successive images are formed by them.

Let α meet a plane perpendicular both to β and to γ, and through the line of intersection draw a plane perpendicular to α; similarly, let γ meet a plane perpendicular both to α and to β, and through the line of intersection draw a plane perpendicular to γ; the intersection of these two planes (which correspond to DAE and $A'CB'$ respectively,) is the line required.

It may be remarked that there is only one solution to the problem; that is, when the order in which the light falls on the mirrors is fixed, there is in general only one direction of the object which satisfies the required condition.

For though we may take, instead of A, the diametrically opposite point, and so of B and C, still we shall not obtain any other line than OB'. For the great circle through BC will not be affected by the supposed change, neither will the great circle of which A is the pole; the points D and E, and the circle DAE, will therefore always keep the same position; hence the three circles which intersect in B', C', A' will always keep their present positions.

Note. By the above construction we have secured, not only that the third image shall be parallel to the original line, but that corresponding ends shall be turned towards the same parts. For instance, if the object were an arrow, and B' corresponded to the head, and O to the feathered end, then in the first image C' would correspond to the head, in the second A', and in the third B'; so that the third image would point in the same direction as the object.

If it be required that the third image shall be reversed, it may be shewn that the problem is impossible unless the mirrors are all perpendicular to one plane.

For, as before, supposing ABC (fig. 36) to be normals to the mirrors, OB' will be reflected by A into OC', OC' by B into OA', OA' by C into OB''; and it is required to find B' such that B'' (instead of coinciding with B', as before,) shall be at the opposite end of the diameter through B'. Hence $B'C'$ produced will pass through B'', and $B'C'B''$ will be a semicircle. Let A_1 be the middle point of $C'B''$, then AA_1 is a quadrant, and the problem is reduced to the following: Given three points A, B, C, to find a triangle $C'A'B''$ such that the middle points of its sides may be B, C, and a point A_1, whose distance from A shall be a quadrant, and that the side $B''C'$ produced shall pass through A. Now, under these circumstances BC produced meets $B''C'$ in points whose distances from A_1 are quadrants; that is, the great circle through BC passes through A. If this condition be not satisfied, there can be no triangle $C'A'B''$ possessing the above-stated properties, and the problem of the reversed arrow is impossible.

But if the three mirrors be perpendicular to one plane, the above condition is satisfied, and any point A_1 may be chosen on the great circle, of which A is the pole, and a triangle $C'A'B''$ constructed by the method given in pages 72 and 73, whence OB' may be drawn.

20. Shew that, in latitude 60°, on the 21st of March, the setting Sun is visible for about 69 seconds longer from the top than from the bottom of a tower 66 feet high, taking the Earth's radius 4000 miles and neglecting the effect of refraction.

On the 21st of March the Sun is on the equator; and therefore in t seconds of time he describes $15t$ seconds of space: again, in latitude 60° the inclination of the equator to the horizon is 30°; therefore when the Sun has described $15t''$ from the horizon, measured along the equator, his vertical distance below the horizon is $(15t.\sin 30°)'' = \frac{15}{2}t''$. If therefore the setting Sun be seen t seconds longer from the top than from the bottom of a tower, the dip of the horizon as seen from the top must be $\frac{15}{2}t''$; therefore a straight line, drawn from the top of the tower to the horizon, subtends at the centre of the Earth an angle of $\frac{15}{2}t''$ or $\frac{15}{2}t\,\dfrac{\pi}{180 \times 60 \times 60}$ in circular measure;

$$\therefore \text{ secant}\left(\frac{\pi}{180\times 60\times 60}\,\frac{15}{2}t\right) = \frac{\text{Earth's radius} + \text{height of tower}}{\text{Earth's radius}};$$

$$\therefore\ 1+\frac{1}{2}\left(\frac{\pi}{180\times 60+60}\ \frac{15}{2}t\right)^2=1+\frac{22}{4000\times 1760};$$

$$\therefore\ \left(\frac{\pi}{180\times 60\times 8}\cdot t\right)^2=\frac{1}{4000\times 40}=\left(\frac{1}{400}\right)^2;$$

$$\therefore\ t=\frac{216}{\pi}=\frac{216}{3\frac{1}{7}}=69\text{ nearly.}$$

21. Shew how to determine graphically the path of the centre of graduation of a mural circle, by observing the differences between the readings of any three microscopes, (severally corrected for runs,) for various positions of the instrument.

Let AO (fig. 37) be the direction of the axis of the first microscope, inclined at an angle A to the horizon,

C_0 the position of the centre of graduation when the circle is in a certain position, chosen as a standard position: C_0 will serve as an origin to which the path of the centre of graduation may be referred,

C its position when the instrument has been turned through a certain angle ϕ,

$C_0C=\rho$; inclination of C_0C to the horizon $=\theta$.

If the values of ρ and θ be ascertained for a great number of successive positions of the instrument, the path of the centre of graduation may be laid down on paper.

Now, let the three microscopes be read off first in the standard position of the circle, and secondly, after the circle has turned through the angle ϕ, and the centre of graduation has arrived at C: the difference of the readings at A will give the value of ϕ, affected with an error, in consequence of the displacement of the centre of graduation; this error would be removed if, before taking the second reading, the circle were moved parallel to itself till C coincided with C_0.

Let A be the point of the limb actually viewed at the second reading, A' the point which would be at A'', and be there viewed if C coincided with C_0; the second reading at A is therefore too great by $A'A$. Now $A''A'$ is equal and parallel to C_0C, and the angle $AA''A'$ equal to $(A-\theta)$;

∴ error in seconds in the value of ϕ assigned by first microscope

$$=\rho \sin(A-\theta) \cdot \frac{180 \times 60 \times 60}{\pi R},$$

R being the radius of the circle.

Similarly, error in the value of ϕ assigned by second microscope

$$=\rho \sin(B-\theta) \cdot \frac{180 \times 60 \times 60}{\pi R}.$$

Now, as we do not know the true value of ϕ, we cannot determine from observation the error at a single microscope; but by subtracting the value of ϕ given by A from that given by B, we shall obtain the difference of the two errors: let ϕ_0 be the value given by A, $\phi_0 + \beta''$ the value given by B;

$$\therefore \ \beta = \rho\{\sin(B-\theta) - \sin(A-\theta)\} \cdot \frac{180 \times 60 \times 60}{\pi R}$$

$$= 2\rho \cos\left(\frac{B+A}{2} - \theta\right) \cdot \sin\frac{B-A}{2} \cdot \frac{180 \times 60 \times 60}{\pi R}.$$

Similarly, if $\phi_0 + \gamma''$ be the value of ϕ given by the third microscope,

$$\gamma = 2\rho \cos\left(\frac{C+A}{2} - \theta\right) \cdot \sin\frac{C-A}{2} \cdot \frac{180 \times 60 \times 60}{\pi R}.$$

Hence ρ and θ may be obtained; and if this operation be repeated for successive positions of the circle, the path of the centre of graduation may be laid down on paper with any required degree of accuracy.

THURSDAY, *Jan.* 19, 1854. 9...12.

1. Two circles of radii r, r', touch a straight line at the same point on opposite sides: a circle, of which the radius is R and of which the straight line is a chord, touches both the former circles. Prove that the length of the chord is equal to

$$\frac{4R}{\left(\frac{r}{r'}\right)^{\frac{1}{2}} + \left(\frac{r'}{r}\right)^{\frac{1}{2}}}.$$

Let AB (fig. 38) be the straight line, E the point in which it is touched by the two circles, the centres of which are O, O'. Let C be the centre of the third circle. Draw CH at right angles to AB. Join OO', OC, $O'C$.

Let $CH = a$, $HE = b$, $\angle OCH = \theta$.

From the geometry,

$$(r + a) \sin\theta = b \cos\theta \text{ .... } (1),$$

also
$$r + a = (R - r) \cos\theta \text{ .... } (2).$$

$(2)^2 - (1)^2$ gives

$$(r + a)^2 \cos^2\theta = \{(R - r)^2 - b^2\} \cos^2\theta,$$

and therefore
$$b^2 = R^2 - 2Rr - a^2 - 2ar \text{ .... } (3).$$

Similarly, putting $-a$ for a, and r' for r,

$$b^2 = R^2 - 2Rr' - a^2 + 2ar' \text{ .... } (4).$$

From (3) and (4), $\quad a = R \cdot \dfrac{r' - r}{r' + r}.$

Hence
$$AH^2 = R^2 - a^2 = R^2 \cdot \frac{4rr'}{(r+r')^2},$$

and therefore $$AB = \frac{4R}{\left(\frac{r}{r'}\right)^{\frac{1}{2}} + \left(\frac{r'}{r}\right)^{\frac{1}{2}}}.$$

2. Prove that, n being any positive integer, and e the base of Napier's logarithms,

$$e^n > \frac{(n+1)^n}{1.2.3 \ldots n}.$$

LEMMA. For any value of x, except zero, between the limits -1 and $+\infty$,

$$x > \log(1+x).$$

Put $$y = x - \log(1+x):$$

then $$\frac{dy}{dx} = \frac{x}{1+x}.$$

Hence, as x increases from -1 to 0, $\frac{dy}{dx}$ is always negative, and therefore y keeps always decreasing. Again, as x increases from 0 to ∞, $\frac{dy}{dx}$ is always positive, and therefore y keeps always increasing. But $y=0$ when $x=0$: hence the truth of the lemma.

Since, when x is any positive quantity,

$$x > \log(x+1),$$

it follows that $$e^x > x + 1.$$

Put $x = \frac{1}{n}$: then $$e^{\frac{1}{n}} > \frac{n+1}{n},$$

$$e.n^n > (n+1)^n.$$

Writing for n, successively, 1, 2, 3, ... n, we have

$$e.1^1 > 2^1,$$
$$e.2^2 > 3^2,$$
$$e.3^3 > 4^3,$$
$$e.4^4 > 5^4,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$e.(n-1)^{n-1} > n^{n-1},$$
$$e.n^n > (n+1)^n.$$

Multiplying these inequalities together and casting out factors common to both sides of the resulting equation, we have

$$e^n . 1.2.3 \ldots n > (n+1)^n, \quad \text{or} \quad e^n > \frac{(n+1)^n}{1.2.3 \ldots n}.$$

3. From a focus S of a conic section $ARQPA$ (fig. 39), three radii vectores SR, SQ, SP, are drawn, the angles PSQ, QSR, being invariable. Prove that the tangent at P intersects the chord RQ produced in a point of which the locus is another conic section.

Supposing e to be the eccentricity of the original conic section and e' of the conical locus, shew that, if $\angle RSQ = 2\alpha$, and $\angle QSP = \beta$,

$$\frac{e'^2}{e^2} = \frac{\sin^4 \frac{\alpha}{2}}{\sin^2 \frac{\alpha+\beta}{2}} + \frac{\cos^4 \frac{\alpha}{2}}{\cos^2 \frac{\alpha+\beta}{2}}.$$

Let $\angle ASQ = \lambda$. Then the equation to the chord RQT is

$$\frac{c}{r} = \sec\alpha \cos(\theta - \lambda + \alpha) + e\cos\theta,$$

and that to the tangent PT is

$$\frac{c}{r} = \cos(\theta - \lambda - \beta) + e\cos\theta.$$

At the intersection of the chord and tangent, subtracting and adding the equations

$$\left(\frac{c}{r} - e\cos\theta\right).\cos\alpha = \cos(\theta - \lambda + \alpha),$$

$$\frac{c}{r} - e\cos\theta = \cos(\theta - \lambda - \beta),$$

we get

$$\left(\frac{c}{r} - e\cos\theta\right).\sin^2\frac{\alpha}{2} = \sin\frac{\alpha+\beta}{2}.\sin\left(\theta - \lambda + \frac{\alpha-\beta}{2}\right),$$

$$\left(\frac{c}{r} - e\cos\theta\right).\cos^2\frac{\alpha}{2} = \cos\frac{\alpha+\beta}{2}.\cos\left(\theta - \lambda + \frac{\alpha-\beta}{2}\right).$$

Hence $$\left(\frac{c}{r} - e\cos\theta\right)^2 \cdot \left\{\frac{\sin^4\frac{\alpha}{2}}{\sin^2\frac{\alpha+\beta}{2}} + \frac{\cos^4\frac{\alpha}{2}}{\cos^2\frac{\alpha+\beta}{2}}\right\} = 1,$$

$$\frac{c}{r} = e\cos\theta \pm \frac{1}{\left\{\frac{\sin^4\frac{\alpha}{2}}{\sin^2\frac{\alpha+\beta}{2}} + \frac{\cos^4\frac{\alpha}{2}}{\cos^2\frac{\alpha+\beta}{2}}\right\}^{\frac{1}{2}}},$$

a result which establishes the proposition.

4. Tangents PP', PP'', are drawn from a point P to touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

at points P', P''. Supposing the harmonic mean between the abscissæ of the points P', P'', to be equal to that between their ordinates, shew that the locus of P consists of four arcs of a curve of the third order.

Trace the curve and shew that, when $a = b$, the curve degenerates into a straight line and an ellipse.

Let h, k, be the coordinates of P; $x_{\prime}$, $y_{\prime}$, of P'; $x_{\prime\prime}$, $y_{\prime\prime}$, of P''. The equation to $P'P''$ is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1.$$

At the intersections of this line with the ellipse,

$$x^2\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right) - 2hx + a^2\left(1 - \frac{k^2}{b^2}\right) = 0.$$

Hence $$x_{\prime} + x_{\prime\prime} = \frac{2h}{\frac{h^2}{a^2} + \frac{k^2}{b^2}}, \quad x_{\prime}x_{\prime\prime} = a^2 \cdot \frac{1 - \frac{k^2}{b^2}}{\frac{h^2}{a^2} + \frac{k^2}{b^2}},$$

and therefore $$\frac{1}{x_{\prime}} + \frac{1}{x_{\prime\prime}} = \frac{2h}{a^2} \cdot \frac{1}{1 - \frac{k^2}{b^2}}.$$

By symmetry, $$\frac{1}{y_{,}} + \frac{1}{y_{,,}} = \frac{2k}{b^2} \cdot \frac{1}{1 - \frac{h^2}{a^2}}.$$

Hence, by the condition of the problem,

$$\frac{h}{a^2} \cdot \left(1 - \frac{h^2}{a^2}\right) = \frac{k}{b^2} \cdot \left(1 - \frac{k^2}{b^2}\right),$$

or, replacing h, k, by x, y, we have for the equation to the curve in which P always lies,

$$\frac{x}{a^2} + \frac{y^3}{b^4} = \frac{y}{b^2} + \frac{x^3}{a^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The shape of the curve is $IEBA'OAB'E'I'$, (fig. 40), IOI' being an asymptote.

The equation to IOI' is

$$y = \left(\frac{b}{a}\right)^{\frac{4}{3}} x.$$

The curve at O is inclined to the axis of x at an angle $\tan^{-1}\left(\frac{b^2}{a^2}\right)$.

The locus of P consists of the four arcs

$$IE,\ BA',\ AB',\ E'I'.$$

At the intersections of the ellipse and curve

$$\begin{matrix} B \\ B' \end{matrix} \begin{pmatrix} x = 0 \\ y = \pm b \end{pmatrix}, \quad \begin{matrix} A \\ A' \end{matrix} \begin{pmatrix} y = 0 \\ x = \pm a \end{pmatrix}, \quad E \begin{Bmatrix} x = \dfrac{ab}{(a^2+b^2)^{\frac{1}{2}}} \\ y = \dfrac{ab}{(a^2+b^2)^{\frac{1}{2}}} \end{Bmatrix}, \quad E' \begin{Bmatrix} x = -\dfrac{ab}{(a^2+b^2)^{\frac{1}{2}}} \\ y = -\dfrac{ab}{(a^2+b^2)^{\frac{1}{2}}} \end{Bmatrix}.$$

If $a = b$, then the equation (1) becomes

$$(x - y) \cdot (x^2 + xy + y^2 - a^2) = 0,$$

which represents a straight line EE' and an ellipse $A\alpha B'\beta' A'\alpha' B\beta$, (fig. 41), the semi-axes of which are

$$O\alpha = a\sqrt{2} = O\alpha',$$

and $$O\beta = a\sqrt{\tfrac{2}{3}} = O\beta'.$$

The locus of P consists of the lines

$$EF,\ E'F',\ B\alpha' A',\ B'\alpha A.$$

5. The distances of the successive angular points of a plane polygon from a given point O within its area are given. Supposing the polygonal area to be the greatest possible, prove that, C_{r-1}, C_r, C_{r+1}, being any three consecutive angular points, no two of which are in a line with O, the line $C_{r-1}C_{r+1}$ is perpendicular to the distance OC_r.

Let OC_1, OC_2, OC_3, ... (fig. 42) be denoted respectively by $c_1, c_2, c_3, \ldots\ldots$, and the angles C_1OC_2, C_2OC_3, C_3OC_4, ... by θ_1^2, θ_2^2, θ_3^2, ... Then, u denoting the area of the polygon,

$$2u = c_1c_2 \sin\theta_1^2 + c_2c_3 \sin\theta_2^2 + c_3c_4 \sin\theta_3^2 + \ldots,$$

and $$2\pi = \theta_1^2 + \theta_2^2 + \theta_3^2 + \ldots$$

Differentiating these equations and putting $du = 0$, we have, λ being an arbitrary multiplier,

$$\lambda\theta_{r-1} = c_{r-1}c_r \cos\theta^2_{r-1} \,.\, \theta_{r-1},$$

$$\lambda\theta_r = c_rc_{r+1} \cos\theta_r^2 \,.\, \theta_r.$$

Hence, supposing neither θ_{r-1} nor θ_r to be zero,

$$c_{r-1} \cos\theta^2_{r-1} = c_{r+1} \cos\theta_r^2.$$

This result establishes the proposition.

Aliter. Let E (fig. 42) be the intersection of C_2O with C_1C_3 Then, the positions of all the radial lines except OC_2 being assigned, the triangular area $C_1C_2C_3$, and therefore the whole polygonal area, will be greatest, when the distance of C_2 from C_1C_3 is greatest, which, since OC_2 is given, will evidently be the case when C_2O is at right angles to C_1C_3, unless C_2 lie in the lines OC_1, OC_3, or these lines produced. Like remarks are applicable to all the other radial lines: hence the truth of the proposition.

6. A rectangular column is formed by placing a number of smooth cubical blocks one above another, the base of the column resting upon a horizontal plane. All the blocks above the lowest are then twisted in the same direction about an edge of the column, first the highest, then the two highest, and so on, in each case as far as is consistent with equilibrium. Prove that the sum of the sines of the inclinations of a diagonal

of the base of any block to the like diagonals of the bases of all the blocks above it is equal to the sum of the cosines.

Take the two sides of the base of any block, which terminate in the edge, as axes of x and y: let this be the n^{th} block from the top. Let $ABCO$ (fig. 43) be the projection of the base of the highest block upon the plane of x, y. Let $\angle AOx = \theta_1$ Then, $(\bar{x}, \bar{y})$ being the projection of the centre of gravity of this block, and $2a$ denoting the length of an edge of any block,

$$\bar{x} = a(\cos\theta_1 - \sin\theta_1).$$

Similarly for the projections of the centres of gravity of all other blocks. Hence, X being the abscissa of the projection of the centre of gravity of all the blocks above the n^{th},

$$(n-1)\frac{X}{a} = \cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \dots + \cos\theta_{n-1}$$
$$- (\sin\theta_1 + \sin\theta_2 + \sin\theta_3 + \dots + \sin\theta_{n-1}).$$

But, under the conditions of the problem, the point (X, Y) must lie in Oy. Hence $X = 0$, and therefore

$$\sin\theta_1 + \sin\theta_2 + \sin\theta_3 + \dots + \sin\theta_{n-1}$$
$$= \cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \dots + \cos\theta_{n-1}.$$

7. A uniform chain of length l hangs over two fixed points, which are in a horizontal line: from its middle point is suspended by one end another chain of equal thickness and of length l'. Supposing each of the two tangents of the former chain at its middle point to make an angle θ with the vertical, find the distance between the two fixed points.

Shew that the value of θ can never exceed that given by the equation

$$\tan^2\frac{\theta}{2} = \frac{l-l'}{l+l'}.$$

Complete the catenary of which BC (fig. 44) is a portion. Let τ = the tension at B. BK is the suspended chain.

Let $CF = z$, $AO = c$, $OE = \alpha$, $OF = \beta$, $CC' = 2EF = u$, m = the mass of a unit of length of the chain.

Then $$\tfrac{1}{2}l - z = \tfrac{1}{2}c\left(e^{\frac{\beta}{c}} - e^{-\frac{\beta}{c}} - e^{\frac{\alpha}{c}} + e^{-\frac{\alpha}{c}}\right) \text{ .... (1)},$$

$$\beta - \alpha = \tfrac{1}{2}u \text{ .... (2)}.$$

Also $$2\tau \cos\theta = mgl',$$

$$\tau = mgBE,$$

and therefore $$2\cos\theta \,.\, BE = l',$$

or $$c\cos\theta \,.\left(e^{\frac{\alpha}{c}} + e^{-\frac{\alpha}{c}}\right) = l' \text{ ....(3)}.$$

Again, putting $\cot\theta = \dfrac{dy}{dx}$ at B,

$$\cot\theta = \tfrac{1}{2}\left(e^{\frac{\alpha}{c}} - e^{-\frac{\alpha}{c}}\right) \text{ .... (4)}.$$

Also $$z = \tfrac{1}{2}c\left(e^{\frac{\beta}{c}} + e^{-\frac{\beta}{c}}\right) \text{ .... (5)}.$$

From (1) and (5),

$$\tfrac{1}{2}l - c\,.\,e^{\frac{\beta}{c}} = -\tfrac{1}{2}c\left(e^{\frac{\alpha}{c}} - e^{-\frac{\alpha}{c}}\right) \text{ .... (6)}.$$

From (4),

$$4\operatorname{cosec}^2\theta = \left(e^{\frac{\alpha}{c}} + e^{-\frac{\alpha}{c}}\right)^2, \quad 2\operatorname{cosec}\theta = e^{\frac{\alpha}{c}} + e^{-\frac{\alpha}{c}},$$

and therefore $$e^{\frac{\alpha}{c}} = \cot\theta + \operatorname{cosec}\theta = \cot\frac{\theta}{2} \text{ .... (7)}.$$

From (3) and (7),

$$c\cos\theta\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right) = l', \quad c = \tfrac{1}{2}l'\tan\theta \text{ ...... (8)}.$$

From (2) and (7),

$$\frac{\beta}{c} = \frac{u}{2c} + \log\left(\cot\frac{\theta}{2}\right)$$

by (8), $$= \frac{u}{l'\tan\theta} + \log\left(\cot\frac{\theta}{2}\right) \text{ .... (9)}.$$

From (4) and (6),

$$\tfrac{1}{2}l - c\,e^{\frac{\beta}{c}} = -c\cot\theta,$$

and therefore, by (8) and (9),

$$\tfrac{1}{2}l - \tfrac{1}{2}l' \,.\, \tan\theta \,.\, \cot\frac{\theta}{2} \,.\, e^{\frac{u}{l'\tan\theta}} = - \tfrac{1}{2}l',$$

$$u = l' \tan\theta \,.\, \log\left\{\frac{l+l'}{l'} \,.\, \frac{\tan\frac{\theta}{2}}{\tan\theta}\right\} \quad\text{............ (10).}$$

From (9) and (10),

$$\frac{\beta}{c} = \log\left(\frac{l+l'}{l'\tan\theta}\right),$$

and therefore, by (5) and (8),

$$z = \tfrac{1}{4}l' \tan\theta \,.\, \left(\frac{l+l'}{l'\tan\theta} + \frac{l'\tan\theta}{l+l'}\right)$$

$$= \frac{(l+l')^2 + l'^2\tan^2\theta}{4(l+l')} \quad\text{....................... (11).}$$

In order that u, given by (10), may have a positive value, we must have

$$(l+l')\tan\frac{\theta}{2} > l'\tan\theta, \quad 2l' < l + l' - (l+l')\tan^2\frac{\theta}{2},$$

$$\tan^2\frac{\theta}{2} < \frac{l-l'}{l+l'}.$$

If $$\tan^2\frac{\theta}{2} = \frac{l-l'}{l+l'}, \quad\text{or}\quad \tan^2\theta = \frac{l^2-l'^2}{l'^2},$$

then, from (10) and (11),

$$u = 0 \quad\text{and}\quad z = \tfrac{1}{4}(l+l'+l-l') = \tfrac{1}{2}l,$$

and therefore B, C, C', will coincide.

8. If $\dfrac{a^2x^2}{(v^2-a^2)^2} + \dfrac{b^2y^2}{(v^2-b^2)^2} = 1$, and if, for any assigned values of x and y, the expression

$$v^4 \,.\, \left\{\frac{x^2}{(v^2-a^2)^2} + \frac{y^2}{(v^2-b^2)^2}\right\}$$

has only one value, prove that

$$a^2x^2 + b^2y^2 = 4(a^2-b^2)^2$$

Write, for the sake of brevity, a, b, x, y, v, r, instead of a^2, b^2, x^2, y^2, v^2, r^2, respectively, r^2 denoting the value of the expression. Then

$$1 = \frac{ax}{(v-a)^2} + \frac{by}{(v-b)^2} \text{ .... (1)},$$

$$r = v^2 \left\{ \frac{x}{(v-a)^2} + \frac{y}{(v-b)^2} \right\},$$

and therefore, by (1),

$$r = v + x + y + \frac{ax}{v-a} + \frac{by}{v-b} \text{ ....(2)}.$$

Let $v_{,}$, $v_{,,}$ be two of the roots of (1): then, r possessing only one value, we have, putting $v_{,}$, $v_{,,}$ successively in (2), subtracting, and dividing out by $v_{,,} - v_{,}$

$$1 = \frac{ax}{(v_{,}-a)\,(v_{,,}-a)} + \frac{by}{(v_{,}-b)\,(v_{,,}-b)} \text{ ....(3)}.$$

Now (1) has four roots: hence, Σ denoting summation in regard to all its roots, of which there are six pairs, we have, from (3),

$$6 = ax\,.\,\Sigma \left\{ \frac{1}{(v_{,}-a)\,(v_{,,}-a)} \right\} + by\,.\,\Sigma \left\{ \frac{1}{(v_{,}-b)\,(v_{,,}-b)} \right\} \text{ ...(4)}.$$

Putting $v - a = w$, (1) may be transformed into

$$w^4 + \dots + \{(a-b)^2 - ax - by\}\, w^2 \dots - ax\,(a-b)^2 = 0.$$

Hence

$$w_1 w_2 w_3 w_4 = -\,ax\,(a-b)^2, \quad \Sigma\,(w_1 w_2) = (a-b)^2 - ax - by,$$

and therefore

$$\Sigma \left\{ \frac{1}{(v_{,}-a)\,(v_{,,}-a)} \right\} = \Sigma \left(\frac{1}{w_1 w_2} \right) = \frac{ax + by - (a-b)^2}{ax\,(a-b)^2}.$$

From (4), attending to symmetry,

$$ax + by = 4\,(a-b)^2,$$

or, restoring a^2, b^2, ... for a, b, ...,

$$a^2x^2 + b^2y^2 = 4\,(a^2 - b^2)^2.$$

9. A great circle of a sphere intersects two given great circles, drawn through a point O, in points A, B, such that the product of $\tan OA$, $\tan OB$, is invariable. If P be the intersection of this circle with the consecutive one of the series of circles described according to the same law, prove that

$$\cot^2 OP \propto \sin POA \,.\, \sin POB.$$

We will first find the polar equation to AB. Let $\angle XOY = \omega$, (fig. 45), $OA = \alpha$, $OB = \beta$, $\angle POA = \theta$, $\angle POB = \phi$, $\angle BAO = i$. From the triangle AOP we have

$$\cot r \,.\, \sin\alpha = \cot i \,.\, \sin\theta + \cos\alpha \,.\, \cos\theta.$$

Since β, ω, are simultaneous values of r, θ, we see that

$$\cot\beta \,.\, \sin\alpha = \cot i \,.\, \sin\omega + \cos\alpha \,.\, \cos\omega.$$

Eliminating $\cot i$ we get

$$\sin\alpha \,.\, (\cot r \,.\, \sin\omega - \cot\beta \,.\, \sin\theta) = \cos\alpha \,.\, (\sin\omega \,.\, \cos\theta - \cos\omega \,.\, \sin\theta),$$

$$\cot r \,.\, \tan\alpha \,.\, \sin\omega = \tan\alpha \,.\, \sin\theta \,.\, \cot\beta + \sin(\omega - \theta),$$

$$\frac{\sin\omega}{\tan r} = \frac{\sin\theta}{\tan\beta} + \frac{\sin\phi}{\tan\alpha} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the polar equation to AB.

Put $m = \tan\alpha$, $n = \tan\beta$: then, from (1),

$$\frac{\sin\omega}{\tan r} = \frac{\sin\theta}{n} + \frac{\sin\phi}{m} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and, by hypothesis,

$$c^2 = mn \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Differentiating (2) and (3) with regard to the parameters m and n, and using an indeterminate multiplier λ, we have

$$m = \frac{\lambda \sin\theta}{n^2}, \quad c^2 = \frac{\lambda \sin\theta}{n}, \quad c^2 = \frac{\lambda \sin\phi}{m},$$

and therefore

$$c^6 = \lambda^2 \,.\, \sin\theta \,.\, \sin\phi, \quad c^3 = \lambda (\sin\theta \sin\phi)^{\frac{1}{2}}.$$

Hence $$m = c\left(\frac{\sin\phi}{\sin\theta}\right)^{\frac{1}{2}}, \quad n = c\left(\frac{\sin\theta}{\sin\phi}\right)^{\frac{1}{2}}.$$

Hence, from (2), $$\frac{\sin\omega}{\tan r} = \frac{2}{c}(\sin\theta \,.\, \sin\phi)^{\frac{1}{2}},$$

and therefore $$\cot^2 OP \propto \sin POA \,.\, \sin POB.$$

10. Investigate an equation for the form of the floats in the paddle-wheels of a steam vessel, in order that they may enter the water without splashing.

If $u = h\omega$, where u = the velocity of the vessel, ω = the angular velocity of the wheels, and h = the height of the centres of the wheels above the water, shew that the floats of each wheel must have the forms of arcs of involutes of a concentric circle touching the water level.

Let O (fig. 46) be the centre of one of the wheels, KS the line of its intersection with the water-level.

Let the dark line at P, a point in KS, indicate one of the floats entering the water, the dotted line from this dark line to A' pointing out the curve of which the float is an arc. Let OA' be the prime radius vector. Let

$r\omega$ = the velocity of the impact of the water, due to revolution, perpendicular to OP,

u = the velocity of the impact of the water, due to translation, parallel to SK.

Then

$r\omega - u\cos i$ = the whole velocity of the impact of the water, perpendicular to OP,

$u\sin i$ = the whole velocity of the impact of the water along PO.

Consequently, that there may be no splashing, we must have, ϕ being the angle between OP and the curve at P,

$$0 = (r\omega - u\cos i)\,.\cos\phi - u\sin i\,.\sin\phi,$$

$$r\omega - u\cos i = u\sin i\,.\tan\phi = u\sin i\,.\frac{rd\theta}{dr},$$

$$r\omega - u\,.\frac{h}{r} = u\,.\left(1 - \frac{h^2}{r^2}\right)^{\frac{1}{2}}.\frac{rd\theta}{dr},$$

$$d\theta = \frac{\omega r dr}{u(r^2 - h^2)^{\frac{1}{2}}} - \frac{hdr}{r(r^2 - h^2)^{\frac{1}{2}}},$$

whence, supposing OA' to be equal to h, which it will be if its position be properly chosen,

$$\theta = \frac{\omega}{u}(r^2 - h^2)^{\frac{1}{2}} - \sec^{-1}\frac{r}{h},$$

the equation to the curve of which the float is an arc.

If $u = \omega h$, then

$$\theta = \frac{1}{h}(r^2 - h^2)^{\frac{1}{2}} - \sec^{-1}\frac{r}{h},$$

the equation to the involute of a circle described round O such as to touch the water level.

11. A hollow vertical polygonal prism, open at both ends, rests upon a horizontal plane. Every two contiguous faces are moveable about their common edge. Supposing the prism to be in equilibrium, when filled with fluid, prove that

$$\frac{c_1}{\sin\alpha_1} = \frac{c_2}{\sin\alpha_2} = \frac{c_3}{\sin\alpha_3} = \dots\dots,$$

$\alpha_1, \alpha_2, \alpha_3, \dots$ being the angles of a transverse section $A_1A_2A_3\dots A_nA_1$, and $c_1, c_2, c_3, \dots$ denoting the lines A_nA_2, A_1A_3, $A_2A_4, \dots$

Thence shew that there will be equilibrium when the points $A_1, A_2, A_3, \dots$ lie all in the circumference of a circle.

The actions of the faces $A_{n-1}A_n$, A_1A_2, (fig. 47), upon the face A_nA_1, must evidently be equal to each other and inclined at the same angle θ to A_nA_1.

For the equilibrium of A_nA_1, putting $A_nA_1 = a$,

$$2P\sin\theta = \text{the fluid pressure on } A_nA_1$$
$$= \mu a \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Similarly, for the equilibrium of A_1A_2, putting $A_1A_2 = b$,

$$2P\sin(\theta + \alpha_1) = \mu b \dots\dots\dots\dots\dots\dots (2).$$

From (1) and (2),

$$\sin\alpha_1 . \cos\theta = \frac{\mu}{2P}(b - a\cos\alpha_1) \dots\dots\dots\dots (3).$$

From (1),

$$\sin\alpha_1 . \sin\theta = \frac{\mu}{2P} a \sin\alpha_1 \dots\dots\dots\dots\dots\dots (4).$$

$(3)^2 + (4)^2$ gives

$$\sin^2\alpha_1 = \frac{\mu^2}{4P^2}(a^2 + b^2 - 2ab\cos\alpha_1) = \frac{\mu^2 c_1^2}{4P^2},$$

$$\frac{\mu}{2P} = \frac{\sin\alpha_1}{c_1}.$$

Hence, by symmetry,

$$\frac{c_1}{\sin\alpha_1} = \frac{c_2}{\sin\alpha_2} = \frac{c_3}{\sin\alpha_3} = \dots$$

COR. $\dfrac{c_1}{\sin\alpha_1} = \dfrac{A_1A_2}{\sin A_1A_nA_2}$, $\dfrac{c_2}{\sin\alpha_2} = \dfrac{A_1A_2}{\sin A_1A_3A_2}$;

and therefore there will be equilibrium if

$$\angle A_1A_nA_2 = \angle A_1A_3A_2,$$

that is if the circle, passing through $A_3A_2A_1$, passes also through A_n; if that through $A_2A_1A_n$ passes also through A_{n-1}, and so on. Thus we see that there will be equilibrium when the polygon is in a circle.

12. A filament of fluid oscillates in a thin cycloidal tube of uniform bore, the axis of the cycloid being vertical and its vertex downwards. Supposing the filament to be placed initially with its lower end at the lowest point of the tube, find the pressure at any point of the filament at any time.

Shew that the pressure is a maximum, during the whole motion, at the middle point of the filament.

Let P', P'', (fig. 48), be the ends of the filament at any time, $l =$ the whole length $P'AP''$ of the filament, $AP = s$, $AP' = s'$, $AP'' = s''$, whence also $s' + s'' = l$.

For the motion of the filament

$$l\frac{d^2s'}{dt^2} = -g\int\frac{dx}{ds}\,.\,ds = -g(x' - x'') = -\frac{g}{8a}(s'^2 - s''^2) = -\frac{gl}{8a}(s' - s'')$$

$$= -\frac{gl}{8a}(2s' - l),$$

$$\frac{d^2s'}{dt^2} + \frac{g}{4a}(s' - \tfrac{1}{2}l) = 0, \quad s' - \tfrac{1}{2}l = A\sin\left\{\left(\frac{g}{4a}\right)^{\frac{1}{2}}.\,t + \varepsilon\right\}.$$

Initially, $s' = l$, $\frac{ds'}{dt} = 0$. Hence $\frac{1}{2}l = A \sin \epsilon$, $0 = A \cos \epsilon$, and therefore $\epsilon = \frac{1}{2}\pi$, $A = \frac{1}{2}l$. Hence

$$s' = \tfrac{1}{2}l\{1 + \cos nt\}, \quad \text{where } n = \left(\frac{g}{4a}\right)^{\frac{1}{2}}.$$

The equation for the pressure p at any point is

$$(C+s)\frac{du'}{dt} + \tfrac{1}{2}u'^2 = -gx - \frac{1}{\rho}p,$$

u' denoting $\frac{ds'}{dt}$ and C a constant.

Let Π be the atmospheric pressure: then

$$(C+s')\frac{du'}{dt} + \tfrac{1}{2}u'^2 = -gx' - \frac{1}{\rho}\Pi;$$

hence

$$(s-s')\frac{du'}{dt} = g(x'-x) - \frac{1}{\rho}(p-\Pi)$$

$$= \frac{g}{8a}(s'^2 - s^2) - \frac{1}{\rho}(p-\Pi),$$

$$(s-s')\frac{d^2s'}{dt^2} = \tfrac{1}{2}n^2(s'^2 - s^2) - \frac{1}{\rho}(p-\Pi),$$

$$\frac{1}{\rho}(p-\Pi) = (s'-s).\left\{\frac{d^2s'}{dt^2} + \tfrac{1}{2}n^2(s'+s)\right\}$$

$$= \{\tfrac{1}{2}l + \tfrac{1}{2}l\cos nt - s\}.\frac{n^2}{2}.\{s + \tfrac{1}{2}l - \tfrac{1}{2}l\cos nt\}$$

$$= \frac{n^2}{8}\{l^2 - (2s - l\cos nt)^2\},$$

which gives the value of p at every point of the filament at any time.

It is evident from the result that p is greatest when

$$s = \tfrac{1}{2}l\cos nt:$$

but

$$s' = \tfrac{1}{2}l\{1 + \cos nt\};$$

hence

$$s' - s = \tfrac{1}{2}l,$$

or the point of greatest pressure coincides with the middle point of the filament.

This problem may also be solved as if the fluid filament were a string. Putting $PP' = l_1$, $P P'' = l_2$, $P =$ the reaction between the two parts of the string at P, we have

$$l_1 \frac{d^2s'}{dt^2} = P - g\int_s^{s'} \frac{dx}{ds}\, ds = P - \frac{g}{8a}(s'^2 - s^2),$$

$$l_2 \frac{d^2s'}{dt^2} = -P - g\int_{s''}^{s} \frac{dx}{ds}\, ds = -P - \frac{g}{8a}(s^2 - s''^2),$$

and therefore $$l\frac{d^2s'}{dt^2} = -\frac{g}{8a}(s'^2 - s''^2).$$

As in the former solution, we have $s' = \frac{1}{2}l(1 + \cos nt)$, and therefore

$$-\tfrac{1}{2}n^2 l \,.\, \cos nt \,.\, (s' - s) = P - \frac{g}{8a}(s'^2 - s^2):$$

differentiating with regard to s,

$$\tfrac{1}{2}n^2 l \cos nt = \frac{dP}{ds} + \frac{g}{4a}s, \quad 0 = \frac{d^2P}{ds^2} + \frac{g}{4a}.$$

Put $\dfrac{dP}{ds} = 0$: then

$$s = \frac{2a}{8}\,.\, n^2 \,.\, l \,.\, \cos nt = \tfrac{1}{2}l \cos nt, \quad s' - s = \tfrac{1}{2}l.$$

Thus the middle point of $P'P''$ is the one of maximum reaction.

13. A ray experiences a series of reflections between two plane inclined mirrors. Prove that all the segments of the ray, produced indefinitely, are tangents to every one of an infinite series of spheres.

LEMMA. If a ray incur reflection at a plane mirror, the incident and reflected rays are equally inclined to any straight line in the mirror.

Let PO, OQ (fig. 49), be the incident and reflected rays at a point O of the mirror, EF the intersection of the plane POQ with the mirror. Through O draw any line HOK in the plane of the mirror. Now PO, HO, are in precisely the same attitude on one side as QO, KO, on the other. Hence $\angle POH = \angle QOK$. Q. E. D.

Let A_1B_1, B_1A_2, A_2B_2 (fig. 50), be any three consecutive segments of the ray. Let O be any point in the line of intersection of the mirrors. With O as a centre describe a sphere to touch A_1B_1 (produced if necessary) in the point C_1. Join OB_1. Now, by the lemma, B_1A_2 and B_1C_1 make equal angles with OB_1: hence evidently B_1A_2 must touch the sphere which A_1B_1 touches, in some point D_1. Again, joining OA_2, and observing that $\angle B_2A_2O = \angle D_1A_2O$, we see that A_2B_2 (produced if necessary) will touch in some point C_2 the same sphere which B_1A_2 touches. So on indefinitely. Thus we see that all the segments are tangents to one sphere described about O. But O is any point in the line of intersection of the mirrors. Hence the number of such spheres is infinite.

14. A narrow self-luminous rectangular lamina is placed with one end at the edge of a circular plate: the lamina is at right angles to the plate, and its plane passes through the centre of the plate: find the whole illumination on the plate.

If the length of the lamina be equal to the diameter of the plate, its intrinsic brightness and breadth being given, prove that the illumination varies as the diameter of the plate.

Let c = the length of the lamina, τ = its breadth, a = the radius of the plate. Let u = the distance of any point P (fig. 51) in the plate from any point Q in the lamina, r = the distance of P from the point O where the lamina touches the plate; and $QO=z$. Let the axis of x coincide with the diameter through O, the axis of y being perpendicular to it in the plane of the plate. Let ϕ be the inclination of u to the plate and ψ to the axis of y, θ the inclination of r to the axis of x, I the illumination on the plate. Then, $d_zd_rd_\theta I$ denoting the illumination on the element $rd\theta dr$ of the plate, derived from an element τdz of the lamina, and μ a constant quantity,

$$d_zd_rd_\theta I = rd\theta dr \,.\, \frac{\mu}{u^2} \,.\, \sin\phi \,.\, \cos\psi \,.\, \tau dz:$$

but $$\sin\phi = \frac{z}{u}, \quad u\cos\psi = r\sin\theta,$$

$$u^2 = r^2 + z^2.$$

Hence $$d_z d_r d_\theta I = \mu\tau\, dz\, dr\, d\theta \,.\, \frac{zr^2 \sin\theta}{(z^2+r^2)^2}\,.$$

Integrating with regard to z from $z=0$ to $z=c$,

$$d_r d_\theta I = \frac{1}{2}\mu\tau r^2 dr\, d\theta \sin\theta \left(\frac{1}{r^2} - \frac{1}{r^2+c^2}\right),$$

$$= \frac{1}{2}\mu\tau c^2 \,.\, \frac{dr\, d\theta \sin\theta}{r^2+c^2}\,.$$

Integrating with regard to r, from $r=0$ to $r=2a\cos\theta$,

$$d_\theta I = \frac{1}{2}\mu\tau c^2 \sin\theta\, d\theta \,.\, \frac{1}{c}\tan^{-1}\frac{r}{c}$$

$$= \frac{1}{2}\mu\tau c \,.\, \sin\theta d\theta \,.\, \tan^{-1}\left(\frac{2a\cos\theta}{c}\right).$$

Now $\int_0^{\frac{1}{2}\pi} \sin\theta d\theta \tan^{-1}\left(\frac{2a\cos\theta}{c}\right)$

$$= \int_0^1 \tan^{-1}\left(\frac{2av}{c}\right) dv = v\tan^{-1}\left(\frac{2av}{c}\right) - \frac{c}{2a}\int \frac{v\,dv}{v^2+\frac{c^2}{4a^2}}$$

$$= \tan^{-1}\left(\frac{2a}{c}\right) - \frac{c}{4a}\log\left(v^2+\frac{c^2}{4a^2}\right) = \tan^{-1}\left(\frac{2a}{c}\right) - \frac{c}{4a}\log\frac{4a^2+c^2}{c^2}\,.$$

Hence $I = \frac{1}{2}\mu\tau c \,.\left\{\tan^{-1}\left(\frac{2a}{c}\right) - \frac{c}{4a}\log\left(\frac{4a^2+c^2}{c^2}\right)\right\}$,

the illumination of one half of the plate.

COR. Let $2a=c$: then the whole illumination is equal to

$$\mu\tau c\left\{\frac{\pi}{4} - \frac{1}{2}\log 2\right\}$$

$$= \frac{1}{4}\mu\tau c(\pi - \log 4) = \frac{1}{4}\mu\tau c \log\left(\frac{e^\pi}{4}\right),$$

or the illumination varies as the diameter of the plate.

15. Prove that an infinite number of plane centric sections of an hyperboloid of one sheet may be drawn, each possessing the following property, viz. that the normals to the surface at

the curve of section all pass through two straight lines lying in the same plane with the two possible axes.

Shew that these centric planes envelope the asymptotic cone, while the two straight lines envelope an ellipse.

Let the hyperboloid be denoted by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ .... } (1).$$

Let the equation to a centric section be

$$z = mx + ny \text{ .... } (2).$$

At the intersection of (1) and (2),

$$\left(\frac{1}{a^2} - \frac{m^2}{c^2}\right)x^2 - \frac{2mnxy}{c^2} + \left(\frac{1}{b^2} - \frac{n^2}{c^2}\right)y^2 = 1 \text{ ...... } (3).$$

If x', y', be the point in which the plane of x, y, is intersected by a normal at x, y, z,

$$(x' - x) \cdot \frac{a^2}{x} = (y' - y) \cdot \frac{b^2}{y} = c^2,$$

whence $$x = \frac{a^2 x'}{a^2 + c^2}, \quad y = \frac{b^2 y'}{b^2 + c^2}.$$

Substituting these values of x and y in (3), we see that

$$(c^2 - m^2 a^2) \cdot \frac{a^2}{c^2} \cdot \frac{x'^2}{(a^2 + c^2)^2} - 2mn \cdot \frac{a^2 b^2}{c^2} \cdot \frac{x'y'}{(a^2 + c^2)(b^2 + c^2)}$$
$$+ (c^2 - n^2 b^2) \cdot \frac{b^2}{c^2} \cdot \frac{y'^2}{(b^2 + c^2)^2} = 1 \text{ ...... } (4).$$

In order that this equation may denote two straight lines, we must have

$$(c^2 - m^2 a^2) . (c^2 - n^2 b^2) = m^2 n^2 a^2 b^2, \quad c^2 = m^2 a^2 + n^2 b^2 \text{ ... } (5).$$

From (4) and (5) we get

$$\frac{nx'}{a^2 + c^2} - \frac{my'}{b^2 + c^2} = \pm \frac{c}{ab} \text{ .... } (6),$$

the equation to two straight lines.

It is readily ascertained that the envelope of (2), under the condition (5), is denoted by the equation

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \text{ the asymptotic cone,}$$

and the envelop of (6), under the same condition, by

$$\frac{a^2x'^2}{(a^2+c^2)^2} + \frac{b^2y'^2}{(b^2+c^2)^2} = 1, \text{ an ellipse.}$$

16. Prove that the envelope of a sphere, of which any one of one series of circular sections of an ellipsoid is a diametral plane, is a spheroid touching a sphere, described on the mean axis of the ellipsoid as diameter, in a plane perpendicular to any one of the same series of circular sections.

Let α, 0, γ, be the coordinates of the centre of any one of the series of circular sections; the radius of the section will be equal to the square root of

$$b^2\left(1 - \frac{\alpha^2}{a^2} - \frac{\gamma^2}{c^2}\right).$$

Thus the equation to the corresponding sphere will be

$$(x-\alpha)^2 + y^2 + (z-\gamma)^2 = b^2\left(1 - \frac{\alpha^2}{a^2} - \frac{\gamma^2}{c^2}\right),$$

or $$(\alpha - x)^2 + (\gamma - z)^2 + b^2\left(\frac{\alpha^2}{a^2} + \frac{\gamma^2}{c^2}\right) = b^2 - y^2 \text{ ...... (1)},$$

α and γ being, as we know, subject to the equation

$$\frac{\alpha}{a}(b^2-c^2)^{\frac{1}{2}} - \frac{\gamma}{c}(a^2-b^2)^{\frac{1}{2}} = 0 \text{ .... (2).}$$

Differentiating (1) and (2) with relation to α and γ, and making use of an arbitrary multiplier λ, we get

$$\frac{\lambda}{a}(b^2-c^2)^{\frac{1}{2}} = \alpha - x + \frac{b^2}{a^2}\alpha,$$

and $$-\frac{\lambda}{c}(a^2-b^2)^{\frac{1}{2}}=\gamma-z+\frac{b^2}{c^2}\gamma.$$

Multiplying these two equations by α, γ, adding, and attending to (2), we get

$$0=\alpha(\alpha-x)+\gamma(\gamma-z)+b^2\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}\right),$$

and, adding this last equation to (1), we see that

$$\alpha x+\gamma z=x^2+y^2+z^2-b^2 \;\ldots\ldots\ldots\ldots\ldots\; (3).$$

Multiplying (3) by 2, and adding the result to (1), we have

$$\frac{\alpha^2}{a^2}(a^2+b^2)+\frac{\gamma^2}{c^2}(b^2+c^2)=x^2+y^2+z^2-b^2 \;\ldots\ldots(4).$$

From (2) and (3) we easily ascertain that

$$\frac{\alpha}{a}\{ax(a^2-b^2)^{\frac{1}{2}}+cz(b^2-c^2)^{\frac{1}{2}}\}=(x^2+y^2+z^2-b^2).(a^2-b^2)^{\frac{1}{2}},$$

and $$\frac{\gamma}{c}\{ax(a^2-b^2)^{\frac{1}{2}}+cz(b^2-c^2)^{\frac{1}{2}}\}=(x^2+y^2+z^2-b^2).(b^2-c^2)^{\frac{1}{2}},$$

and therefore (4) becomes

$$(x^2+y^2+z^2-b^2)^2.(a^4-c^4)$$
$$=(x^2+y^2+z^2-b^2).\{ax(a^2-b^2)^{\frac{1}{2}}+cz(b^2-c^2)^{\frac{1}{2}}\}^2,$$

and therefore the required envelop has for its equation

$$\{ax(a^2-b^2)^{\frac{1}{2}}+cz(b^2-c^2)^{\frac{1}{2}}\}^2=(a^4-c^4).(x^2+y^2+z^2-b^2),$$

the form of which establishes the truth of the proposition.

The factor $x^2+y^2+z^2-b^2$ has been rejected, because, if $x^2+y^2+z^2-b^2=0$, we get, from (1) and (3),

$$\alpha^2-2\alpha x+\gamma^2-2\gamma z+b^2\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}\right)=0,\quad \alpha x+\gamma z=0,$$

and therefore

$$\alpha^2+\gamma^2+b^2\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}\right)=0,\quad \text{whence } \alpha=0,\ \gamma=0,$$

whereas α, γ, are by the hypothesis *variable* parameters.

17. The Sun's centre, in proceeding from Aries to the Summer Solstice, passes, when at a distance ϕ from the Solstice, through the zenith of a certain place. Prove that, supposing the Earth's orbit circular and the plane of the equator invariable in position, it will not again pass exactly through the zenith of this place in moving from the Solstice to Libra, unless

$$\frac{\tan n\phi}{\tan\phi} = \sec\omega,$$

n denoting the ratio of the Earth's angular velocity about its axis to its angular velocity about the Sun.

Let P (fig. 52) be the pole of the equator, ♈, L, Aries and Libra, E the summer solstice, Z and Z' the positions of the Sun when in the zenith of the place before and after the summer solstice.

Join PZ, PE, PZ', by arcs of great circles, and produce PE to cut the equator in I.

Then $ZE = \phi = Z'E$, $\angle ZPE = n\phi - 2r\pi = \angle Z'PE$, where r is an integer. By the right-angled triangle ZPE or $Z'PE$, we see that

$$\frac{\tan n\phi}{\tan\phi} = \operatorname{cosec} PE = \sec EI = \sec\omega.$$

18. Determine $u_{x,t}$ from the equation

$$c^2 \frac{d^2}{dt^2} u_{x+2,\,t} = \Delta^2 u_{x,t},$$

where Δ affects x only; and, having given the expressions for $u_{x,0}$, $\frac{d}{d0} u_{x,0}$, shew how to determine the values of the arbitrary functions which appear in the result.

If $u_{x,0} = ax + b$, and $\frac{d}{d0} u_{x,0} = a'r^x$, shew from your formulæ that

$$\frac{d}{dt} u_{x,t} = \tfrac{1}{2} a'.r^x.(\mu^t + \mu^{-t}),$$

μ being a constant quantity.

$$c^2 \frac{d^2}{dt^2} u_{x+2,t} = \Delta^2 u_{x,t}, \quad c^2 \frac{d^2}{dt^2} (1+\Delta)^2 u_{x,t} = \Delta^2 u_{x,t},$$

$$\left\{c^2 \frac{d^2}{dt^2} (1+\Delta)^2 - \Delta^2\right\} u_{x,t} = 0,$$

$$u_{x,t} = \frac{1}{2\Delta} \left\{\frac{1}{c\frac{d}{dt}(1+\Delta) - \Delta} - \frac{1}{c\frac{d}{dt}(1+\Delta)+\Delta}\right\} 0,$$

$$2c\Delta(1+\Delta)\, u_{x,t} = \left\{\frac{1}{\frac{d}{dt} - \frac{\Delta}{c(1+\Delta)}} - \frac{1}{\frac{d}{dt} + \frac{\Delta}{c(1+\Delta)}}\right\} 0$$

$$= e^{\frac{t}{c}\frac{\Delta}{1+\Delta}} \phi_1(x) + e^{-\frac{t}{c}\frac{\Delta}{1+\Delta}} \psi_1(x),$$

$$u_{x,t} = e^{\frac{t}{c}\frac{\Delta}{1+\Delta}} v_x + e^{-\frac{t}{c}\frac{\Delta}{1+\Delta}} w_x,$$

v_x and w_x being arbitrary functions of x.

Then $$u_{x,0} = v_x + w_x \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{d}{d0} u_{x,0} = \frac{1}{c}\frac{\Delta}{1+\Delta}(v_x - w_x) = \frac{1}{c}\Delta(v_{x-1} - w_{x-1}),$$

and therefore $$c\Sigma \frac{d}{d0} u_{x+1,0} = v_x - w_x \;\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we get

$$2v_x = u_{x,0} + c\frac{d}{d0}\Sigma u_{x+1,0}, \quad 2w_x = u_{x,0} - c\frac{d}{d0}\Sigma u_{x+1,0}.$$

These formulæ determine the arbitrary functions.

Let $u_{x,0} = ax + b$, $\dfrac{du_{x,0}}{d0} = a'r^x$. Then

$$2v_x = ax + b + ca'\Sigma r^{x+1} = ax + b + \frac{ca'r}{r-1} \cdot r^x,$$

$$2w_x = ax + b - \frac{ca'r}{r-1} \cdot r^x.$$

Hence $$2e^{\frac{t}{c}\frac{\Delta}{1+\Delta}} v_x = e^{\frac{t}{c}\frac{\Delta}{1+\Delta}}\left(ax+b+\frac{ca'r}{r-1}\,.\,r^x\right)$$

$$= ax+b+\frac{at}{c}+\frac{ca'r}{r-1}\left\{1+\frac{t}{c}\,\frac{\Delta}{1+\Delta}+\frac{t^2}{1.2.c^2}\left(\frac{\Delta}{1+\Delta}\right)^2 + \frac{t^3}{1.2.3.c^3}\left(\frac{\Delta}{1+\Delta}\right)^3+\ldots\right\}r^x$$

$$= ax+b+\frac{at}{c}+\frac{ca'r}{r-1}\left\{r^x+\frac{t}{c}\,\frac{r-1}{r}\,r^x+\frac{t^2}{1.2.c^2}\left(\frac{r-1}{r}\right)^2 r^x + \frac{t^3}{1.2.3.c^3}\left(\frac{r-1}{r}\right)^3 r^x+\ldots\right\}$$

$$= ax+b+\frac{at}{c}+\frac{ca'r^{x+1}}{r-1}\,.\,e^{\frac{t(r-1)}{cr}}.$$

Similarly, putting $-c$ for c,

$$2e^{-\frac{t}{c}\frac{\Delta}{1+\Delta}} w_x = ax+b-\frac{at}{c}-\frac{ca'r^{x+1}}{r-1}\,.\,e^{-\frac{t(r-1)}{cr}}.$$

Hence $$u_{x,t} = ax+b+\frac{ca'r^{x+1}}{2(r-1)}\left\{e^{\frac{t}{cr}(r-1)}-e^{-\frac{t}{cr}(r-1)}\right\},$$

$$\frac{d}{dt}\,u_{x,t} = \frac{a'r^x}{2}\left\{e^{\frac{t}{cr}(r-1)}+e^{-\frac{t}{cr}(r-1)}\right\}$$

$$= \tfrac{1}{2}a'\,.\,r^x\,.\,(\mu^t+\mu^{-t}),$$

μ being a constant.

19. Determine the differential equation to a family of curves which possess the following property: if we take in one of the curves any three points P, P', P'', so related that C', C'', the centres of curvature at P', P'', lie respectively in the ordinates PM, $P'M'$, produced if necessary, the ratio of $M'M''$ to MM' shall be invariable.

Shew from your result that the Elastica, the equation to which is

$$dy = \frac{x^2dx}{(c^4-x^4)^{\frac{1}{2}}},$$

is an individual of the family.

Let x, y, be the coordinates of P (fig. 53), and let $f(x)$ denote the length MM'. Then, the abscissa of P' being $x+f(x)$, $M'M''$ will be equal to $f\{x+f(x)\}$.

Hence, by the condition of the problem,

$$f\{x+f(x)\} = \lambda f(x), \text{ where } \lambda \text{ is a constant.}$$

Let $x+f(x) = \psi(x)$, or $f(x) = \psi(x) - x$: then

$$f\{x+f(x)\} = f\{\psi(x)\} = \psi^2(x) - \psi(x).$$

Thus

$$\psi^2(x) - \psi(x) = \lambda\psi(x) - \lambda x, \quad \psi^2(x) - (1+\lambda)\psi(x) + \lambda x = 0.$$

Assume $\psi(x) = \beta x$: then

$$\beta^2 - (1+\lambda)\beta + \lambda = 0:$$

hence $$\beta = 1 \quad \text{or} \quad \beta = \lambda,$$

and therefore $$\psi(x) = x \quad \text{or} \quad \psi(x) = \lambda x,$$

whence $$f(x) = 0 \quad \text{or} \quad f(x) = (\lambda - 1)x.$$

The former value of $f(x)$ must evidently be rejected: the latter shews that

$$MM' = (\lambda-1)x = (\lambda-1)(x' - MM') = \frac{\lambda-1}{\lambda}x',$$

and therefore, by the differential calculus,

$$\frac{\lambda-1}{\lambda}x' = \frac{dy'}{dx'} \cdot \frac{1+\dfrac{dy'^2}{dx'^2}}{\dfrac{d^2y'}{dx'^2}},$$

or, dropping accents,

$$\frac{\lambda}{\lambda-1} \cdot \frac{1}{x} = \frac{\dfrac{d^2y}{dx^2}}{\dfrac{dy}{dx}\left(1+\dfrac{dy^2}{dx^2}\right)} = \frac{\dfrac{d^2y}{dx^2}}{\dfrac{dy}{dx}} - \frac{\dfrac{dy}{dx} \cdot \dfrac{d^2y}{dx^2}}{1+\dfrac{dy^2}{dx^2}},$$

$$\frac{\lambda}{\lambda-1}\log(ax) = \log\left(\frac{dy}{dx}\right) - \log\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}, \ a \text{ being a constant,}$$

$$a^{\frac{\lambda}{\lambda-1}} \cdot x^{\frac{\lambda}{\lambda-1}} = \frac{\dfrac{dy}{dx}}{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}}.$$

Put $\dfrac{\lambda}{\lambda-1}=n,\ a=\dfrac{1}{c}$: then

$$\frac{x^n}{c^n}=\frac{p}{(1+p^2)^{\frac{1}{2}}},\quad p^2=\frac{x^{2n}}{c^{2n}-x^{2n}},\quad dy=\frac{x^n dx}{(c^{2n}-x^{2n})^{\frac{1}{2}}}.$$

The constant ratio $\lambda=\dfrac{n}{n-1}$. If $n=2$, the curve is the Elastica and the ratio $=2$.

Verification. $\dfrac{dy}{dx}=\dfrac{x^n}{(c^{2n}-x^{2n})^{\frac{1}{2}}},\quad \dfrac{d^2y}{dx^2}=\dfrac{nc^{2n}x^{n-1}}{(c^{2n}-x^{2n})^{\frac{3}{2}}},$

and therefore, ξ denoting the abscissa of C,

$$x-\xi=\frac{dy}{dx}\cdot\frac{1+\dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}}=\frac{x}{n}.$$

Hence $\qquad MM'=\dfrac{OM'}{n},\quad M'M''=\dfrac{OM''}{n},$

and therefore

$$M'M''-MM'=\frac{M'M''}{n},\quad M'M''=\frac{n}{n-1}.MM'.$$

20. A small heavy insect, placed at an end of the horizontal diameter of a thin heavy motionless ring, which is moveable about its centre in a vertical plane, starts off to crawl round the ring so as to describe in space equal angles in equal times about its centre. Determine its velocity relatively to the ring in any position.

Let P (fig. 54) be the insect at any time after starting, O the centre of the ring, Ox a horizontal line.

Let $a=$ the radius of the ring, $m=$ its mass, $\mu=$ the mass of the insect, $\theta=$ the inclination of OP to Ox, $\omega=$ the constant value of $\dfrac{d\theta}{dt}$. Let N, T, denote the normal and tangential actions respectively between the ring and the insect.

Then, for the motion of the insect,

$$\mu \frac{d^2x}{dt^2} = N\cos\theta - T\sin\theta,$$

$$\mu \frac{d^2y}{dt^2} = N\sin\theta + T\cos\theta - \mu g.$$

But $\quad x = a\cos\theta, \quad \frac{dx}{dt} = -a\omega\sin\theta, \quad \frac{d^2x}{dt^2} = -a\omega^2\cos\theta.$

$$y = a\sin\theta, \quad \frac{dy}{dt} = \quad a\omega\cos\theta, \quad \frac{d^2y}{dt^2} = -a\omega^2\sin\theta.$$

Hence
$$-\mu a\omega^2\cos\theta = N\cos\theta - T\sin\theta,$$
$$-\mu a\omega^2\sin\theta = N\sin\theta + T\cos\theta - \mu g.$$

From these two equations we see that

$$T = \mu g\cos\theta, \quad N = \mu(g\sin\theta - a\omega^2).$$

For the motion of the ring, Ω denoting its angular velocity,

$$ma^2\frac{d\Omega}{dt} = -Ta = -\mu ag\cos\theta.$$

Let α = the angular velocity of the insect relatively to the ring: then

$$\Omega + \alpha = \omega, \quad \text{and therefore } ma\frac{d\alpha}{dt} = \mu g\cos\theta.$$

Let the time be dated from the instant of the insect's being in Ox: then

$$ma\frac{d\alpha}{dt} = \mu g\cos\omega t, \quad ma\alpha = \frac{\mu g}{\omega}\sin\omega t + C.$$

Let α' be the value of α when t is zero: then

$$a\alpha = \frac{\mu g}{m\omega}\sin\omega t + a\alpha',$$

or the relative velocity of the insect at P = its relative velocity at $A + \frac{\mu g}{m\omega}\sin\theta$.

Suppose the ring to be initially at rest, the insect to be placed at A, and then to start suddenly to move as stated in the problem: then, $\Omega_{\prime}$ being the value of Ω just after the insect starts,

$$ma^2\Omega_{\prime} + \mu a^2\omega = 0, \quad m\Omega_{\prime} + \mu\omega = 0.$$

Hence, attending to the equation $\Omega + \alpha = \omega$, we have

$$\Omega_{\prime} + \alpha' = \omega, \quad -\mu\omega + m\alpha' = m\omega, \quad \alpha' = \frac{m+\mu}{m}\omega.$$

Hence

$$a\alpha = \frac{\mu g}{m\omega} \cdot \sin\omega t + a\frac{m+\mu}{m}\omega = \text{the relative velocity of the insect.}$$

It may be observed that

$$T = \mu g \cos\omega t, \quad N = \mu(g \sin\omega t - a\omega^2).$$

21. A series of perfectly rough semicylinders are fixed, side by side, upon their flat faces directly across a straight road of constant inclination. Determine the inclination of the road in order that a rough circular inelastic hoop, just started downwards from the summit of one of the cylindrical ridges, may travel directly along the road with a uniform mean velocity.

Let a = the radius of the hoop, a_1 = that of one of the cylinders, m = the mass of the hoop, u = the velocity of the hoop's centre just before and u' just after collision: let ω, ω', denote the angular velocities of the hoop just before and just after.

Then, see (fig. 55),

$$mu' = mu\cos 2\alpha + R,$$

and
$$ma^2\omega' = ma^2\omega - Ra,$$

whence
$$u' + a\omega' = u\cos 2\alpha + a\omega.$$

But $a\omega = u$, $a\omega' = u'$: hence

$$2u' = u(1 + \cos 2\alpha),$$

$$u' = u\cos^2\alpha.$$

But, by the condition of the problem,

$$u^2 + a^2\omega^2 = 2u^2 = 2g(a+a_1)(1-\cos\theta) = 4g(a+a_1)\sin^2\frac{\theta}{2},$$

$$u = \{2g(a+a_1)\}^{\frac{1}{2}}\sin\frac{\theta}{2}.$$

Similarly, $$u' = \{2g(a+a_1)\}^{\frac{1}{2}}\sin\frac{\theta'}{2},$$

since u' is lost in ascending the next ridge.

Hence $$\sin\frac{\theta'}{2} = \sin\frac{\theta}{2}\,.\cos^2\alpha.$$

But $$\theta = \alpha+\beta, \quad \theta' = \alpha-\beta.$$

Hence $$\cos^2\alpha = \frac{\sin\dfrac{\alpha-\beta}{2}}{\sin\dfrac{\alpha+\beta}{2}},$$

an equation which determines β, α being given by the equation

$$\sin\alpha = \frac{a_1}{a+a_1}.$$

22. A brittle rod AB, attached to smooth hinges at A and B, is attracted towards a centre of force C according to the law of nature. Supposing the absolute force to be indefinitely augmented, prove that the rod will eventually snap at a point E, the position of which is defined by the equation

$$\cos\angle AEC = \frac{\sin\dfrac{\alpha-\beta}{2}}{\sin\dfrac{\alpha+\beta}{2}},$$

where α, β, denote the angles BAC, ABC, respectively.

Draw CM, CN, (fig. 56), bisecting the angles ACE, BCE. The force of C on AE is, by a known proposition, equal to $\frac{2\mu}{c}\sin\frac{\theta}{2}$, and acts along MC, c being the perpendicular distance of C from AB.

Let $\kappa =$ the wrench impressed upon AE by BE to preserve the equilibrium of AE. Then, Y denoting the action at E perpendicular to AB, for the equilibrium of AE there is

$$\kappa = Y.AE + \frac{2\mu}{c}\sin\frac{\theta}{2}.AM.\sin\left(\alpha+\frac{\theta}{2}\right).$$

Similarly, for the equilibrium of BE,

$$\kappa = -Y.BE + \frac{2\mu}{c}\sin\frac{\phi}{2}.BN.\sin\left(\beta+\frac{\phi}{2}\right).$$

Hence $$\kappa.AB = \frac{2\mu}{c}\left\{AM.BE.\sin\frac{\theta}{2}.\sin\left(\frac{\theta}{2}+\alpha\right)\right.$$
$$\left.+ BN.AE.\sin\frac{\phi}{2}.\sin\left(\frac{\phi}{2}+\beta\right)\right\}.$$

Now $AM.\sin\left(\frac{\theta}{2}+\alpha\right) = AC.\sin\frac{\theta}{2}$, and $BE = BC.\dfrac{\sin\phi}{\sin(\phi+\beta)}$.

Hence $$AM.BE.\sin\frac{\theta}{2}.\sin\left(\frac{\theta}{2}+\alpha\right) = AC.BC.\frac{\sin^2\frac{\theta}{2}.\sin\phi}{\sin(\phi+\beta)}.$$

Similarly, $$BN.AE.\sin\frac{\phi}{2}.\sin\left(\frac{\phi}{2}+\beta\right) = AC.BC.\frac{\sin^2\frac{\phi}{2}.\sin\theta}{\sin(\theta+\alpha)}.$$

Hence $$\frac{c.\kappa.AB}{2\mu.AC.BC} = \frac{\sin^2\frac{\phi}{2}.\sin\theta}{\sin(\theta+\alpha)} + \frac{\sin^2\frac{\theta}{2}.\sin\phi}{\sin(\phi+\beta)}.$$

But $\theta+\alpha+\phi+\beta=\pi$: hence the left-hand member is equal to

$$2\sin\frac{\theta}{2}.\sin\frac{\phi}{2}.\frac{\sin\frac{\theta+\phi}{2}}{\sin(\theta+\alpha)} = 2\cos\frac{\alpha+\beta}{2}.\frac{\sin\frac{\theta}{2}.\sin\frac{\phi}{2}}{\sin(\theta+\alpha)}$$

$$= \cos\frac{\alpha+\beta}{2}.\frac{\cos\frac{\theta-\phi}{2} - \cos\frac{\theta+\phi}{2}}{\sin(\theta+\alpha)}$$

$$= \cos\frac{\alpha+\beta}{2}.\frac{\sin\left(\frac{\alpha+\beta}{2}+\theta\right) - \sin\frac{\alpha+\beta}{2}}{\sin(\theta+\alpha)}.$$

This is a positive quantity, because

$$\sin\left(\frac{\alpha+\beta}{2}+\theta\right)-\sin\frac{\alpha+\beta}{2}=2\sin\frac{\theta}{2}.\cos\frac{\alpha+\beta+\theta}{2},$$

$$\text{and}\quad \frac{\alpha+\beta+\theta}{2}<\frac{\pi}{2}.$$

When κ is a maximum, $\dfrac{\sin\left(\frac{\alpha+\beta}{2}+\theta\right)-\sin\frac{\alpha+\beta}{2}}{\sin(\theta+\alpha)}$ is a maximum.

Hence $f(\theta)=\sin(\theta+\alpha).\cos\left(\frac{\alpha+\beta}{2}+\theta\right)$

$$-\cos(\theta+\alpha).\left\{\sin\left(\frac{\alpha+\beta}{2}+\theta\right)-\sin\frac{\alpha+\beta}{2}\right\}$$

$$=\sin\frac{\alpha-\beta}{2}+\sin\frac{\alpha+\beta}{2}.\cos(\theta+\alpha)=0.$$

$$f'(\theta)=-\sin\frac{\alpha+\beta}{2}.\sin(\theta+\alpha)=\text{a negative quantity.}$$

Hence the point, where fracture will take place, is given by the equation

$$\cos(\theta+\alpha)=-\frac{\sin\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}},$$

or
$$\cos\angle AEC=\frac{\sin\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}}.$$

23. A vessel, of given capacity, in the form of a surface of revolution with two circular ends, is just filled with inelastic fluid which revolves about the axis of the vessel, and is supposed to be free from the action of gravity: investigate the form of the vessel that the whole pressure which the fluid exerts upon it may be the least possible, the magnitudes of the circular ends being given.

Shew that, for a certain relation between the radii of the circular ends, the generating curve of the surface of revolution is the common catenary.

Let the axis of the vessel be taken as the axis of x (fig. 57). Then

Pressure on curve surface $= \int 2\pi y ds . \frac{1}{2}\rho\omega^2 y^2 = \pi\rho\omega^2 \int y^3 (1+p^2)^{\frac{1}{2}} dx.$

$$\text{Volume} = \pi \int y^2 dx.$$

Hence $$V = y^3(1+p^2)^{\frac{1}{2}} - ay^2.$$

But $V = Pp + C$, C being a constant: hence

$$y^3(1+p^2)^{\frac{1}{2}} - ay^2 = \frac{y^3p^2}{(1+p^2)^{\frac{1}{2}}} + C,$$

$$\frac{y^3}{(1+p^2)^{\frac{1}{2}}} = ay^2 + C.$$

The values of y at the circular ends being given, there is $\delta y_{,} = 0$, $\delta y_{,,} = 0$: thus the equation for the limits becomes

$$(V_{,,} - P_{,,}p_{,,})\,\delta x_{,,} - (V_{,} - P_{,}p_{,})\,\delta x_{,} = 0, \quad \text{or} \quad C\delta x_{,,} - C\delta x_{,} = 0:$$

but $\delta x_{,}$, $\delta x_{,,}$ are independent of each other: hence $C = 0$, and the equation for the generating curve becomes

$$y = a(1+p^2)^{\frac{1}{2}}, \quad y^2 - a^2 = a^2\frac{dy^2}{dx^2}, \quad \frac{dx}{a} = \frac{dy}{(y^2-a^2)^{\frac{1}{2}}} \;...(1),$$

$$\frac{x+c}{a} = \log\{y + (y^2-a^2)^{\frac{1}{2}}\}, \; c \text{ being a constant:}$$

$$y + (y^2-a^2)^{\frac{1}{2}} = e^{\frac{x+c}{a}}, \quad y - (y^2-a^2)^{\frac{1}{2}} = a^2e^{-\frac{x+c}{a}}, \quad \frac{2y}{a} = me^{\frac{x}{a}} + m^{-1}e^{-\frac{x}{a}} \;...(2),$$

a and m being unknown constants.

Let the origin be in one of the ends, of which the radius $= b$:

then $$2\frac{b}{a} = m + m^{-1} \;......................(3).$$

Also let k^3 denote the capacity of the vessel: then, from (1), b' denoting the radius of the other end,

$$k^3 = \pi \int y^2 dx = \pi a \int_b^{b'} \frac{y^2 dy}{(y^2 - a^2)^{\frac{1}{2}}}$$

$$= \pi a \left\{ \tfrac{1}{2} b' (b'^2 - a^2)^{\frac{1}{2}} - \tfrac{1}{2} b (b^2 - a^2)^{\frac{1}{2}} + \tfrac{1}{2} a^2 \log \frac{b' + (b'^2 - a^2)^{\frac{1}{2}}}{b + (b^2 - a^2)^{\frac{1}{2}}} \right\} \ldots (4).$$

Thus (3) and (4) determine the constants a and m. The equation (2) defines the generating curve.

COR. If $m = 1$, the curve is the common catenary, the conditions being, from (3) and (4),

$$a = b, \quad k^3 = \tfrac{1}{2} \pi b \left\{ b' (b'^2 - b^2)^{\frac{1}{2}} + b^2 \log \frac{b' + (b'^2 - b^2)^{\frac{1}{2}}}{b} \right\}.$$

24. If α, β, γ, be the direction-cosines of one of the two lines of vibration of the plane front of a wave in a biaxal crystal, and α', β', γ', those of either of the two lines of vibration of a plane front intersecting the former plane front at right angles and passing through the line (α, β, γ), prove that

$$\frac{\alpha'}{\alpha} (b^2 - c^2) + \frac{\beta'}{\beta} (c^2 - a^2) + \frac{\gamma'}{\gamma} (a^2 - b^2) = 0,$$

and that
$$\frac{(b^2 - c^2)^2}{\alpha\alpha'} + \frac{(c^2 - a^2)^2}{\beta\beta'} + \frac{(a^2 - b^2)^2}{\gamma\gamma'} = 0.$$

Let (α, β, γ) be a line of vibration in the front

$$lx + my + nz = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to a plane front perpendicular to (1) and passing through (α, β, γ), is

$$\frac{x}{\alpha}\left(\frac{m}{\beta} - \frac{n}{\gamma}\right) + \frac{y}{\beta}\left(\frac{n}{\gamma} - \frac{l}{\alpha}\right) + \frac{z}{\gamma}\left(\frac{l}{\alpha} - \frac{m}{\beta}\right) = 0 \ldots\ldots (2).$$

Then, (α, β, γ) being a line of vibration in (1),

$$l\alpha + m\beta + n\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and
$$\frac{l}{\alpha}(b^2 - c^2) + \frac{m}{\beta}(c^2 - a^2) + \frac{n}{\gamma}(a^2 - b^2) = 0 \ldots\ldots\ldots (4).$$

Also, $(\alpha', \beta', \gamma')$ being a line of vibration in (2),

$$\frac{\alpha'}{\alpha}\left(\frac{m}{\beta} - \frac{n}{\gamma}\right) + \frac{\beta'}{\beta}\left(\frac{n}{\gamma} - \frac{l}{\alpha}\right) + \frac{\gamma'}{\gamma}\left(\frac{l}{\alpha} - \frac{m}{\beta}\right) = 0 \ldots\ldots (5),$$

and

$$\frac{b^2-c^2}{\alpha\alpha'}\left(\frac{m}{\beta}-\frac{n}{\gamma}\right)+\frac{c^2-a^2}{\beta\beta'}\left(\frac{n}{\gamma}-\frac{l}{\alpha}\right)+\frac{a^2-b^2}{\gamma\gamma'}\left(\frac{l}{\alpha}-\frac{m}{\beta}\right)=0\ \dots (6).$$

From (3) and (4) we have

$$\frac{\frac{l}{\alpha}}{(a^2-b^2)\beta^2-(c^2-a^2)\gamma^2}=\frac{\frac{m}{\beta}}{(b^2-c^2)\gamma^2-(a^2-b^2)\alpha^2}=\frac{\frac{n}{\gamma}}{(c^2-a^2)\alpha^2-(b^2-c^2)\beta^2},$$

and therefore $\frac{m}{\beta}-\frac{n}{\gamma}$, $\frac{n}{\gamma}-\frac{l}{\alpha}$, $\frac{l}{\alpha}-\frac{m}{\beta}$,

are proportional to

$$b^2-c^2,\quad c^2-a^2,\quad a^2-b^2.$$

The equations (5) and (6) become therefore

$$\frac{\alpha'}{\alpha}(b^2-c^2)+\frac{\beta'}{\beta}(c^2-a^2)+\frac{\gamma'}{\gamma}(a^2-b^2)=0,$$

and
$$\frac{(b^2-c^2)^2}{\alpha\alpha'}+\frac{(c^2-a^2)^2}{\beta\beta'}+\frac{(a^2-b^2)^2}{\gamma\gamma'}=0.$$

TUESDAY, *Jan.* 3, 1854. 9...12.

1. THE complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

If K be the common angular point of these parallelograms, and BD the other diameter, the difference of the parallelograms is equal to twice the triangle BKD.

Since BK (fig. 58) bisects EG, and KD bisects HF, the two GBK, FKD are together equal to EBK, HKD; to these equals add the unequals GF, EH; then the difference of the parallelograms GF, EH is equal to the difference of the figures $CBKD$, $ABKD$: but the latter difference is evidently equal to twice the triangle KBD; for $CBKD$ exceeds CBD or ABD by the triangle KBD, and CBD or ABD exceeds $ABKD$ by the triangle KBD; therefore the difference of the parallelograms GF, EH is equal to twice the triangle KBD.

2. Divide a given straight line into two parts so that the rectangle contained by the whole line and one of the parts shall be equal to the square of the other part.

Produce a given straight line to a point such that the rectangle contained by the whole line thus produced and the part produced shall be equal to the square of the given straight line.

In Euclid's figure, the rectangle contained by CF and FA is proved to be equal to the square on CA.

If therefore CA be the given line, describe a square on CA, and proceed as in Euclid: F will be the point required.

3. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

If the opposite sides of the quadrilateral be produced to meet in P, Q, and about the triangles so formed without the quadrilateral circles be described meeting again in R; P, R, Q will be in one straight line.

Let AB and DC meet in P, AD and BC in Q (fig. 59); then the circles described about the triangles PBC, QDC meet in C and R. Join PR, CR, QR.

The angles CRP, CBP are together equal to CBP and CBA (Euc. I. 13; III. 22); therefore CRP is equal to CBA; similarly CRQ is equal to CDA; therefore the two CRP and CRQ are together equal to the two CBA and CDA, that is to two right angles; therefore PRQ is a straight line.

5. AE, EA', are diameters of two circles touching each other externally at E: a chord AB of the former circle, when produced, touches the latter at C', while a chord $A'B'$ of the latter touches the former at C. Prove that the rectangle contained by AB, $A'B'$, is four times as great as that contained by BC', $B'C$.

Euc. bk. VI., prop. 2,

$$AB : BC' :: 2OE : EO', \text{ (fig. 60)}.$$

Euc. bk. V., prop. 4,

$$AB : 2BC' :: 2OE : 2EO'.$$

Similarly, $$A'B' : 2B'C :: 2O'E : 2EO.$$

Hence, Euc. bk. V., prop. 11,

$$AB : 2BC' :: 2B'C : A'B'.$$

Hence, Euc. bk. VI., prop. 16,

$$\begin{aligned}\text{rect.}(AB, A'B') &= \text{rect.}(2BC', 2B'C)\\ &= 4\text{ rect.}(BC', B'C).\end{aligned}$$

6. Within the area of a given triangle is described a triangle, the sides of which are parallel to those of the given one. Prove that the sum of the angles subtended by the sides of the interior triangle at any point not in the plane of the triangles

is less than the sum of the angles subtended at the same point by the sides of the exterior triangle.

Let ABC (fig. 61) be the exterior, and abc the interior triangle. Produce the sides of the interior triangle to intersect those of the exterior, in the points α', α''; β', β''; γ', γ''; and join $c\gamma'$.

Let the angles subtended by any line in the plane of the triangles at the external point be denoted by the line itself.

Then $$bc \angle b\gamma' + c\gamma',$$
$$bc \angle \alpha''B + c\beta'' + \gamma'\beta'',$$
$$bc \angle \alpha''B + C\alpha' + \gamma'\beta''.$$
Similarly $$ca \angle \beta''C + A\beta' + \alpha'\gamma'',$$
$$ab \angle \gamma''A + B\gamma' + \beta'\alpha''.$$

Adding together these inequalities, we have
$$bc + ca + ab \angle BC + CA + AB.$$

8. If NP be the ordinate of any point P of an ellipse, Y and Z the points where the tangent at P meets the perpendiculars from the foci,
$$NY : NZ :: PY : PZ.$$

Circles may be described about $NPYS$, $NPZH$, (fig. 62).

But $$\angle SPY = \angle HPZ:$$
hence $$\angle SNY = \angle HNZ;$$
hence $$\angle YNP = \angle ZNP;$$
and therefore, by Euclid, bk. VI., 3,
$$NY : NZ :: PY : PZ.$$

11. Parallelograms, whose sides touch an hyperbola and its conjugate, and are parallel to conjugate diameters, have the same area.

If CP, CD be conjugate semi-diameters, and through C a straight line be drawn parallel to either focal distance of P, the perpendicular let fall from D on this straight line will be equal to half the minor axis.

Let DM (fig. 63) be the perpendicular let fall from D on the line through C parallel to SP or HP; draw PF perpendicular to CD, and produce SP, CD to meet in E; then $PE = AC$.

The triangles MDC, FPE are similar, for $\angle DCM = PEF$, and $DMC = PFE$;

$$\therefore\ MD : CD :: FP : PE;$$

$$\therefore\ MD.PE = CD.FP,$$

$$MD.AC = AC.BC,$$

$$MD = BC.$$

TUESDAY, *Jan.* 3. $1\frac{1}{2}$...4.

10. Find the value of $\sin 18°$.

In Euclid's construction for determining an isosceles triangle, the angles at whose base are double of the angle at the vertex, shew that the common chord of the two circles is equal to the base of the triangle.

Let E be the other point of intersection of the two circles,

$$AE = AC$$

are chords of segments containing equal angles ADC or BDC;

$$\therefore\ \angle ACE = \angle BDC = \angle ABC;$$

$$\therefore\ CE = BC.$$

11. Find A from the equation, $\tan 2A = 8\cos^2 A - \cot A$.

$$\operatorname{Tan} 2A = 8\cos^2 A - \cot A,$$

$$8\cos^2 A = \tan 2A + \cot A,$$

$$= \frac{\cos(2A - A)}{\cos 2A \sin A};$$

$$\therefore\ \cos A = 0 \text{..............................} (1);$$

or $$1 = 8\cos A \sin A \cos 2A$$
$$= 4\sin 2A \cos 2A = 2\sin 4A,$$
$$\sin 4A = \tfrac{1}{2} = \sin 30° \ldots\ldots (2),$$
by (1), $$A = (2n+1)\,90°,$$
by (2), $$4A = n.180° + (-1)^n\, 30°,$$
$$A = n.45° + (-1)^n\, 7°\,30',$$
n being any positive or negative integer or zero.

If $\sin 3A = n \sin A$ be true for any values of A besides 0 or a multiple of 90°, shew that n must be less than 3 and not less than $-$ 1. Solve the equation when $n = 2$.

If $$\sin 3A = n \sin A,$$
or $$3\sin A - 4\sin^3 A = n \sin A,$$
be true for other values of A than 0 or multiples of 90°, so is
$$3 - 4\sin^2 A = n,$$
or $$\sin^2 A = \frac{3-n}{4},$$
in which case n is < 3, and $3 - n < 4$, or $n > -1$.

If $$n = 2,$$
$$\sin A = 0 \ldots\ldots (1),$$
and $$\sin^2 A = \tfrac{1}{4},$$
or $$\sin A = \sin(\pm 30°) \ldots\ldots (2);$$
therefore by (1), $$A = m180°,$$
by (2), $$A = m180° \pm 30°,$$
$m = 0$, or any integer.

If $$\cos\theta \cos\phi = \sin(\alpha - \beta) \sin(\alpha + \beta),$$
and $\sin(\theta - \phi) \sin(\theta + \phi) = 4\cos\alpha \cos\beta$; find $\cos\theta$, and $\cos\phi$.

By the second equation,
$$\cos^2\phi - \cos^2\theta = 4\cos\alpha \cos\beta,$$
by the first, $$\cos\phi \cos\theta = \cos^2\beta - \cos^2\alpha;$$
$$\therefore\ (\cos^2\phi + \cos^2\theta)^2 = 16\cos^2\alpha \cos^2\beta + 4(\cos^2\beta - \cos^2\alpha)^2,$$
$$= 4(\cos^2\beta + \cos^2\alpha)^2;$$

$$\therefore \cos^2\phi + \cos^2\theta = 2(\cos^2\beta + \cos^2\alpha);$$

$$\therefore \cos^2\phi = (\cos\beta + \cos\alpha)^2,$$

$$\cos^2\theta = (\cos\beta - \cos\alpha)^2;$$

$$\therefore \cos\phi = \pm(\cos\beta + \cos\alpha),$$

$$\cos\theta = \pm(\cos\beta - \cos\alpha).$$

The radical must be taken with the same sign in each, the additional roots obtained by taking different signs having been introduced by squaring in the third step.

12. In any triangle ABC, prove that

$$AB^2 = BC^2 + CA^2 - 2BC.CA\cos C.$$

AD (fig. 64) is drawn to meet BC, or BC produced, in D, so that AD is equal to AC; shew that if the sum of AB and AC is n times BC, their difference is $\frac{1}{n}$th of BD.

$$AB + AC = n.BC,$$

$$AB^2 - AC^2 = BC^2 - 2BC.CA\cos C,$$

and $$CD = 2CA\cos C, \text{ or } 2CA\cos(180° - C);$$

$$\therefore (AB - AC)(AB + AC) = BC(BC \mp CD);$$

$$\therefore AB - AC = \frac{1}{n}BD.$$

13. Find the radius of the circle described about a triangle whose sides are given.

Shew that the radius of the circle inscribed in an isosceles triangle can never be greater than one half of that of the circumscribed circle.

The radius of the circumscribing circle (R)

$$= \frac{abc}{\sqrt{\{(a+b+c)(a+b-c)(a+c-b)(b+c-a)\}}}.$$

The radius of the inscribed circle (r)

$$= \frac{1}{2}\sqrt{\left\{\frac{(a+b-c)(a+c-b)(b+c-a)}{a+b+c}\right\}};$$

$$\therefore\ \frac{r}{R} = \tfrac{1}{2}\,.\,\frac{(a+b-c)\,(a+c-b)\,(b+c-a)}{abc}\,.$$

$$\text{If } b = c,\ \frac{r}{R} = \tfrac{1}{2}\,.\,\frac{a^2\,(2c-a)}{ac^2} = \tfrac{1}{2}\left\{1-\left(1-\frac{a}{c}\right)^2\right\},$$

which can never be greater than $\frac{1}{2}$.

$\frac{r}{R}$ is equal to $\frac{1}{2}$ when $a = c$, or the triangle is equilateral.

14. Two posts AB and CD (fig. 65) are placed at the edge of a river at a distance AC equal to AB, the height of CD being such that AB and CD subtend equal angles at E a point on the other bank exactly opposite to A; shew that the square of the breadth of the river is equal to $\dfrac{AB^4}{CD^2 - AB^2}$; and that AD and BC subtend equal angles at E.

$$\angle BEA = \angle DEC = \alpha,$$

$$\angle BAE = \angle CAE = 90^\circ,$$

and $$BA = AC;$$

therefore BAE, CAE are equal in all respects.

$$AB = AE\tan\alpha,$$

$$CD = CE\tan\alpha,$$

$$AB^2 = CE^2 - AE^2$$

$$= (CD^2 - AB^2)\cot^2\alpha;$$

$$\therefore\ AB^4 = (CD^2 - AB^2)\,AE^2;$$

$$\therefore\ AE^2 = \frac{AB^4}{CD^2 - AB^2}.$$

Also we have two solid angles at E,

one contained by AEC, BEA, and BEC,

and the other by AEC, CED, and DEA;

AEC is common to both,

$BEA = CED$ in planes perpendicular to AEC;

$$\therefore\ \angle BEC = \angle DEA,$$

or AD, BC subtend equal angles at E.

Otherwise: $\cos BEC = 1 - 2\sin^2(\tfrac{1}{2}BEC)$

$$= 1 - 2 \cdot \left\{\frac{\frac{BC}{2}}{BE}\right\}^2$$

$$= 1 - \frac{AB^2}{BE^2}, \text{ since } BC^2 = 2AB^2,$$

$$= \frac{AE^2}{BE^2}$$

$$= \frac{AE}{BE} \cdot \frac{CE}{ED} = \frac{AE}{ED}, \quad \text{since } CE = BE,$$

$$= \cos AED;$$

$$\therefore\ BEC = AED.$$

WEDNESDAY, *Jan.* 4. 9...12.

1. TWO unequal forces act in parallel lines and in opposite directions upon a rigid body moveable about a fixed point in their plane; shew that, if there be equilibrium, the moments of the forces with respect to the fixed point are equal.

Three straight tobacco-pipes rest upon a table, with their bowls, mouth downwards, in the angles of an equilateral triangle, the tubes being supported in the air by crossing symmetrically, each under one and over the other, so as to form another equilateral triangle; shew that the mutual pressure of the tubes varies inversely as the side of the last triangle.

Let ABC (fig. 66) be the positions of the bowls of the three pipes Aa, Bb, Cc.

The mutual actions on a, b, c, are the same.

Let W = weight of each pipe,

G the centre of gravity of the pipe Aa,

R the mutual action.

Aa is kept at rest by the action of the table, the weight, and the couple whose moment is $R.ab$;
therefore, taking moments round A,

$$W.AG = R.ab;$$

$$\therefore R \propto \frac{1}{ab}.$$

2. If three forces acting upon a particle keep it at rest, shew that the forces are respectively in the ratio of the sines of the angles contained by the other two.

A smooth circular ring is fixed in a horizontal position, and a small ring sliding upon it is in equilibrium when acted on by two strings in the direction of the chords PA, PB; shew that, if PC be a diameter of the circle, the tensions of the strings are in the ratio of BC to AC.

If A and B be fixed points, is the equilibrium stable?

The ring is kept in equilibrium at P (fig. 67) by the reactions in direction CP, and the tensions in direction PA, PB;

$$\therefore \text{tension of } PA : \text{tension of } PB :: \sin BPC : \sin APC$$
$$:: BC : AC.$$

If P be displaced to P', the tensions of the strings remaining the same, the effect of the tension of the string towards B is diminished in the ratio of $\cos P'CB : \cos PCB$, and that of the tension to A is increased; similarly, if P be displaced to P''; therefore the equilibrium is unstable.

3. Define the centre of gravity of a system of heavy particles, and shew that in every case there exists one and only one such point.

From this fact deduce the property that the lines joining the middle points of opposite sides of any quadrilateral bisect each other.

Let equal masses be placed in the angular points of the quadrilateral $ABCD$.

The centre of gravity of these four masses is that of the masses of A, B, collected at a the middle point of AB, and that of C, D, collected at c the middle point of CD, and is therefore in O the middle point of ac.

Similarly, it is at the middle point of bd bisecting BC and DA.

Therefore, since a system has only one centre of gravity, ac and bd bisect each other.

4. Find the ratio of P to W in the single moveable pulley, when the strings are not parallel.

If a weight W be supported by a weight P hanging over a fixed pulley, the strings being parallel, shew that, in whatever position they hang, the position of their centre of gravity is the same.

If W be depressed through a space α, P is raised through a space 2α, and the centre of gravity is moved through a space $\dfrac{W.\alpha - P.2\alpha}{W+P}$; and since $W = 2P$, the centre of gravity is stationary.

5. Describe the construction and graduation of the common steelyard.

Shew that, if a steelyard be constructed with a given rod, whose weight is inconsiderable compared with that of the sliding weight, the sensibility varies inversely as the sum of the sliding weight and the greatest weight which can be weighed.

Let P at M (fig. 68) balance the weight W,

P at N.................................. W';

therefore $$P.CM = W.AC,$$

$$P.CN = W'.AC;$$

therefore $$P.MN = (W - W')\,AC.$$

Now the sensibility of the instrument varies as the distance through which P must move in order to detect a given difference of weight; therefore the sensibility $\propto MN \propto \frac{AC}{P}$;

and if $Q =$ the greatest weight which can be weighed,

$$Q.AC = P.BC;$$

therefore $$(Q+P)\,AC = P.AB;$$

therefore the sensibility $\propto \frac{1}{Q+P}$, since AB is given.

8. What is meant by a unit, and what is usually taken as the unit of accelerating force?

If the force of gravity be taken as the unit of force, and a rate of ten miles an hour as the unit of velocity, what must be the units of time and space?

Suppose a feet to be the unit of space,

and b seconds .... time;

a and b are numbers whose values it is now our object to ascertain.

A velocity of 10 miles an hour is the same as a velocity of

$$\frac{10 \times 1760 \times 3}{60 \times 60} \text{ feet per } 1'',$$

or $$\frac{10 \times 1760 \times 3}{60 \times 60}\, b \text{ feet per } b''.$$

But a velocity of 10 miles an hour is the unit of velocity, and is therefore a velocity of a feet per b'';

therefore $$a = \frac{10 \times 1760 \times 3}{60 \times 60}\, b$$

$$= \frac{44}{9}\, b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, the force of gravity

generates in $1''$ a velocity of 32.2 feet per $1''$;

therefore it .... $1''$ .... $32.2b$......... b'';

therefore it .... b'' .... $32.2b^2$......... b'':

but the force of gravity is the unit of force;

therefore it generates in b'' the unit of velocity,

that is, ... a velocity of a feet per b'';

therefore $$a = 32.2b^2 \text{ .... (2).}$$

From (1) and (2) the values of a and b may be found;

$$b = \frac{44}{9 \times 32.2}, \quad a = \frac{(44)^2}{9^2 \times 32.2};$$

the unit of space is therefore $\dfrac{(44)^2}{9^2 + 32{\cdot}2}$ feet, and the unit of time $\dfrac{44}{9 \times 32{\cdot}2}$ seconds.

11. Two balls of given masses and given elasticity are moving with given velocities in the same direction; determine their motion after impact.

Two balls are moving in the same straight line, one of them only being acted on by a force; if the force be constant and tend towards the other ball, shew that the times which elapse between consecutive impacts decrease in geometrical progression.

Let m, m' be the masses of the balls,

f the force acting on the former,

v_{n-1}, v'_{n-1} their velocities after the $(n-1)^{\text{th}}$ impact,

u_n, v'_{n-1} .... before the n^{th} impact,

v_n, v'_n .... after the n^{th} impact,

R_n, eR_n the impulsive forces of compression and restitution at the n^{th} impact,

t_{n-1} the time between the $(n-1)^{\text{th}}$ and the n^{th} impact.

Since the spaces described by the two balls between the $(n-1)^{\text{th}}$ and n^{th} impacts are equal,

$$v_{n-1}t_{n-1} + \tfrac{1}{2}ft^2_{n-1} = v'_{n-1}t_{n-1};$$

$$\therefore\ t_{n-1} = 2\,\frac{v'_{n-1} - v_{n-1}}{f} \text{ .... (1).}$$

Similarly, $$t_n = 2\frac{v'_n - v_n}{f};$$

$$\therefore\ \frac{t_n}{t_{n-1}} = \frac{v'_n - v_n}{v'_{n-1} - v_{n-1}} \quad\text{.................(2).}$$

Also the force f acting for the time t_{n-1} changes the velocity of m from v_{n-1} to u_n;

therefore $$u_n = v_{n-1} + ft_{n-1}$$
$$= v_{n-1} + 2(v'_{n-1} - v_{n-1}),\quad\text{by (1);}$$

therefore $$u_n - v'_{n-1} = v'_{n-1} - v_{n-1}\quad\text{.................(3).}$$

Again, since the two balls have the same velocity at the instant of greatest compression,

$$u_n - \frac{R_n}{m} = v'_{n-1} + \frac{R_n}{m'},$$

$$v_n + e\frac{R_n}{m} = v'_n - e\frac{R_n}{m'};$$

therefore $$v'_n - v_n = e(u_n - v'_{n-1})$$
$$= e(v'_{n-1} - v_{n-1}),\quad\text{by (3);}$$

therefore (2) becomes $\dfrac{t_n}{t_{n-1}} = e.$

Hence the times decrease in geometrical progression.

12. Prove that the time of falling in a straight line from the highest point of a vertical circle to any point in the circumference is less than to any point outside; and give a geometrical construction for the straight line of quickest descent to the circumference of a vertical circle from a given point within it.

Shew that the circumference of two circles contains all points from which the time of quickest descent to a given vertical circle is the same.

Let R be the radius of the given circle, and construct any number of equal circles touching the given circle, and let r be the radius of one of them; then the time of quickest descent to the given circle from the highest point of any of these circles is the same: and it may be shewn that those circles which touch the given circle externally have their highest points in the circumference of a circle whose radius is $R+r$, and whose centre is a point Q at a distance r vertically above the centre of the given circle; and that those circles which touch the given circle internally have their highest points in the circumference of another circle whose centre is Q and radius $R \sim r$: hence the circumference of these two circles contain, &c.

WEDNESDAY, *Jan.* 4. 1½...4.

4. A rod of length a and density ρ, is moveable freely about one end, which is fixed at a depth c below the surface of a fluid of density σ: prove that the rod may remain at rest, when inclined to the vertical, provided that

$$\frac{\sigma}{\rho} > 1 \text{ and } < \frac{a^2}{c^2}.$$

Shew that such a position is one of stable equilibrium.

It is evident that the rod cannot rest obliquely when entirely immersed within the fluid.

For equilibrium, supposing the rod partially out of the water, O being its lower and A its higher end, and P its intersection with the surface of the water, (fig. 69),

$$G = \left(\rho . a . \frac{a}{2} - \frac{\sigma . c}{\cos\theta} \cdot \frac{c}{2\cos\theta}\right) . \sin\theta = 0 \ldots\ldots\ldots (1),$$

where θ denotes the inclination of OP to the vertical line OC.

Hence, for oblique equilibrium,

$$\cos^2\theta = \frac{\sigma . c^2}{\rho . a^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

But, from the geometry,

$$\cos^2\theta = \frac{c^2}{OP^2} > \frac{c^2}{a^2}:$$

hence, from (2), $\sigma > \rho$.

Also, $\sigma.c^2 > \rho.a^2$, in order that θ in (2) may be possible. Thus there will be oblique equilibrium, defined by (2), under the conditions $\sigma > \rho$, $\sigma.c^2 < \rho.a^2$.

Putting β for the value of θ given by (2), we have, by (1),

$$G = \tfrac{1}{2}\rho a^2 \left(1 - \frac{\cos^2\beta}{\cos^2\theta}\right) = \pm, \text{ as } \theta < \beta \text{ or } \theta > \beta.$$

Hence, the oblique equilibrium is stable.

8. A pencil of rays diverging from a point at a given distance from the centre, is incident directly on a concave spherical refracting surface, determine the distance of the geometrical focus of the refracted pencil from the centre.

An eye is placed close to the surface of a sphere of glass ($\mu = \frac{3}{2}$), which is silvered at the back; shew that the image which the eye sees of itself is $\frac{3}{5}$ of the natural size.

Let ACB (fig. 70) be the diameter of the eye placed close to the surface of the sphere, so that rays proceeding from it are unaffected by refraction at entering the sphere, and after reflection at the back form the image acb. Let COM be the diameter of the sphere, O the centre. Then

$$\frac{1}{OC} + \frac{1}{Oc} = -\frac{2}{OC},$$

or

$$Oc = -\frac{OC}{3},$$

the negative sign signifying on which side of O c lies.

Let the image of acb, formed by refraction out of the sphere, be $a'c'b'$, which is the image which the eye sees. Then

$$\frac{1}{\mu . Oc} - \frac{1}{Oc'} = \frac{\frac{1}{\mu} - 1}{-OC},$$

or

$$\frac{2}{3\,Oc} - \frac{1}{Oc'} = \frac{1}{3\,OC},$$

$$\frac{2}{OC} - \frac{1}{3\,OC} = \frac{1}{Oc'},$$

$$Oc' = \frac{3\,OC}{5}.$$

Now the ratio of $Oc' : OC$ is the ratio of $a'c'b'$ to ACB, therefore the ratio of the image to the natural size of the eye is $3 : 5$.

9. A rod, inclined at any angle to a plate of glass, is seen by an eye on the opposite side of the plate; shew that the length of the image of the rod, formed by geometrical foci, is equal to the length of the rod. Is the image, formed by refraction at the first surface, of the same magnitude as either?

Let PQ (fig. 71) be the rod, $P'Q'$ its image after refraction at the first surface AB of the plate, and $P''Q''$ after refraction at the second surface CD. Draw QR, $Q'R'$, $Q''R''$, at right-angles to BP'.

Then $$Q'A = \mu.AQ,$$
$$P'B = \mu.PB,$$
and therefore $$P'R' = \mu.PR:$$
hence $$P'R' > PR,$$
and therefore $$P'Q' > PQ.$$

Again, $$DP'' = \frac{1}{\mu} DP'$$
$$= \frac{1}{\mu}(t + \mu.PB)$$
$$= \frac{t}{\mu} + PB;$$
similarly, $$CQ'' = \frac{t}{\mu} + QA;$$
hence $$P''R'' = PR,$$
and therefore $$P''Q'' = PQ.$$

10. Find the deviation of a ray of light refracted through a prism in a plane perpendicular to the edge.

If rays in this plane are incident at one point of the prism in all directions, shew that, if the refracting angle be greater than $\sin^{-1}\frac{1}{\mu}$, rays incident from that side of the normal which is towards the edge of the prism will not pass through, and examine what rays will pass through.

If the refracting angle of a prism be $> \sin^{-1}\frac{1}{\mu}$, and rays be incident at a point of one face of the prism in all directions, lying in a plane perpendicular to the edge of the prism, shew that no ray will pass through which is incident from the side of the normal towards the edge, and examine what rays will pass through.

Let the plane of the paper be the plane of incident rays, A (fig. 72) the point of incidence, V the trace of the edge, QAR the course of a ray incident at A and refracted to R, NAn normal at A,

$$\angle VRA = \frac{\pi}{2} - \angle RAn - \angle V,$$

$$< \frac{\pi}{2} - \angle V;$$

therefore the angle of incidence at $A > \angle V$, *a fortiori*, $> \sin^{-1}\frac{1}{\mu}$, or the ray cannot emerge at R.

If $Q'AR'$ be a ray incident from the side of the normal which is from the edge,

$$\angle VR'A = \frac{\pi}{2} + \angle R'An - V;$$

and if the ray be capable of emergence,

$$\angle VR'A \nless \frac{\pi}{2} - \sin^{-1}\frac{1}{\mu};$$

therefore $$\frac{\pi}{2} - \sin^{-1}\frac{1}{\mu} \ngtr \frac{\pi}{2} + R'An - V,$$

or $$\angle R'AN \nless V - \sin^{-1}\frac{1}{\mu},$$

$$\sin Q'AN \nless \mu \sin\left(V - \sin^{-1}\frac{1}{\mu}\right),$$

or all rays on that side of the normal, incident at an angle not less than $\sin^{-1}\left\{\mu \sin\left(V - \sin^{-1}\frac{1}{\mu}\right)\right\}$, will pass through.

12. A short-sighted person moves his eye-glass gradually from his eye towards a small object: shew that the linear magnitude of the image will keep increasing during the motion, and that the angle subtended by the image at the eye will be least when the eye-glass has advanced half way towards the object.

Let PQ (fig. 73) be the object, which we may suppose to be at right angles to the axis EAP of the eye-glass, A the centre of the eye-glass, and E the place of the eye.

Let pq be the image of PQ; join QE, qE. Let $AP = u$, $AE = d$, $\angle PEQ = \alpha$, $\angle pEq = \theta$, $Ap = v$.

Then, as is proved in elementary treatises on Optics,

$$\frac{\tan\theta}{\tan\alpha} = \frac{u+d}{u+d+\dfrac{du}{f}}.$$

Now $u+d$ is constant: hence $\tan\theta$ is least when du is greatest, that is, when $u = d$.

Again, $$pq = PQ \cdot \frac{v}{u} = \frac{PQ}{1+\dfrac{u}{f}}.$$

Hence pq increases as u diminishes.

THURSDAY, *Jan.* 5. 9...12.

1. EXPLAIN what is meant by the limit of a varying quantity or ratio, and enunciate and prove Newton's first Lemma.

Two triangles CAB, $C'AB'$ have a common angle A, and the sum of their sides about that angle the same in each; if CB, $C'B'$ intersect in D, and B' move up to B, then in the limit $DC : DB :: AB : AC$.

From (fig. 74) C draw CE parallel to AB meeting $B'C'$ in E.
Because $AB + AC = AB' + AC'$,

$$BB' = CC';$$

by similar triangles, DCE, DBB',

$$DC : DB :: CE : BB';$$

and by similar triangles $AB'C'$, CEC',

$$AC' : CC' :: AB' : CE,$$

or alt. $$AC' : AB' :: CC' : CE$$

$$:: BB' : CE;$$

but $$DC : DB :: CE : BB';$$

therefore comp. $$AC' : AB' :: DC : DB;$$

therefore in the limit $AC : AB :: DC : DB$.

2. Define the circle of curvature at any point of a curve. If PQ be an arc, and QR a subtense, the chord of the circle of curvature at P parallel to QR is equal to the limit of the third proportional to QR and PQ. Find the chord of curvature through the focus of an ellipse.

EF is a chord of a given circle and S its middle point; construct the ellipse of which E is one point, S one focus, and the given circle the circle of curvature at E.

The chord of curvature (fig. 75) through the focus $= 2\,\dfrac{SE.HE}{AC}$, if H be the second focus and AC the semi-major axis.

But in this case the chord is equal to $2SE$. Hence $HE = AC$, and E is the extremity of the minor axis of the ellipse.

Draw through E the chord EG making the same angle with the tangent at E that EF does. The middle point of this chord will be the second focus H, and the ellipse is constructed.

3. Shew that, in an orbit described under the action of a force tending to a fixed point, the velocity at any point is inversely proportional to the perpendicular from the centre of force on the tangent at that point.

A body is describing a parabola under the action of a force which always tends to the focus, and a straight line is drawn from the focus perpendicular to the tangent, and proportional to the velocity, at any point; shew that the extremity of this straight line will lie in a certain circle.

Draw SY (fig. 76) perpendicular from the focus S on the tangent at P. Produce it to Q, so that SQ bears a certain ratio to the velocity at P, and in the axis take a point B, such that SB bears the same ratio to the velocity at the vertex A; and join BQ.

Then $$SQ \propto \text{velocity at } P \propto \frac{1}{SY},$$

or $$SY.SQ = \text{a constant quantity} = SA.SB,$$

or $$SY : SA :: SB : SQ.$$

And the triangles ASY, QSB, having a common angle at S, and their sides about that angle proportional, are similar. Hence the angle $BQS =$ the angle $SAY =$ a right angle, and Q will always lie on the circle whose diameter is SB.

4. Given the velocities and the directions of motion at any three points of an orbit described under the action of a central force: find the centre of force.

If the velocities at the three points be respectively parallel and proportional to the opposite sides of the triangle of which they are the angular points, the centre of force is the centre of gravity of the triangle.

Let P, Q, R (fig. 77) be three points of a central orbit, at each of which the velocity is parallel and proportional to the opposite side of the triangle PQR: produce the tangents at P, Q, R so as to form a new triangle $P'Q'R'$, having its sides parallel and proportional to those of PQR.

Join PP': because the perpendiculars from the centre of force on $P'Q'$, $P'R'$ are inversely proportional to the velocities at R, Q, they are inversely proportional to the sides $P'Q'$, $P'R'$; therefore the triangles, whose common vertex is the centre of force, and whose bases are the sides $P'Q'$, $P'R'$, will be equal, and therefore the centre of force will lie in the line PP': so also it will lie in the line QQ', and will be the centre of gravity of the triangle PQR, for the lines PP', QQ' bisect respectively the sides QR, RP.

5. An ellipse is described under the action of a force tending to the focus; find the law of force and the velocity at any point.

If, without changing the velocity, the direction of motion of the body receive a very slight alteration, shew that the position of the major axis will be altered, unless the body be at one extremity of the latus-rectum through the focus to which the force does not tend.

Let S (fig. 78) be the centre of force, PY the tangent, H the second focus: let the direction of motion be altered through the indefinitely small angle YPY', and let H' be the position of the second focus of the new orbit.

Then, since the focal distances make equal angles with the tangent

$$\angle HPH' = \text{twice } \angle YPY',$$

and because the velocity is unaltered, the major axis is unaltered in length,

$$SP + HP = SP + H'P,$$

or

$$HP = H'P,$$

and the position of the major axis will be altered, unless S, H, H' be in one straight line. Let them be in one straight line, then PH, PH' make equal angles with this line; that is, since HPH' is indefinitely small, PH, PH' are each at right angles to SH, or the particle is at the extremity of the latus-rectum through H.

6. Enumerate the principal steps which led Newton to conclude that the Moon is retained in her orbit by the force of gravity.

Assuming that the Moon is retained in her orbit by the Earth's attraction alone, and that, approximately, her orbit is circular, her period about the Earth 27 days, the accelerating effect of gravity at the Earth's surface 32 feet per second, and the Earth's radius 4000 miles, find the distance of the Moon from the Earth's centre.

Let $4000r$ be the distance of the Moon from the Earth's centre in miles, f the accelerating effect on the Moon by reason of the Earth's attraction, in feet per second, then

$$f : 32 :: 1 : r^2,$$

or
$$f = \frac{32}{r^2};$$

but the periodic time of the Moon is $27 \times 24 \times 60 \times 60$ seconds, and therefore its velocity is $\frac{2\pi . 4000r \times 1760 \times 3}{27 \times 24 \times 60 \times 60}$ in feet per second: hence the accelerating force on it tending to its centre is

$$\frac{(2\pi)^2}{(27 \times 24 \times 60 \times 60)^2} . 4000r \times 1760 \times 3.$$

If the Moon be under the influence of the Earth's attraction only, this must be equal to f,

or
$$\frac{32}{r^2} = \frac{(2\pi)^2}{(27 \times 24 \times 60 \times 60)^2} . 4000r \times 1760 \times 3,$$

or
$$r^3 = \frac{32 \times (27 \times 24 \times 60 \times 60)^2}{4\pi^2 . 4000 \times 1760 \times 3}$$

$$= \frac{8 \times 3^6 . 2^6 . 3^2 . 2^4 . 3^4 . (10)^4}{\pi^2 . 2^6 . 11 \times 3 \times 10^4}$$

$$= \frac{2^7 . 3^{11}}{\pi^2 . 11} = \frac{2^6 . 3^{12}}{\pi^2 \times 16{\cdot}5},$$

or
$$r = \frac{324}{(\pi^2 \times 16{\cdot}5)^{\frac{1}{3}}} = 60 \text{ very nearly.}$$

11. Explain the aberration of light, and shew in what direction the error of aberration takes place.

What limit is there to the position of a place in order that at some time in the day a star in the ecliptic may have its error of aberration in a vertical plane?

The aberration of a star always taking place towards a point in the ecliptic 90° behind the sun, if a star be in the ecliptic, its aberration will take place in the ecliptic: the question is therefore equivalent to this, At what places is the ecliptic ever vertical? The answer is, At every place whose zenith is not more than 23° 28′ from the celestial equator, that is, at every place within the tropics.

MONDAY, *Jan.* 16. 9...12.

1. A SYSTEM of rigid bodies is under the action of no forces but their weights, mutual reactions, tensions of inextensible strings, and pressures on smooth fixed surfaces; prove that if the height of the centre of gravity above a fixed horizontal plane be a maximum or a minimum, the system will be in equilibrium.

Apply this principle to determine the position of equilibrium of two equal uniform rods, connected by a smooth hinge at one extremity and resting symmetrically on two smooth pegs in the same horizontal line.

Let A, B (fig. 79) be the pegs, C the middle point of AB, P the hinge connecting the rods, which will be in the vertical line through C; Q the centre of gravity of the two rods, which will be the middle point of the straight line joining the middle points of the rods, and will therefore also be in the vertical line passing through C.

$$\text{The depth of } Q \text{ below } C = PQ - CP$$

$$= a \cos\theta - b \cot\theta,$$

if the length of each rod be $2a$, $AB = 2b$, angle at $P = 2\theta$.

For this depth to be a maximum,

$$0 = -a \sin\theta + \frac{b}{\sin^2\theta},$$

or $$\sin^3\theta = \frac{b}{a},$$

which determines the position of equilibrium.

It is manifest that b is always $< a$, and therefore the position is possible.

2. Determine the necessary and sufficient conditions that a system of forces acting on a rigid body may have a single resultant.

A portion of a curve surface of continuous curvature is cut off by a plane, and at a point in each element of that portion, a force proportional to the element is applied in the direction of the normal; shew that, if all the forces act inwards or all outwards, they will in the limit have a single resultant.

Let ΔS represent an element of the surface, whose coordinates are x, y, z; the bounding plane being taken as that of xy, l, m, n the direction-cosines of the normal. Then, if $P\Delta S$ be the force applied in the direction of the normal, the resolved parts of this force are $Pl\Delta S$, $Pm\Delta S$, $Pn\Delta S$ parallel to the axes of x, y, z, and the moments of this force about the axes of x, y, z are respectively $P\Delta S\,(ny - mz)$, $P\Delta S\,(lz - nx)$, $P\Delta S\,(mx - ly)$. But if A_x, A_y, A_z, represent the projections of the surface on the coordinate planes of yz, zx, xy respectively, we shall have

$$l\Delta S = \Delta A_x,$$
$$m\Delta S = \Delta A_y,$$
$$n\Delta Z = \Delta A_z;$$

or if $\Sigma(X)$, $\Sigma(Y)$, $\Sigma(Z)$ represent the sums of all the resolved forces, and L, M, N the sums of all the moments,

$$\Sigma(X) = P\Sigma(\Delta A_x) = P.A_x = 0,$$
$$\Sigma(Y) = P\Sigma(\Delta A_y) = P.A_y = 0,$$
$$\Sigma(Z) = P\Sigma(\Delta A_y) = P.A_z = P.A,$$

A being the area of the curve bounding the section by the plane of xy, and $\bar{x}, \bar{y}$ the coordinates of its centre of gravity.

Also
$$L = P\Sigma(y\Delta A_z - z\Delta A_y) = P.A.\bar{y},$$
$$M = P\Sigma(z\Delta A_x - x\Delta A_z) = -P.A.\bar{x},$$
$$N = P\Sigma(x\Delta A_y - y\Delta Ax) = 0;$$
$$\therefore\ L\Sigma X + M.\Sigma Y + N.\Sigma Z = 0.$$

3. A particle under the action of any forces rests on a surface whose equation is given; determine the conditions of equilibrium, (1) when the surface is smooth, (2) when it is rough.

Find the least coefficient of friction between a given elliptic cylinder and a particle, in order that, for all positions of the cylinder in which the axis is horizontal, the particle may be capable of resting at any point vertically over the axis.

Let APA' (fig. 80) be a section of the cylinder made by a plane perpendicular to the axis, and passing through the particle, C the centre, CA the semi-axis-major of the elliptic section, P the particle vertically above C, $\angle PCA = \theta$, $2a$, $2b$ the axes of the elliptic section.

Then, in order that the particle may be capable of resting for all values of θ, the greatest angle which the tangent at P can make with the horizon must be not greater than $\tan^{-1}\mu$, μ being the coefficient of friction. Let the tangent at P be produced to meet CA produced in T: then CPT must not be greater than $\frac{\pi}{2} + \tan^{-1}\mu$,

and
$$\tan CPT = -\tan(\theta + CTP)$$
$$= -\frac{\tan\theta + \tan CTP}{1 - \tan\theta . \tan CTP}.$$

Now, if ϕ be the eccentric angle of P,
$$\tan\theta = \frac{b}{a} . \tan\phi,$$
and
$$\tan CTP = \frac{b}{a} . \cot\phi;$$

therefore $$\tan CPT = -\frac{\frac{b}{a}\cdot(\tan\phi + \cot\phi)}{1 - \frac{b^2}{a^2}}$$

$$= -\frac{2ab}{a^2 - b^2}\cdot\frac{1}{\sin 2\phi};$$

therefore the angle which PT makes with the horizon

$$= CPT - \frac{\pi}{2} = \tan^{-1}\frac{(a^2 - b^2)\sin 2\phi}{2ab};$$

therefore the greatest angle which PT makes with the horizon is $\tan^{-1}\dfrac{a^2 - b^2}{2ab}$, and this must not be greater than $\tan^{-1}\mu$: hence μ must not be less than $\dfrac{a^2 - b^2}{2ab}$.

4. A heavy elastic string is suspended from one extremity, and stretched by its own weight; determine its length when it is at rest.

If a heavy elastic string rest upon the convex side of a smooth curve in a vertical plane, shew how to determine the tension at any point.

Let a heavy elastic string (fig. 81) rest in a vertical plane on the smooth curve APQ, beginning at A. Take Ox, Oy horizontal and vertical axes, and let x, y be the coordinates of a point P, Q a contiguous point, $AP = s$, $PQ = \delta s$; then the coordinates of Q will be

$$x + \frac{dx}{ds}\cdot\delta s + \ldots\ldots,$$

$$y + \frac{dy}{ds}\cdot\delta s + \ldots\ldots;$$

and if t be the tension at P, $t + \dfrac{dt}{ds}\cdot\delta s + \ldots\ldots$ will be the tension at Q.

Let the natural length of AP be s', of AQ, $s' + \delta s'$, e the coefficient of elasticity.

Then $$\delta s = \delta s'(1 + et'),$$
where t' is intermediate to the tensions at P and Q, provided PQ be taken sufficiently small;

therefore, taking the limit, $\dfrac{ds}{ds'} = 1 + et$.

Also resolving the forces which act on PQ along the tangent at P, μ being the mass of a unit of length of the string in its natural state, we have ultimately

$$\frac{dt}{ds} = -\mu g \frac{dy}{ds} \cdot \frac{ds'}{ds},$$

$$(1 + et)\frac{dt}{ds} = -\mu g \frac{dy}{ds},$$

an equation from which t may be determined.

5. If a particle be moving in any path, straight or curved, and, at the time t, s be its distance measured along its path from a fixed point; shew that $\dfrac{d^2s}{dt^2}$ is a measure of the accelerating force in the direction of motion.

If the position of a particle moving in a plane be determined by the coordinates ρ and ϕ, ρ being measured from a fixed circle along a tangent which has revolved through an angle ϕ from a fixed tangent, investigate the following expressions for the components of the accelerating force along and perpendicular to ρ respectively, (the latter being considered positive when it tends to increase ϕ):

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2 + a\frac{d^2\phi}{dt^2}, \quad \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right) + a\left(\frac{d\phi}{dt}\right)^2.$$

After proving the first part of the question we may state at once, that if x and y be rectangular coordinates of a particle, the accelerating forces parallel to the axes of x and y are $\dfrac{d^2x}{dt^2}$ and $\dfrac{d^2y}{dt^2}$ respectively.

Let the centre of the fixed circle be the origin, and a line parallel to the fixed tangent be the axis of x: then

$$x = a\sin\phi + \rho\cos\phi, \quad y = -a\cos\phi + \rho\sin\phi;$$

therefore $$\frac{dx}{dt} = \left(a\frac{d\phi}{dt} + \frac{d\rho}{dt}\right)\cos\phi - \rho\frac{d\phi}{dt}.\sin\phi,$$

$$\frac{d^2x}{dt^2} = \left\{a\frac{d^2\phi}{dt^2} + \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2\right\}\cos\phi$$
$$-\left\{a\left(\frac{d\phi}{dt}\right)^2 + 2\frac{d\rho}{dt}\frac{d\phi}{dt} + \rho\frac{d^2\phi}{dt^2}\right\}\sin\phi.$$

Similarly,

$$\frac{d^2y}{dt^2} = \left\{a\frac{d^2\phi}{dt^2} + \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2\right\}\sin\phi$$
$$+\left\{a\left(\frac{d\phi}{dt}\right)^2 + 2\frac{d\rho}{dt}\frac{d\phi}{dt} + \rho\frac{d^2\phi}{dt^2}\right\}\cos\phi;$$

therefore force along $\rho = \dfrac{d^2x}{dt^2}\cos\phi + \dfrac{d^2y}{dt^2}\sin\phi$

$$= a\frac{d^2\phi}{dt^2} + \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2,$$

force perpendicular to $\rho = \dfrac{d^2y}{dt^2}\cos\phi - \dfrac{d^2x}{dt^2}\sin\phi$

$$= a\left(\frac{d\phi}{dt}\right)^2 + 2\frac{d\rho}{dt}\frac{d\phi}{dt} + \rho\frac{d^2\phi}{dt^2}$$

$$= a\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right).$$

6. State the laws which regulate the magnitude and the direction of statical and of sliding friction.

Two equal bodies lie on a rough horizontal table, and are connected by a string which passes through a fine ring on the table; if the string be stretched, find the greatest velocity with which one of the bodies can be projected in a direction perpendicular to its portion of the string without moving the other body.

Dynamical friction acts in the direction in which the body is moving; statical friction acts in the direction in which the body tends to move, that is, the direction in which the body would begin to move if there were no friction.

Suppose that when the one body is projected the other remains at rest; friction will act on the former in the direction perpendicular to the string, on the latter in the direction of the string.

Let m be the mass of each body,

μ the coefficient of friction,

r the length of string between the ring and the projected body,

V the velocity of projection,

T the tension of the string:

since one body is at rest the other body describes a circle, and since friction acts on it in a direction perpendicular to the string,

$$T = m\frac{V^2}{r};$$

but in order that the other may continue at rest, the tension and friction must be equal, therefore the tension must not exceed the greatest possible friction, that is,

$$m\frac{V^2}{r} \ngtr \mu mg;$$

therefore $$V^2 \ngtr \mu gr.$$

The velocity is always decreasing, in consequence of the friction on the moving body; if therefore the other body do not move at first it will not move at all.

9. Having given the index of refraction between the two media A and B, and also between the two A and C, shew how to find that between B and C.

The index of refraction (μ) in a medium varies from point to point, being a function of the distances x and y from two planes at right angles to each other; a ray traverses the medium in a plane perpendicular to these two planes; if $\log\mu = f(x, y)$, prove that the curvature of the path of the ray varies as

$$f'(x)\frac{dy}{ds} - f'(y)\frac{dx}{ds}.$$

Let P (fig. 82) be a point in the path of the ray, PT its direction at that point, $AN = x$, $NP = y$; at P let the ray pass from a medium whose index is μ, to one whose index is $\mu+\delta\mu$, and the direction be consequently changed to PT': let PG be the normal at P to the surface of equal density passing through P. Then

$$\frac{\sin GPT}{\sin GPT'}=\frac{\mu+\delta\mu}{\mu},$$

$$\cos TPT'+\cot GPT'.\sin TPT'=1+\frac{\delta\mu}{\mu};$$

or, taking the limit,

$$\cot GPT.\frac{d}{dx}.\left(\tan^{-1}\frac{dy}{dx}\right)=\frac{1}{\mu}.\frac{d\mu}{dx}$$

$$=\frac{d}{dx}(\log\mu)$$

$$=f'(x)+f'(y)\frac{dy}{dx}.$$

Now, $\cot GPT=\cot(PGx-PTx)$,

and $\tan PGx=\dfrac{f'(y)}{f'(x)}$, $\tan PTx=\dfrac{dy}{dx}$;

therefore
$$\cot GPT=\frac{1+\dfrac{f'(y)}{f'(x)}.\dfrac{dy}{dx}}{\dfrac{f'(y)}{f'(x)}-\dfrac{dy}{dx}};$$

$$\therefore\ \frac{f'(x)+f'(y)\dfrac{dy}{dx}}{f'(y)-f'(x)\dfrac{dy}{dx}}.\frac{\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}=f'(x)+f'(y)\frac{dy}{dx},$$

$$\frac{\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}=f'(y)-f'(x)\frac{dy}{dx},$$

$$\frac{1}{\rho}=\frac{-\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}=f'(x)\frac{dy}{ds}-f'(y).\frac{dx}{ds},$$

or the curvature varies as $f'(x)\dfrac{dy}{ds}-f'(y)\dfrac{dx}{ds}$.

12. Describe the reading microscope of the mural circle. What are 'Runs'? Shew that the effects of the eccentricity and irregular form of the pivot are eliminated by taking the sum of opposite Microscope-readings corrected for Runs.

The effects of eccentricity and of the irregular form of the pivot in the mural circle are of exactly the same kind, viz. displacing the circle, parallel to itself, from the position which it would have had, when the telescope was pointed to the same heavenly body, if its pivot had been truly conical, and the axis of the pivot had passed through the centre of graduation.

Now, if there be two microscopes opposite to each other, that is, having their axes in the same straight line, the two points on the limb actually observed through the microscopes will be at the extremities of a certain chord of the circle, while the points which ought to be observed are at the extremities of a parallel chord; but the two arcs contained between parallel chords of a circle are equal; therefore the error of one reading in excess is equal to the error of the other in defect; these errors are therefore eliminated by taking the sum of opposite readings.

The necessity of correcting for runs arises from the fact, that the errors of runs at two microscopes have no tendency to compensate each other. The error of runs may be kept within convenient limits by properly adjusting the distance of the microscope from the limb; but so long as there is any eccentricity, or any irregularity of the form of the pivot, this distance will necessarily vary, and the error of runs will consequently exist. And when the error exists, the value of the correction to be applied depends upon the number of minutes and seconds which are read off at the microscope in question: for example, if the error for $5'$ be $5''$, the error will be $1''$ for $1'$, $2''$ for $2'$, and so on. Hence it is impossible to give any method for eliminating these errors: they must therefore be separately corrected for.

14. What is the greatest value of the inclination of the Moon's orbit to the ecliptic, for which there would have been a lunar eclipse at every opposition?

Find the lunar ecliptic limits; and determine whether there was or was not an eclipse of the Moon on the 31st of March 1847, from the following data, selected from the Nautical Almanac:

1847.	The Sun's		The Moon's			
	Semidiam.	Longitude.	Semidiam.	Parallax.	Longitude.	Latitude.
Mar. 31. Noon	16′1″.3	10°9′18″.3	14′44″.3	54′ 5″.0	185°56′16″.2	1°10′27″.1
Midnight	......	......	14′45″.8	54′10″.5	191°53′11″.3	0°37′55″.6
Apr. 1. Noon	16′1″.0	11°8′26″.1	....	....	....	....

Apr. 1. Sun's parallax 8″.58, longitude of Moon's ascending node 199° 26′.2.

In order that there might be a lunar eclipse at every opposition, it would be necessary that

the greatest distance of the Moon's centre from the ecliptic $\not>$ the least value of the sum of the Moon's radius and the radius of the Earth's shadow:

but the greatest distance of the Moon's centre from the ecliptic = the inclination of the Moon's orbit to the ecliptic;

therefore the inclination of the Moon's orbit to the ecliptic must not exceed the sum of the Moon's radius and the radius of the Earth's shadow.

From the numerical data the difference of the longitudes of the Sun and Moon was

175° 46′ 57″.9 at noon
181° 14′ 19″.1 at midnight } on March 31.

Hence the separation in longitude during twelve hours was 5° 27′ 21″.2.

In this interval the Moon's longitude increased by 4° 35′ 53″.3, and the Moon's longitude at opposition was 190° 32′ 9″.5. But the longitude of her node was 199° 26′ 12″; therefore the distance of the Moon from her node was 8° 53′ 52″.5.

Now (Hymers' *Astron.*, Art. 425) if the Moon's distance from her node when she is in opposition be less than 9°, there must be an eclipse; therefore on the 31st of March, between noon and midnight, the Moon was eclipsed.

If the distance of the Moon from her node had been between 9° and 12° 36′, we should have been obliged to calculate the exact value, under the given circumstances, of the quantity whose greatest and least values are 9° and 12° 36′.

MONDAY, *Jan.* 16. $1\frac{1}{2}$...4.

2. SHEW that all the roots of the following equation are possible:

$$\frac{A_1^2}{x-a_1}+\frac{A_2^2}{x-a_2}+\frac{A_3^2}{x-a_3}+\dots\dots+\frac{A_r^2}{x-a_r}=1.$$

If possible, let $x = u + v\sqrt{(-1)}$; then

$$\frac{A_1^2}{u-a_1+v\sqrt{(-1)}}+\frac{A_2^2}{u-a_2+v\sqrt{(-1)}}+\dots\dots=1,$$

or
$$\frac{A_1^2\{u-a_1-v\sqrt{(-1)}\}}{(u-a_1)^2+v^2}+\dots\dots\dots\dots\dots\dots=1,$$

and therefore

$$v\sqrt{(-1)}\,.\left\{\frac{A_1^2}{(u-a_1)^2+v^2}+\frac{A_2^2}{(u-a_2)^2+v^2}+\dots\dots\right\}=0,$$

which shews that $v = 0$, and therefore establishes the proposition.—Liouville: *Journal de Mathématiques*, 1838, p. 337.

The same thing is true in relation to the equation

$$\frac{A_1^2}{x-a_1}+\frac{A_2^2}{x-a_2}+\frac{A_3^2}{x-a_3}+\dots\dots+\frac{A_r^2}{x-a_r}=\lambda+\mu^2x.$$

Liouville: *Ibid.*

3. If $\alpha + \beta\sqrt{(-1)}$ be a root of the equation

$$x^3+qx+r=0,$$

prove that α is a root of the equation

$$8x^3+2qx-r=0.$$

Since $\alpha + \beta\sqrt{(-1)}$ is a root of the proposed equation, we have

$$\{\alpha+\beta\sqrt{(-1)}\}^3 + q\{\alpha+\beta\sqrt{(-1)}\} + r = 0,$$

whence

$$\alpha^3 + 3\alpha^2\beta\sqrt{(-1)} - 3\alpha\beta^2 - \beta^3\sqrt{(-1)} + q\alpha + q\beta\sqrt{(-1)} + r = 0.$$

Hence $$\alpha^3 - 3\alpha\beta^2 + q\alpha + r = 0,$$

and $$3\alpha^2 - \beta^2 + q = 0.$$

Eliminating β between these equations, we see that

$$\alpha^3 - 3\alpha(3\alpha^2+q) + q\alpha + r = 0,$$

$$8\alpha^3 + 2q\alpha - r = 0,$$

or that α is a root of the cubic

$$8x^3 + 2qx - r = 0.$$

5. Prove that the series $\tan\alpha - \frac{1}{3}\tan^3\alpha + \frac{1}{5}\tan^5\alpha - \dots$ *ad inf.* is equal to $n\pi + \alpha$, where n is zero or such a positive or negative integer as will make $n\pi + \alpha$ lie between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

Shew that, whatever positive integer m be, if

$$\phi = \frac{2}{(2m+1)\pi}, \quad \frac{1}{\phi} - \phi - \frac{2}{3}\phi^3$$

is a very approximate solution of the equation $\tan\theta = \theta$.

As θ changes from $m\pi$ to $m\pi + \frac{\pi}{2}$,

$\tan\theta$ .... 0 to ∞ continually;

therefore, at some intermediate value of θ, $\tan\theta = \theta$; and since in this case the arc subtended is equal to the linear tangent, the angle must be nearly $m\pi + \frac{\pi}{2}$, and more nearly the larger integer m is.

Let $$\theta = m\pi + \frac{\pi}{2} - \alpha$$

$$= \frac{1}{\phi} - \alpha, \quad \phi \text{ and } \alpha \text{ being small};$$

$$\therefore \frac{1}{\phi} - \alpha = \tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha;$$

$$\therefore\ \tan\alpha = \phi + \alpha\phi\tan\alpha,$$

$$\alpha = \phi, \text{ for a first approximation;}$$

$$\therefore\ \tan\alpha = \phi + \phi^3, \text{ nearly,}$$

and

$$\alpha = \tan\alpha - \tfrac{1}{3}\tan^3\alpha, \ldots\ldots\ldots$$

$$= \phi + \phi^3 - \tfrac{1}{3}\phi^3;$$

$$\therefore\ \theta = \frac{1}{\phi} - \phi - \tfrac{2}{3}\phi^3, \text{ nearly.}$$

6. Investigate the condition of perpendicularity of two straight lines whose equations are

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0.$$

Shew that, if the axes be inclined at an angle ω, the condition that the straight lines may be equally inclined to the axis of x in opposite directions, is

$$\frac{B}{A} + \frac{B'}{A'} = 2\cos\omega.$$

If, besides being equally inclined to the axis of x, the straight lines pass through the origin and be perpendicular to one another, the equation of the straight lines is

$$x^2 + 2xy\cos\omega + y^2\cos 2\omega = 0.$$

Let θ, $\pi - \theta$ be the inclinations of the straight lines to the axis of x.

$$\frac{B}{A} = -\frac{\sin(\omega-\theta)}{\sin\theta}$$

$$= -\sin\omega\cot\theta + \cos\omega,$$

$$\frac{B'}{A'} = \sin\omega\cot\theta + \cos\omega;$$

$$\therefore\ \frac{B}{A} + \frac{B'}{A'} = 2\cos\omega.$$

If the straight lines be perpendicular and pass through the origin $C = 0 = C'$, and

$$AA' + BB' - (AB' + BA')\cos\omega = 0;$$

$$\therefore\ \frac{BB'}{AA'} + 1 - \left(\frac{B'}{A'} + \frac{B}{A}\right)\cos\omega = 0,$$

and
$$\frac{BB'}{AA'} = -1 + 2\cos^2\omega = \cos 2\omega;$$

$$\therefore \quad \frac{B}{A} \text{ and } \frac{B'}{A'} \text{ are roots of the equation}$$

$$z^2 - 2\cos\omega z + \cos 2\omega = 0;$$

$\therefore$ replacing z by $-\dfrac{x}{y}$,

$$x^2 + 2xy\cos\omega + y^2\cos 2\omega = 0$$

is the equation of the two lines.

7. Investigate the equations to the tangents at the extremities of two conjugate diameters of an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the coordinates of the extremity of one of the diameters being given.

In an ellipse SQ and HQ, drawn perpendicularly to a pair of conjugate diameters, intersect in Q; prove that the locus of Q is a concentric ellipse.

Let CP, CD be semi-conjugate diameters of an ellipse, x', y' the coordinates of P.

Then the equations to the tangents at P and D will be

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \quad \text{and} \quad yx' - xy' = ab;$$

therefore the equations to the perpendiculars drawn from S and H on CD, CP, which are parallel to these lines, are

$$(x + ae)\frac{y'}{b^2} - y\frac{x'}{a^2} = 0,$$

and
$$(x - ae)x' + yy' = 0.$$

And eliminating $x'y'$ to find the locus of Q,

$$\frac{x^2 - a^2e^2}{b^2} + \frac{y^2}{a^2} = 0,$$

the equation to a concentric ellipse.

8. Shew that the locus of the poles of all tangents to a given circle, with respect to another fixed circle, is a conic section, whose directrix is the polar of the centre of the first circle.

Employ the method of reciprocal polars to shew that, if three ellipses have one common focus, and pairs of common tangents be drawn to the ellipses taken two together, the three points of intersection of these pairs of tangents lie in a straight line.

Corresponding to each ellipse is a circle,

.... a tangent to each ellipse is a point in the corresponding circle,

.... each common tangent to two ellipses is a point of intersection of the two corresponding circles,

.... intersection of the two common tangents is the common chord of the two circles;

and since three common chords intersect in one point, therefore the three points of intersection of the pairs of common tangents lie in one straight line.

10. Investigate formulæ for the transformation of coordinates in passing from one system of three rectangular axes to another having the same origin.

Shew that the equation of a surface $yz + zx + xy = a^2$ may be reduced to the form

$$x^2 - \frac{y^2 + z^2}{2} = a^2.$$

The surface is evidently symmetrically placed with respect to a line equally inclined to the three coordinate axes; if therefore such a line be one of the axes of a new system, the equation will assume a symmetrical form with respect to the axis.

This will be effected if we first turn the axis of y and z through 45° in their own plane,

or for y write $\qquad \dfrac{y - z}{\sqrt{2}}$,

and for z write $\qquad \dfrac{y + z}{\sqrt{2}}$, and then turn the axis of x and y through the angle $\cos^{-1}\frac{1}{\sqrt{3}}$, or $\sin^{-1}\sqrt{\frac{2}{3}}$, in their own plane.

i.e. for x write $$\frac{x - y\sqrt{2}}{\sqrt{3}},$$

and for y write $$\frac{x\sqrt{2} + y}{\sqrt{3}}.$$

The result of the first substitution is

$$\frac{y^2 - z^2}{2} + xy\sqrt{2} = a^2,$$

and of the second

$$\frac{(y + x\sqrt{2})^2}{6} + \frac{2(x^2 - y^2) - xy\sqrt{2}}{3} - \frac{z^2}{2} = a^2,$$

or $$x^2 - \frac{y^2 + z^2}{2} = a^2.$$

11. If A, B, C, be extremities of the axes of an ellipsoid, and AC, BC be the principal sections containing the least axis, find the equations of the two cones, whose vertices are A, B, and bases BC, AC respectively: shew that they have a common tangent plane, and a common parabolic section, the plane of the parabola and the tangent plane intersecting the ellipsoid in ellipses the area of one of which is double that of the other; and, if l be the latus-rectum of the parabola, l_1, l_2 of the sections AC, BC, prove that

$$\frac{1}{l^2} = \frac{1}{l_1^2} + \frac{1}{l_2^2}.$$

The equations of the base BC are

$$x = 0, \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let the equations of a generating line be

$$\frac{x - a}{l} = \frac{y}{m} = \frac{z}{n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

at the point of intersection with BC (1) and (2) are simultaneous;

$$\therefore\ -\frac{a}{l}=\frac{y}{m}=\frac{z}{n},$$

$$\therefore\ \frac{m^2}{b^2}+\frac{n^2}{c^2}=\frac{l^2}{a^2};$$

therefore the equation of the cone, vertex A, is

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{(x-a)^2}{a^2}.$$

Similarly, the equation of the cone, vertex B, is

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=\frac{(y-b)^2}{b^2}.$$

These surfaces intersect where

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-\left(\frac{x-a}{a}\right)^2+\left(\frac{y-b}{b}\right)^2$$

$$=\left(\frac{y}{b}-\frac{x}{a}\right)\left(\frac{y}{b}+\frac{x}{a}-2\right);$$

therefore, where $\dfrac{x}{a}=\dfrac{y}{b}$ .... (3),

or $\dfrac{x}{a}+\dfrac{y}{b}=1$.......................... (4),

where $\dfrac{x}{a}=\dfrac{y}{b},\quad \dfrac{z^2}{c^2}=1-\dfrac{2x}{a},$

or the projection of the curve of intersection is a parabola, so is therefore the curve itself, where

$$\frac{x}{a}+\frac{y}{b}=1,\quad z^2=0,$$

or the plane denoted by (4) is a tangent plane, since it only meets either surface in a generating line. Whence the two properties are established.

The planes (3) and (4) are inclined at the same angle to the plane of xz, and the equations of the projections of their intersection with the ellipsoid, whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

are $$\frac{2x^2}{a^2}+\frac{z^2}{c^2}=1 \text{ .... } (5),$$

and $$\frac{2x^2}{a^2}-\frac{2x}{a}+1+\frac{z^2}{c^2}=1,$$

or $$z^2=\frac{2c^2}{a^2}(ax-x^2) \text{ .... } (6).$$

The half-axes of (5) are $\frac{a}{\sqrt{2}}$, c, and of (6) $\frac{a}{2}$, $\frac{c}{\sqrt{2}}$; therefore one area is double of the other, and the same is true of the areas of the equally inclined curves of intersection.

The secant (fig. 83) of the inclination of the plane (3) to that of xz is

$$\frac{\sqrt{(a^2+b^2)}}{a},$$

and if DC be the parabolic section, MP an ordinate,

$$DM=\left(\frac{a}{2}-x\right)\frac{\sqrt{(a^2+b^2)}}{a};$$

$$\therefore\ PM^2=\frac{2c^2}{\sqrt{(a^2+b^2)}}DM;$$

$$\therefore\ \frac{1}{l^2}=\frac{a^2+b^2}{4c^4}=\frac{1}{l_1^2}+\frac{1}{l_2^2}.$$

Or we can shew the geometrical properties thus: The cones have a common generating line AB, and the plane through AB parallel to OC, which touches both sections BC and AC, is a tangent plane to each.

Also, all plane sections parallel to DOC, since OD is parallel to a generating line of each cone, are parabolic sections; therefore, DOC must be a parabolic section.

Again, all parallel sections of an ellipsoid are similar ellipses; therefore, the section of the ellipsoid by DOC being equal in all respects to the sections through OC parallel to BA, the areas of the sections by DOC and the tangent plane through AB are as $AD^2 : \alpha^2$, α being the half-diameter conjugate to OD, and $AD^2 = \alpha^2 - AD^2$, since the conjugate diameters are equal;

$$\therefore\ \alpha^2 = 2AD^2,$$

and the areas are one double of the other.

12. Prove that, if p, q, r, be the lengths of arcs of great circles drawn from the angles A, B, C, of a spherical triangle perpendicularly to the opposite sides,

$$\sin a \sin p = \sin b \sin q = \sin c \sin r$$
$$= (1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c)^{\frac{1}{2}}.$$

By the formulæ of spherical trigonometry,

$$\sin p = \sin c \,.\, \sin B \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$\cos b = \cos c \,.\, \cos a + \sin c \,.\, \sin a \,.\, \cos B \ldots\ldots\ldots (2).$$

From (1) and (2)

$$(\cos b - \cos c \,.\, \cos a)^2 + \sin^2 p \,.\, \sin^2 a = \sin^2 c \,.\, \sin^2 a,$$
$$\sin^2 a \,.\, \sin^2 p = \sin^2 c \,.\, \sin^2 a - \cos^2 b + 2\cos a \cos b \cos c - \cos^2 c \cos^2 a$$
$$= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c,$$

whence, by symmetry,

$$\sin a \sin p = \sin b \sin q = \sin c \sin r$$
$$= \{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c\}^{\frac{1}{2}}.$$

The equations $\sin a \sin p = \sin b \sin q = \sin c \sin r$ may be proved also thus.

From (1) and the analogous equation

$$\sin b = \sin a \sin C,$$

we have $$\frac{\sin p}{\sin q} = \frac{\sin c}{\sin a} \cdot \frac{\sin B}{\sin C} = \frac{\sin c}{\sin a} \cdot \frac{\sin b}{\sin c} = \frac{\sin b}{\sin a},$$

and therefore, by symmetry,

$$\sin a \sin p = \sin b \sin q = \sin c \sin r.$$

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4. Determine the motion of a planet in geocentric longitude, and shew that all planets will sometimes appear stationary to an observer on the earth.

If m be the ratio of the radius of the Earth's orbit to that of an inferior planet, n the ratio of their motions in longitude considered uniform, shew that the elongation of the planet as seen from the Earth, when the planet appears stationary, is equal to

$$\tan^{-1}\left(\sqrt{\frac{1-m^2n^2}{m^2-1}}\right).$$

Let S, E, P, (fig. 84) be the positions of the Sun, Earth, and planet at the time when the planet appears stationary, E', P' the positions of the Earth and planet immediately afterwards: then EE', PP' may be considered coincident with the tangents at E, P; and since the planet appears stationary from the Earth, EP is parallel to $E'P'$: also the orbits must be considered circular, since the motion in longitude is uniform: produce the tangents at E, P to meet in T.

$$TP : TE :: PP' : EE' :: \angle PSP' : m \,.\, \angle ESE'$$
$$:: 1 : mn.$$

But $TP : TE :: \sin TEP : \sin TPE :: \cos SEP : -\cos SPE$,

$$mn = \frac{-\cos SPE}{\cos SEP};$$

therefore
$$1 - m^2n^2 = \frac{\cos^2 SEP - \cos^2 SPE}{\cos^2 SEP}.$$

Also
$$m = \frac{SE}{SP} = \frac{\sin SPE}{\sin SEP},$$

$$m^2 - 1 = \frac{\sin^2 SPE - \sin^2 SEP}{\sin^2 SEP}$$

$$= \frac{\cos^2 SEP - \cos^2 SPE}{\sin^2 SEP},$$

or
$$\frac{1-m^2n^2}{m^2-1} = \frac{\sin^2 SEP}{\cos^2 SEP},$$

and SEP the elongation $= \tan^{-1}\sqrt{\left(\frac{1-m^2n^2}{m^2-1}\right)}$.

5. **Determine the motion of a particle acted on by given forces, and constrained to remain on a given surface.**

A particle is in motion on the surface whose equation is $z = \phi(x, y)$, and is acted on by a constant accelerating force f parallel to the axis of z; if v be the velocity of the particle, and its path be always perpendicular to the direction of the force, shew that, at any point of its path,

$$\frac{v^2}{f} = \frac{\left\{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^2}{\frac{d^2z}{dx^2}\left(\frac{dz}{dy}\right)^2 - 2\frac{d^2z}{dx\,dy}\frac{dz}{dx}\frac{dz}{dy} + \frac{d^2z}{dy^2}\left(\frac{dz}{dx}\right)}.$$

If R be the pressure of the surface on the particle, in the direction of the normal whose direction-cosines are l, m, n, the equations of motion are, M being the mass of the particle,

$$M\frac{d^2x}{dt^2} = Rl,$$

$$M\frac{d^2y}{dt^2} = Rm,$$

$$M\frac{d^2z}{dt^2} = Rn + Mf;$$

but the path of the particle being always perpendicular to the direction of the force, z is constant throughout the motion, and $\frac{dz}{dt} = 0$, $\frac{d^2z}{dt^2} = 0$, and the equations become

$$\frac{d^2x}{dt^2} = -f\frac{l}{n} = f\frac{dz}{dx},$$

$$\frac{d^2y}{dt^2} = -f\frac{m}{n} = f\frac{dz}{dy}.$$

Also, since $\frac{dz}{dt} = 0$,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v^2 \quad \text{and} \quad \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt} = 0,$$

and therefore

$$\frac{dx}{dt} = \pm\frac{v\frac{dz}{dy}}{\sqrt{\left\{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}}, \quad \frac{dy}{dt} = \mp\frac{v\frac{dz}{dx}}{\sqrt{\left\{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}}.$$

Also, differentiating the equation $\frac{dz}{dx}\frac{dx}{dt}+\frac{dz}{dy}\frac{dy}{dt}=0$, with respect to t,

$$\frac{d^2z}{dx^2}\left(\frac{dx}{dt}\right)^2+2\frac{d^2z}{dx\,dy}\frac{dx}{dt}\frac{dy}{dt}+\frac{d^2z}{dy^2}\left(\frac{dy}{dt}\right)^2$$
$$+\frac{dz}{dx}\frac{d^2x}{dt^2}+\frac{dz}{dy}\frac{d^2y}{dt^2}=0;$$

or substituting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{d^2x}{dt^2}$, and $\frac{d^2y}{dt^2}$,

$$\frac{v^2}{\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\left\{\frac{d^2z}{dx^2}\left(\frac{dz}{dy}\right)^2-2\frac{d^2z}{dx\,dy}\cdot\frac{dz}{dx}\cdot\frac{dz}{dy}+\frac{d^2z}{dy^2}\cdot\left(\frac{dz}{dx}\right)^2\right\}$$
$$+f\left\{\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}=0,$$

or $$\frac{v^2}{f}+\frac{\left\{\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}^2}{\frac{d^2z}{dx^2}\cdot\left(\frac{dz}{dy}\right)^2-2\frac{d^2z}{dx\,dy}\cdot\frac{dz}{dx}\cdot\frac{dz}{dy}+\frac{d^2z}{dy^2}\cdot\left(\frac{dz}{dx}\right)^2}=0.$$

The different sign depends on the direction in which f is estimated.

8. Define the principal axes of a rigid body, and shew that for every point in space there exists a system of such axes.

Shew that in general there is only one point for which the principal axes are parallel to those drawn through a given point; but that, if the given point be in one of the principal planes through the centre of gravity, there is an infinite number of such points lying in an hyperbola which passes through the given point.

Let the rigid body be referred to axes through the centre of gravity, parallel to the principal axes through the point whose coordinates are α, β, γ; therefore x, y, z, being coordinates of a particle in

$$\Sigma m(x-\alpha)\,(y-\beta)=0,$$

therefore, since $$\Sigma(mx)=0=\Sigma(my),$$

$$\Sigma(mxy)=M\alpha\beta.$$

If ξ, η, ζ, be coordinates of a point for which the principal axes are parallel to the given axes,

$$\Sigma(mxy) = M\xi\eta;$$

therefore $\xi\eta = \alpha\beta$
similarly, $\eta\zeta = \beta\gamma$
and $\zeta\xi = \gamma\alpha$

$$\left.\begin{aligned}\xi\eta &= \alpha\beta\\ \eta\zeta &= \beta\gamma\\ \zeta\xi &= \gamma\alpha\end{aligned}\right\} \quad \text{......................} \quad (1).$$

If α, β, and γ be each different from zero,

$$\xi = -\alpha, \quad \eta = -\beta, \quad \text{and} \quad \zeta = -\gamma;$$

or there is only one other point equally distant from G, and in the line joining G and the given point.

But if they be not all different from zero, let $\gamma = 0$;

therefore $$\Sigma(myz) = 0,$$

and $$\Sigma(mzx) = 0;$$

therefore xy is one of the principal planes through G: and in this case the equations (1) are satisfied by

$$\zeta = 0 \quad \text{and} \quad \xi\eta = \alpha\beta;$$

therefore all points in the rectangular hyperbola represented by those equations satisfy the required condition.

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3. If $f(x)$ be a continuous function of x, shew that, when x increases, $f(x)$ increases or diminishes according as $f'(x)$ is positive or negative; deduce tests which are sufficient for distinguishing between the maximum and minimum values of $f(x)$, supposing them to exist for certain values of x.

Find the least triangle which can be described about a given ellipse, having a side parallel to the major axis.

The triangle (fig. 85) is evidently isosceles.

Take the side parallel to the major axis to touch the ellipse on B',

Let a, b, be the ½ axes,

xy the coordinates of P, the point of contact of one of the sides,

E the vertex.

$$CE = \frac{b^2}{y},$$

$$B'D : x :: B'E : \frac{b^2}{y} - y$$

$$:: b + \frac{b^2}{y} : \frac{b^2 - y^2}{y}$$

$$:: b : b - y;$$

$$\text{area of triangle} = B'D \, . \, B'E,$$

$$\propto \frac{x}{b-y} \, . \, \frac{b+y}{y};$$

therefore $$\frac{x^2}{y^2} \, . \left(\frac{b+y}{b-y}\right)^2 \text{ is a minimum,}$$

and $$x^2 : b^2 - y^2 :: a^2 : b^2;$$

therefore $$\frac{(b+y)^3}{y^2(b-y)} \text{ is a minimum;}$$

therefore $$\frac{3}{b+y} - \frac{2}{y} + \frac{1}{b-y}$$

changes sign from $-$ to $+$ as y increases;

therefore $(4b - 2y)\,y - 2\,(b^2 - y^2) = 2b\,(2y - b)$

changes sign from $-$ to $+$,

which happens when $$y = \frac{b}{2},$$

or $$CE = 2BC.$$

Otherwise. Since an equilateral triangle is the least triangle circumscribing a circle, in which case the height of the triangle is 3 times the radius, project both on a plane inclined to the plane of the triangle through one side, and the projection of the triangle is circumscribed round that of the circle, which is an ellipse, and the height of the triangle is 3 times the minor axis.

4. If $x^n f(y)$ contain all the terms involving the highest power of x in the rationalized equation of a curve, shew that $f(y) = 0$ is the equation of all the asymptotes parallel to the axis of x.

If the equation, arranged in the form of a series of homogeneous functions of descending order, be

$$x^n f\left(\frac{y}{x}\right) + x^{n-r}\phi\left(\frac{y}{x}\right) + \dots = 0,$$

and $f(z) = 0$ have two equal roots different from zero, each equal to α; shew that, if $r = 1$, there is a parabolic asymptote whose equation is

$$(y - \alpha x)^2 = x\,\frac{-2\phi(\alpha)}{f''(\alpha)};$$

and, if $r = 2$, there are two parallel rectilinear asymptotes whose equations are

$$y = \alpha x \pm \left\{\frac{-2\phi(\alpha)}{f''(\alpha)}\right\}^{\frac{1}{2}}.$$

Divide the equation by x^n, and the result is $f(y)$ + terms involving negative powers of x.

The curve is satisfied by the system of values

$$x = \infty \text{ and } f(y) = 0;$$

therefore, $f(y) = 0$ is the equation of a series of all straight lines parallel to the axis of x, which meet the curve at an infinite distance, or is the equation of all the asymptotes parallel to the axis of x.

Let α be a root of $f(z) = 0$, and let $y = \alpha x + t$ at every point of the curve;

$$\therefore\ x^n f\left(\alpha + \frac{t}{x}\right) + x^{n-r}\phi\left(\alpha + \frac{t}{x}\right) + \dots\dots = 0,$$

$$\text{and } f(\alpha) + f'(\alpha)\,\frac{t}{x} + f''\left(\alpha + \theta\,\frac{t}{x}\right)\frac{t^2}{2x^2} + \frac{1}{x^r}\,\phi\left(\alpha + \frac{t}{x}\right) + \dots = 0,$$

$$\theta > 0 < 1.$$

If now $f(z) = 0$ have two roots $= \alpha$,

$$f(\alpha) = 0 \text{ and } f'(\alpha) = 0;$$

$$\therefore\ f''\left(\alpha+\theta\frac{t}{x}\right)\frac{t^2}{2}+x^{2-r}\phi\left(\alpha+\frac{t}{x}\right)+\ldots=0.$$

If possible, let x be taken indefinitely large.

I. If
$$r=1,$$
$$f''(\alpha)\frac{t^2}{2}+x\phi(\alpha)=0,$$
and
$$t^2=\frac{-2x\phi(\alpha)}{f''(\alpha)};$$
$$\therefore\ (y-\alpha x)^2=x\,.\,\frac{-2\phi(\alpha)}{f''(\alpha)}$$
is the ultimate relation between x and y at an infinite distance.

Or the curve ultimately coincides with a parabola of which $y=\alpha x$ is the equation of a diameter.

II. If
$$r=2,$$
$$f''(\alpha)\frac{t^2}{2}=-\phi(\alpha);$$
$$\therefore\ y-\alpha x=\pm\left\{\frac{-2\phi(\alpha)}{f''(\alpha)}\right\}^{\frac{1}{2}}$$
is the equation of two straight lines which are asymptotes to the curve.

To trace the curve whose equation is
$$xy(y-x)^2-ay^3=a^4.$$
When
$$x=0,\quad y=-a,$$
x cannot be negative for positive values of y;

$x-a=0$ is the equation of the asymptote parallel to the axis of y (1), $y=0$ to that parallel to the axis of x (2).

Let
$$y-x=t;$$
therefore
$$xyt^2-ay^3=a^4;$$
therefore, when x and y are very large,
$$x^2t^2-ax^3=0;$$
therefore
$$(y-x)^2=ax$$
is the equation of a parabolic asymptote (3).

To find on which side of (1) the curve lies, retain terms to order y^3;

therefore $$(x-a)\,y^3 - 2x^2y^3 = 0,$$

and $$x = a + \frac{2a^2}{y}.$$

Similarly for (2),

$$yx^3 - 2x^2y^2 = a^4,$$

$$y = \frac{a^4}{x^3};$$

therefore the shape of the curve is that which is given in fig. (86).

5. If r, θ be coordinates of a point in a plane curve, and ϕ the angle between the radius-vector and tangent at that point, prove that

$$\cos\phi = \frac{dr}{ds}, \quad \text{and} \quad \sin\phi = r\frac{d\theta}{ds}.$$

S, H are two fixed points, and a curve is described such that, if P be a point in it, the rectangle contained by SP and HP is constant; shew that the straight lines drawn from S at right angles to SP and from H at right angles to HP meet the tangent at P in points equidistant from P.

Let T, T' (fig. 87) be the points in which the straight lines so drawn meet the tangent at P, $SP = r$, $HP = r'$.

Then $$\cos SPT = \frac{dr}{ds}, \quad \cos HPT = \frac{dr'}{ds},$$

and $$PT = \frac{SP}{\cos SPT} = \frac{r}{\frac{dr}{ds}},$$

$$PT' = \frac{HP}{\cos HPT'} = \frac{r'}{-\frac{dr'}{ds}};$$

but, since rr' is constant,

$$r\frac{dr'}{ds} + r'\frac{dr}{ds} = 0,$$

or
$$\frac{r}{\frac{dr}{ds}} = -\frac{r'}{\frac{dr'}{ds}},$$
$$PT = PT'.$$

6. Trace the curve whose equation is
$$\frac{a}{r} = 1 - \tan\theta.$$

We find that
$$r\frac{d\theta}{dr} = \frac{1-\tan\theta}{1+\tan^2\theta} \quad \text{and} \quad r^2\frac{d\theta}{dr} = \frac{a}{1+\tan^2\theta},$$

when $\theta = 0$, $r = a$, $r\dfrac{d\theta}{dr} = 1$, and the curve cuts the prime radius at an angle $\dfrac{\pi}{4}$;

when
$$\theta > 0 < \frac{\pi}{4},$$
r is positive and changes from 0 to ∞, and when $\theta = \dfrac{\pi}{4}$, $r^2\dfrac{d\theta}{dr} = \dfrac{a}{2}$, giving an asymptote;
$$\theta > \frac{\pi}{4} < \frac{\pi}{2},$$
r is negative and changes from ∞ to 0, and when $\theta = \dfrac{\pi}{2}$, $r^2\dfrac{d\theta}{dr} = 0$, or the curve passes through the origin in a direction perpendicular to the prime radius;
$$\theta > \frac{\pi}{2} < \pi,$$
r is positive and changes from 0 to a, when $\theta = \pi$, $r\dfrac{d\theta}{dr} = 1$, and the curve again cuts the prime radius at an angle $\dfrac{\pi}{4}$.

Since $\tan\alpha = \tan(\pi + \alpha)$, the remaining portion of the curve from $\theta = \pi$ to $\theta = 2\pi$ is precisely similar to that already discussed, and the form of the curve is that given in fig. 88.

O being the origin, $OA = OA' = a,$

$$OB = OB' = \frac{a}{2},$$

$$OC = OC' = \frac{a}{\sqrt{2}};$$

AL the branch from $\theta = 0$ to $\theta = \frac{\pi}{4}$,

MO .... $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$,

$OB'A'$ .... $\theta = \frac{\pi}{2}$ to $\theta = \pi$,

and so on for the other quadrants.

8. State between what limits the summation of $dx\,dy\,dz$ should be performed, in order to obtain the volume contained between the conical surface whose equation is $x^2 + y^2 = (a - z)^2$ and the planes whose equations are $x = z$ and $x = 0$; and find the volume by this or any other method.

Integrate

from $z = MQ = x$ to $z = MP = a - \sqrt{(x^2 + y^2)}$, (fig. 89),

from $y = -RN = -\sqrt{(a^2 - 2ax)}$ to $y = +RN = +\sqrt{(a^2 - 2ax)}$,

R being the projection of S on OAB,

and from $x = 0$ to $x = OE = \frac{a}{2}$.

The section of the cone by the plane $z = x$ is a parabola, being parallel to the opposite generating line.

The area of the base of the required volume

$$= \frac{2}{3} \cdot 2a \cdot \frac{a}{\sqrt{2}},$$

$$\text{height} = \frac{a}{\sqrt{2}};$$

therefore the volume required $= \frac{1}{3} \cdot \frac{2}{3} \cdot 2a\, \frac{a}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}}$

$$= \frac{2a^3}{9};$$

or, performing the integrals,

$$\text{volume required} = \iint dx\,dy\,\{a - x - \sqrt{(x^2+y^2)}\}$$

$$= 2\int dx \left\{(a-x)\,y - \frac{y\sqrt{(x^2+y^2)}}{2} - \frac{x^2}{2}\,l_\varepsilon\{y+\sqrt{(x^2+y^2)}\}\right\}_0^{\sqrt{(a^2-2ax)}}$$

$$= \int_0^{\frac{1}{2}a} dx \left\{2(a-x)\sqrt{(a^2-2ax)} - (a-x)\sqrt{(a^2-2ax)} - x^2 l_\varepsilon \frac{a-x+\sqrt{(a^2-2ax)}}{x}\right\};$$

and, if $$1 - \frac{2x}{a} = z^2,$$

or $$-\frac{dx}{a} = z\,dz$$

$$= a^3\int_0^1 z\,dz \left\{\frac{1+z^2}{2}\,z - \frac{(z^2-1)^2}{4}\,l_\varepsilon\,\frac{1+z^2-2z}{1-z^2}\right\}$$

$$= \frac{a^3}{4}\int_0^1 dz \left\{2z^2(1+z^2) - z(z^2-1)^2\,l_\varepsilon\,\frac{1+z}{1-z}\right\},$$

and $$\int z\,dz\,(z^2-1)^2\,l_\varepsilon\,\frac{1+z}{1-z} = \frac{1}{6}(z^2-1)^3\,l_\varepsilon\,\frac{1+z}{1-z} - \frac{1}{6}\int (z^2-1)^3\,.\,\frac{2dz}{1-z^2};$$

$$\therefore \int_0^1 z\,dz\,(z^2-1)^2 l_\varepsilon\frac{1+z}{1-z} = \frac{1}{3}\int_0^1 (1-z^2)^2\,dz = \frac{1}{3}\left(z - \frac{2z^3}{3} + \frac{z^5}{5}\right)_0^1;$$

therefore volume required

$$= \frac{a^3}{4}\left\{\frac{2}{3} + \frac{2}{5} - \frac{1}{3} + \frac{2}{9} - \frac{1}{15}\right\}$$

$$= \frac{a^3}{4}\left\{\frac{1}{3} + \frac{2}{9} + \frac{1}{3}\right\}$$

$$= \frac{a^3}{2}\left\{\frac{1}{3} + \frac{1}{9}\right\} = \frac{2a^3}{9}.$$

9. Give a geometrical interpretation of the singular solution of a differential equation.

Investigate the singular solution of the equation

$$8y^2\left(\frac{dy}{dx}\right)^2 - 2xy\,\frac{dy}{dx} + 9y^2 - x^2 = 0,$$

and shew that it is the envelope of a series of circles described on the subnormal of a rectangular hyperbola as diameter.

$$V = 8y^2p^2 - 2xyp + 9y^2 - x^2 = 0,$$

$$\frac{dV}{dp} = 16y^2p - 2xy = 0;$$

and eliminating p,

$$xyp - 2xyp + 9y^2 - x^2 = 0,$$

$$9y^2 - x^2 - \frac{x^2}{8} = 0,$$

$$x^2 = 8y^2.$$

Also $$\frac{dV}{dy} = 16yp^2 - 2xp + 18y$$

$$= 18y;$$

therefore $x^2 = 8y^2$ is a singular solution.

The equation of a rectangular hyperbola being

$$x^2 - y^2 = a^2,$$

the subnormal $= \xi$ at a point (ξ, η).

The equation of a circle on this subnormal is

$$\left(x - \frac{3\xi}{2}\right)^2 + y^2 = \left(\frac{\xi}{2}\right)^2;$$

therefore, performing the operation for determining the locus of the ultimate intersections,

$$-3\left(x - \frac{3\xi}{2}\right) = \frac{\xi}{2},$$

$$4\xi = 3x;$$

therefore the equation of the locus is

$$\left(\frac{x}{8}\right)^2 + y^2 = \frac{9x^2}{8^2},$$

or $$8y^2 = x^2,$$

the singular solution of the differential equation.

10. Shew that the differential equation of all surfaces which are generated by a circle, whose plane is parallel to the plane of yz, and which passes through the axis of x and through two curves respectively in the planes of zx and xy, is

$$(y^2+z^2)\,t+2\,(z-yq)\,(1+q^2)=0.$$

Let the equations of the two curves be respectively

$$y=0, \quad z=\phi(x) \dots\dots\dots\dots (1),$$
$$z=0, \quad y=\psi(x) \dots\dots\dots\dots (2).$$

The equations of a generating circle

$$y^2+z^2-\beta y-\gamma z=0,$$
$$x=\alpha;$$

and since this circle meets the curve (1),

$$\phi(\alpha)=\gamma,$$

similarly $\quad \psi(\alpha)=\beta;$

therefore the equation of the surface generated is

$$y^2+z^2-\psi(x)\,y-\phi(x)\,z=0 \dots\dots\dots (3);$$

therefore $\quad 2y+2zq-\psi(x)-\phi(x)\,q=0 \dots\dots\dots (4),$

$$2+2q^2+2qzt-\phi(x)\,t=0 \dots\dots\dots (5),$$

and eliminating the functions by cross-multiplication,

$$(y^2+z^2)\,t-2\,(y+zq)\,yt+2\,(1+q^2+zqt)\,(yq-z)=0;$$

therefore $\quad (y^2+z^2)\,t+2\,(1+q^2)\,(z-yq)=0.$

11. Find the general functional equation to surfaces generated by the motion of a straight line which always intersects and is perpendicular to a given straight line.

If the surface whose equation referred to rectangular coordinates is

$$ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy+2a''x+2b''y+2c''z+1=0,$$

be capable of generation in this manner, shew that

$$a+b+c=0, \quad aa'^2+bb'^2+cc'^2=2a'b'c'+abc.$$

If the equation to the given straight line be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n},$$

the general functional equation to such surfaces is

$$lx+my+nz=\phi\left\{\frac{n(y-b)-m(z-c)}{n(x-a)-l(z-c)}\right\}$$

(Gregory's *Solid Geometry*, Art. 206),

$$=\phi\left(\frac{u}{v}\right) \text{ suppose}$$

$$=\frac{A+B\left(\frac{u}{v}\right)+C\left(\frac{u}{v}\right)^2+\ldots}{A'+B'\left(\frac{u}{v}\right)+C'\left(\frac{u}{v}\right)^2+\ldots}.$$

Now, u, v being linear functions of x, y, z, if the surface be of the second order, this must become of the form

$$\frac{A+B\frac{u}{v}}{A'+B'\frac{u}{v}},$$

and the equation to the surface will be

$$(lx+my+nz)\,(A'v+B'u)=Av+Bu,$$

or

$$(lx+my+nz)\,\{A'(nx-lz)+B'(ny-mz)-A'(na-lc)-B'(nb-mc)\}$$
$$=Av+Bu.$$

This being coincident with the given equation of the second order, we must have, λ being some factor,

$$\lambda a=A'nl,\quad \lambda b=B'mn,\quad \lambda c=-A'nl-B'mn;$$

therefore $\quad \lambda(a+b+c)=0,\quad$ or $\quad a+b+c=0,$

since $\lambda=0$ would destroy the whole.

The second condition may be obtained from the values of the coefficients, but may be inferred from the fact of the generators being all parallel to a fixed plane, and successive generators not intersecting, so that the surface must be a hyperbolic paraboloid, the condition for which is

$$aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c' = 0.$$

Obs. The equation to this surface may be reduced to the form $y^2 - z^2 = lx$.

THURSDAY, *Jan.* 19. $1\frac{1}{2}$...4.

5. INTEGRATE the equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{y}{a^2} = 0.$$

Put $y = \frac{v}{x}$: then

$$\frac{dy}{dx} = x^{-1}\frac{dv}{dx} - x^{-2}v,$$

$$\frac{d^2y}{dx^2} = x^{-1}\frac{d^2v}{dx^2} - 2x^{-2}\frac{dv}{dx} + 2x^{-3}v.$$

Hence
$$x^{-1}\frac{d^2v}{dx^2} + \frac{v}{a^2}x^{-1} = 0,$$

$$\frac{d^2v}{dx^2} + \frac{v}{a^2} = 0,$$

$$v = xy = C\sin\left(\frac{x}{a} + C\right),$$

where C and C' are arbitrary constants.

5. Obtain a general expression for $\psi(x)$ from the equation

$$\psi(x) + \psi(1-x) = c.$$

This equation is a particular case of the equation

$$\psi(x) + a\psi(1-x) + (a-1)\phi(x) = c \,\ldots\ldots\ldots\, (1),$$

when $a = 1$, $\phi(x)$ denoting an arbitrary function of x.

In the equation (1) put $1-x$ for x: then

$$\psi(1-x) + a\psi(x) + (a-1)\phi(1-x) = c \,\ldots\ldots\, (2).$$

Eliminating $\psi(1-x)$ between (1) and (2), we have

$$(a^2-1)\,\psi(x) + (a-1)\,\{a\phi(1-x) - \phi(x)\} = (a-1)\,c,$$

$$\psi(x) = \frac{1}{a+1}\,\{c + \phi(x) - a\phi(1-x)\}.$$

Put $a=1$: then

$$\psi(x) = \tfrac{1}{2}\{\phi(x) - \phi(1-x) + c\},$$

which is a general expression for $\psi(x)$.

6. A lamina, in the form of a semi-ellipse bounded by the axis minor, is moveable about the centre as a fixed point, and falls from the position in which its plane is horizontal; find the pressure on the fixed point for any position of the lamina, and determine the impulse which must be applied at the centre of gravity, when the lamina is vertical, in order to reduce it to rest.

If this force be applied perpendicularly to the lamina at the extremity of an ordinate through the centre of gravity, instead of being applied at the centre of gravity itself, about what axis will the lamina begin to revolve?

If the axis minor had been a fixed axis, the pressure of the lamina on the axis would, by symmetry, have passed through the centre; therefore in the actual case, when the centre only is fixed, there will be a pressure at this point, and the lamina will revolve about the axis minor.

Let θ be the angle described at a given time,

h the distance of the centre of gravity from the axis minor,

Mk'^2 the moment of inertia of the lamina about the axis minor,

R, F, the pressures on the fixed point, parallel and perpendicular to the lamina;

then the effective forces on a particle δm, at a distance r from the axis, are

$$\delta m \,.\, r\left(\frac{d\theta}{dt}\right)^2, \text{ and } \delta m \,.\, r\,\frac{d^2\theta}{dt^2},$$

respectively parallel and perpendicular to the lamina; therefore, by D'Alembert's principle,

$$R = Mg\sin\theta + \Sigma\left\{\delta m\, r\left(\frac{d\theta}{dt}\right)^2\right\}$$

$$= Mg\sin\theta + Mh\left(\frac{d\theta}{dt}\right)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$F = Mg\cos\theta - \Sigma\left(\delta m r\,\frac{d^2\theta}{dt^2}\right)$$

$$= Mg\cos\theta - Mh\,\frac{d^2\theta}{dt^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

$$\Sigma\left(\delta m\, g\cos\theta\,.\,r\right) = \Sigma\left(\delta m\, r\,\frac{d^2\theta}{dt^2}\,.\,r\right),$$

$$Mgh\cos\theta = Mk'^2\,\frac{d^2\theta}{dt^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (3) $$\left(\frac{d\theta}{dt}\right)^2 = \frac{2gh}{k'^2}\sin\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

the constant being omitted, because $\frac{d\theta}{dt} = 0$, when $\theta = 0$; therefore, substituting for $\frac{d^2\theta}{dt^2}$ and $\left(\frac{d\theta}{dt}\right)^2$, in (1) and (2), and observing that $h = \frac{4a}{3\pi}$ and $k'^2 = \frac{a^2}{4}$,

$$R = Mg\sin\theta\left(1 + 2\,\frac{h^2}{k'^2}\right) = Mg\sin\theta\left(1 + \frac{128}{9\pi^2}\right),$$

$$F = Mg\cos\theta\left(1 - \frac{h^2}{k'^2}\right) = Mg\cos\theta\left(1 - \frac{64}{9\pi^2}\right).$$

Let X be the impulse which must be applied to the centre of gravity, when the lamina is vertical, in order to reduce it to rest.

Let ω be the angular velocity at this time; then, by (4),

$$\omega^2 = \frac{2gh}{k'^2},$$

and the effective impulsive force on any particle δm at a distance r from the axis minor is $\delta m \,.\, \omega r$; therefore, reversing the effective forces, and taking moments about the axis minor,

$$Xh = \Sigma\,(\delta m\,\omega r\,.\,r)$$

$$= Mk'^2\omega,$$

$$X = M\frac{k'^2}{h}\sqrt{\frac{2gh}{k'^2}} = M\sqrt{\left(2g\,\frac{k'^2}{h}\right)}$$

$$= M\sqrt{\left(\frac{3\pi}{8}\,ga\right)}.$$

Suppose this force to act at the extremity of an ordinate through the centre of gravity, and apply two opposite forces each equal to it at the centre of gravity: one of these will destroy the motion of the lamina, while the other, together with the force acting at the extremity of the ordinate, will form a couple in a plane perpendicular to the axis major of the semi-ellipse. Since the plane of this couple is perpendicular to a principal axis of the rigid body through the centre of gravity, its effect, if the lamina were free, would be to make it revolve about this principal axis; and since the body is constrained only by having a point in this axis fixed, it will in fact begin to revolve about the major axis.

7. A thin uniform smooth tube is balancing horizontally about its middle point, which is fixed: a uniform rod, such as just to fit the bore of the tube, is placed end to end in a line with the tube, and then shot into it with such a horizontal velocity that its middle point shall only just reach that of the tube: supposing the velocity of projection to be known, find the angular velocity of the tube and rod at the moment of the coincidence of their middle points.

Let m denote the mass of the rod, m' that of the tube, and $2a$, $2a'$, their respective lengths. Let v represent the velocity of the rod's projection, ω the required angular velocity.

Then the *vis viva* of the whole system is mv^2 initially: at the moment of the coincidence of the middle points it is

$\frac{1}{3}ma^2\omega^2 + \frac{1}{3}m'a'^2\omega^2$. But the altitude of the centre of gravity is the same in both cases. Hence, by the principle of *vis viva*,

$$mv^2 = \tfrac{1}{3}ma^2\omega^2 + \tfrac{1}{3}m'a'^2\omega^2,$$

$$\omega^2 = \frac{3mv^2}{ma^2 + m'a'^2}.$$

FRIDAY, *Jan.* 20. 9...12.

1. THE position of a point in space being determined by the polar coordinates r', θ', ϕ', where θ' is the angle through which r' has revolved from a fixed line Oz, in a plane which has revolved through an angle ϕ' from a fixed plane zOx: shew that the equation to the tangent plane at a point $r\theta\phi$ of a surface is

$$\frac{r^2}{r'} = \frac{d}{d\theta}[r\{\sin\theta\cos\theta' - \sin\theta'\cos\theta\cos(\phi-\phi')\}] + \sin\theta'\frac{\sin(\phi-\phi')}{\sin\theta}\cdot\frac{dr}{d\phi}.$$

Let the equation to the plane be

$$\frac{1}{r'} = A\sin\theta'\cos\phi' + B\sin\theta'\sin\phi' + C\cos\theta' \;\ldots\ldots\; (1),$$

this being perfectly general.

Since this passes through the point r, θ, ϕ,

$$\frac{1}{r} = A\sin\theta\cos\phi + B\sin\theta\sin\phi + C\cos\theta \;\ldots\ldots\; (2).$$

Also, since the plane has a contact of the first order with the surface at the point r, θ, ϕ, we must have

$$\frac{dr'}{d\theta'} = \frac{dr}{d\theta} \text{ and } \frac{dr'}{d\phi'} = \frac{dr}{d\phi};$$

hence $$-\frac{1}{r^2}\frac{dr}{d\theta} = A\cos\theta\cos\phi + B\cos\theta\sin\phi - C\sin\theta \ldots (3),$$

$$-\frac{1}{r^2}\cdot\frac{dr}{d\phi} = -A\sin\theta\sin\phi + B\sin\theta\cos\phi \;\ldots\ldots\ldots\ldots (4).$$

From the equations (2), (3), (4) we may determine A, B, C, and therefore the plane (1).

By (2) and (3),

$$\frac{1}{r}\sin\theta - \frac{1}{r^2}\frac{dr}{d\theta}\cos\theta = A\cos\phi + B\sin\phi,$$

and by (4), $$-\frac{1}{r^2}\cdot\frac{1}{\sin\theta}\frac{dr}{d\phi} = -A\sin\phi + B\cos\phi;$$

$$\therefore\; A = \frac{1}{r}\sin\theta\cos\phi - \frac{1}{r^2}\cdot\frac{dr}{d\theta}\cos\theta\cos\phi + \frac{1}{r^2}\cdot\frac{\sin\phi}{\sin\theta}\cdot\frac{dr}{d\phi},$$

$$B = \frac{1}{r}\sin\theta\sin\phi - \frac{1}{r^2}\cdot\frac{dr}{d\theta}\cos\theta\sin\phi - \frac{1}{r^2}\cdot\frac{\cos\phi}{\sin\theta}\cdot\frac{dr}{d\phi};$$

and again, by (2) and (3), we get

$$C = \frac{1}{r}\cos\theta + \frac{1}{r^2}\cdot\frac{dr}{d\theta}\cdot\sin\theta.$$

The equation to the plane becomes

$$\begin{aligned}
\frac{r^2}{r'} &= \sin\theta'\cos\phi'\left(r\sin\theta\cos\phi - \frac{dr}{d\theta}\cos\theta\cos\phi + \frac{\sin\phi}{\sin\theta}\cdot\frac{dr}{d\phi}\right)\\
&\quad + \sin\theta'\sin\phi'\left(r\sin\theta\sin\phi - \frac{dr}{d\theta}\cos\theta\sin\phi - \frac{\cos\phi}{\sin\theta}\cdot\frac{dr}{d\phi}\right)\\
&\quad + \cos\theta'\left\{r\cos\theta + \frac{dr}{d\theta}\cdot\sin\theta\right\}\\
&= r\left\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right\}\\
&\quad + \frac{dr}{d\theta}\left\{\sin\theta\cos\theta' - \cos\theta\sin\theta'\cos(\phi-\phi')\right\}\\
&\quad + \frac{dr}{d\phi}\cdot\frac{\sin\theta'}{\sin\theta}\cdot\sin(\phi-\phi')\\
&= \frac{d}{d\theta}\left[r\left\{\sin\theta\cos\theta' - \cos\theta\sin\theta'\cos(\phi-\phi')\right\}\right]\\
&\quad + \frac{dr}{d\phi}\cdot\frac{\sin\theta'}{\sin\theta}\cdot\sin(\phi-\phi').
\end{aligned}$$

2. If x be an integer, shew that

$$\Sigma_1^\infty\left\{\frac{1}{x^{2n}}\right\} \text{ is equal to } \frac{2^{2n-1}B_{2n-1}\pi^{2n}}{1.2.3\ldots 2n},$$

B_{2n-1} being the n^{th} of Bernoulli's numbers.

Shew, by Bernoulli's numbers or otherwise, that

$$\frac{1^2}{1^2+1}\cdot\frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\ldots\ldots \textit{ad inf.} = \frac{2\pi}{e^\pi - e^{-\pi}}.$$

Let $P = \dfrac{1^2}{1^2+1}\cdot\dfrac{2^2}{2^2+1}\ \ldots\ldots\ \textit{ad inf.}$,

$$\log_e P = \log_e\left(\frac{1}{1+1^2}\right) + \log_e\left\{\frac{1}{1+(\frac{1}{2})^2}\right\} + \log_e\left\{\frac{1}{1+(\frac{1}{3})^2}\right\} + \ldots \textit{ad inf.}$$

$$= -\{1^2 - \tfrac{1}{2}.(1)^4 + \tfrac{1}{3}.(1)^6 - \ldots\ldots\},$$

$$-\left(\frac{1}{2^2} - \frac{1}{2}.\frac{1}{2^4} + \frac{1}{3}.\frac{1}{2^6} - \ldots\ldots\right),$$

$$-\left(\frac{1}{3^2} - \frac{1}{2}.\frac{1}{3^4} + \frac{1}{3}.\frac{1}{3^6} - \ldots\ldots\right),$$

$$= -\left(\Sigma_1^\infty \frac{1}{x^2} - \frac{1}{2}.\Sigma_1^\infty\frac{1}{x^4} + \frac{1}{3}.\Sigma_1^\infty\frac{1}{x^6} - \ldots\ldots \textit{ad inf.}\right),$$

$$= -\left\{\frac{2B_1\pi^2}{1.2} - \frac{1}{2}.\frac{2^3B_3\pi^3}{1.2.3.4} + \ldots + \frac{(-1)^{n-1}}{n}.\frac{2^{2n-1}B_{2n-1}\pi^{2n}}{1.2\ \ldots\ 2n} + \ldots\right\}.$$

Let this $= F(\pi)$, taking

$$F(x) = -\frac{2B_1x^2}{1.2} + \ldots + \frac{(-1)^n}{n}.\frac{2^{2n-1}B_{2n-1}x^{2n}}{1.2\ldots2n} + \ldots;$$

$$\therefore\ F'(x) = -\frac{2^2B_1x}{1.2} + \frac{2^4B_3x^3}{1.2.3.4} + \ldots + (-1)^n\frac{2^{2n}B_{2n-1}x^{2n-1}}{1.2\ldots2n} + \ldots,$$

$$\text{or } xF'(x) = -\frac{B_1(2x)^2}{1.2} + \frac{B_3(2x)^4}{1.2.3.4} - \ldots + (-1)^n\frac{B_{2n-1}(2x)^{2n}}{1.2.3\ldots2n} + \ldots\ldots$$

$$= -\frac{2x}{c^{2x}-1} + 1 - x,$$

by the definition of Bernoulli's numbers.

Or

$$F'(x) = -\frac{2}{e^{2x}-1} + \frac{1}{x} - 1$$

$$= -\frac{e^{2x}+1}{e^{2x}-1} + \frac{1}{x}$$

$$= -\frac{e^x+e^{-x}}{e^x-e^{-x}} + \frac{1}{x};$$

$$\therefore\ F(x) = -\log(e^x - e^{-x}) + \log x + \log C$$

$$= -\log\left(\frac{e^x - e^{-x}}{Cx}\right);$$

and when $x = 0$, $F(x) = 0$; $\therefore\ C = 2$;

$$\therefore\ F(\pi) = -\log\frac{e^\pi - e^{-\pi}}{2\pi} = \log P,$$

or
$$P = \frac{2\pi}{e^\pi - e^{-\pi}}.$$

Otherwise,

$$\sin\theta = \frac{1}{2\sqrt{(-1)}}\left(e^{\theta\sqrt{(-1)}} - e^{-\theta\sqrt{(-1)}}\right).$$

Also $\sin\theta = \theta\left\{1 - \left(\frac{\theta}{\pi}\right)^2\right\}\left\{1 - \left(\frac{\theta}{2\pi}\right)^2\right\}\left\{1 - \left(\frac{\theta}{3\pi}\right)^2\right\} \dots$ *ad inf.*;

hence $\dfrac{e^{\theta\sqrt{(-1)}} - e^{-\theta\sqrt{(-1)}}}{2\theta\sqrt{(-1)}}$

$$= \left\{1 + \left(\frac{\theta\sqrt{(-1)}}{\pi}\right)^2\right\}\left\{1 + \left(\frac{\theta\sqrt{(-1)}}{2\pi}\right)^2\right\}\left\{1 + \left(\frac{\theta\sqrt{(-1)}}{3\pi}\right)^2\right\} \dots \textit{ad inf.};$$

put $\theta\sqrt{(-1)} = \pi$, and this gives

$$\frac{e^\pi - e^{-\pi}}{2\pi} = \left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{3^2}\right) \dots \textit{ad inf.},$$

or
$$\frac{2\pi}{e^\pi - e^{-\pi}} = \frac{1^2}{1^2 + 1}\cdot\frac{2^2}{2^2 + 1}\cdot\frac{3^2}{3^2 + 1} \dots \textit{ad inf.}$$

3. Define the terms convergent and divergent when applied to a series of quantities real or imaginary.

Investigate a rule which is ordinarily sufficient to ascertain whether a series is or is not convergent.

Are the following series convergent?

$\frac{3}{2}x + \frac{5}{5}x^2 + \frac{7}{10}x^3 + \frac{9}{17}x^4 + \dots + \frac{2n+1}{n^2+1}x^n + \dots$, where x is real;

$1 + x\cos\alpha + x^2\cos 2\alpha +$ &c., where x is real or imaginary.

Let
$$u_1 + u_2 + \dots + u_n + \dots$$
stand for the series $\frac{3}{2}x + \frac{5}{5}x^2 + \dots + \frac{2n+1}{n^2+1}x^n + \dots$

Then limit $\dfrac{u_{n+1}}{u_n} = \text{limit}\ \dfrac{2n+3}{2n+1}\,\dfrac{n^2+1}{n^2+2n+2}\,x,$

$$= x,$$

the series is therefore convergent if x be less than 1.

Again, the series $1 + x\cos\alpha + x^2\cos 2\alpha + \ldots$

is less than the series $\quad 1 + x + x^2 + \ldots$

Suppose x to be real: this latter series is convergent if x be less than 1; suppose x to be imaginary, and let

$$x = \alpha + \beta\sqrt{(-1)},$$

the series is convergent if $\alpha^2 + \beta^2 < 1$.

In these cases, therefore, the given series is convergent.

7. Solve the differential equation for the vibratory motion of the air contained in an indefinite cylindrical tube; and shew that when such motion is produced by a vibrating plate placed at one end of a finite tube, of which the other end is open, if the period of vibration have a certain relation to the length of the tube, it is possible for the character of the vibrations to remain permanently the same.

If such a tube be sounding its fundamental note, what would be the effect of making a small aperture in the side of the tube, first at its middle point, secondly a little nearer to the open end?

Suppose the fundamental note to be produced by a tube open at one end, and having a vibrating plate at the other; each end of the tube is a loop, and the middle point is a node.

Suppose a small aperture to be made at the middle point; then, in order to maintain small vibrations, we must make the period of the vibrating plate half what it was: there will now be three loops, one at each end and one in the middle, and there will be two nodes between them; the tube will now sound the octave above the fundamental note.

Suppose the aperture to be made a little nearer the open end; and suppose, as before, the period of the vibrating plate to be so adjusted as to maintain small vibrations: there will now be a loop at the plate and a loop at or near the aperture.

The part of the tube between the plate and the aperture will be affected more or less by the part between the aperture and the open end; the loop, which would otherwise have been at the aperture, will be forced to retire a little towards the plate, but not so far as the middle point of the tube, so that the tone of the note produced will be somewhat flatter than the octave of the fundamental note; the regularity of the aerial vibrations will not be so perfect as before, and consequently the note will not be so musical.

The above account is founded on the hypothesis that the open end is a loop; but, as is well known, it is found by experiment that, in the case of a tube open at one end, the whole system of loops and nodes is shifted a little nearer the open end than the places assigned by theory. Hence, in order to allow the perfect octave to be sounded, the aperture would have to be made at a certain point a little nearer to the open end than to the plate; if it be made still nearer the open end, the note produced would be a flat octave as above described: if the aperture were at the middle point of the tube an imperfect note would be produced, somewhat sharper than the octave.

8. Find the difference of retardation of the two waves produced by a thin lamina cut from a uniaxal crystal perpendicular to its axis, when a ray of common light is incident nearly parallel to the axis: describe the rings produced by interposing such a lamina between a polarizing and an analyzing plate, the planes of incidence at the two plates being inclined at an angle of 45° to each other.

If two such laminæ, one cut from a positive and the other from a negative uniaxal crystal, be placed together and interposed, what must be the ratio of their thicknesses in order that neither rings nor brushes may be visible?

In order that neither rings nor brushes may be seen, the difference of retardation of the ordinary and extraordinary rays, after passing through both plates, must be equal to zero;

therefore $$T\frac{c^2-a^2}{2av}\sin^2 i - T'\frac{a'^2-c'^2}{2a'v}\sin^2 i = 0;$$

therefore $$T : T' :: \frac{a'^2-c'^2}{a'} : \frac{c^2-a^2}{a},$$

where T, T' are the thicknesses of the plates,

a, c; a', c' the constants of elasticity respectively.

FRIDAY, *Jan.* 20. 1½...4.

1. If $f(p, q, r, s\,...) = 0$, where $p, q, r, s\,...$ are the distances of any point in a curve from fixed points in its plane, or of any point in a surface from fixed points, and if a set of forces proportional to $f'(p)$, $f'(q)$, $f'(r)\,...$, act on the point, along the distances $p, q, r, ...$, prove that their resultant acts along the normal at that point.

If $\sin\lambda : \sin\mu :: p^n : q^n$, where λ, μ, are the respective inclinations of p, q, to the normal at any point of the curve $f(p, q) = 0$, prove that, c being a constant,

$$p^{1-n} + q^{1-n} = c^{1-n}.$$

Let x, y, z, be the coordinates of the variable point of the surface. Then, if $f(p, q, r, s, ...) = \phi(x, y, z) = u$,

$$\frac{du}{dx} = f'(p)\frac{dp}{dx} + f'(q)\frac{dq}{dx} + \cdots$$

= the sum of the components of $f'(p)$, $f'(q)$, ..., parallel to the axis of x.

Similarly, $\frac{du}{dy}$, $\frac{du}{dz}$, denote the sums of the components of these derivatives parallel to y, z.

Hence the direction-cosines of the resultant are proportional to $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, and therefore the resultant is a normal.

In the example, (see fig. 90),

$$SP = p, \quad TP = q.$$

Take S', T', such points in SP, TP, that

$$S'P : T'P :: f'(p) : f'(q),$$

and complete the parallelogram $PS'GT'$. Then the diagonal PG is the normal at P.

Hence $$\frac{p^n}{q^n} = \frac{\sin\lambda}{\sin\mu} = \frac{f'(q)}{f'(p)};$$

but $$df = f'(p)\,dp + f'(q)\,dq = 0:$$

hence $$p^{-n}dp + q^{-n}dq = 0,$$

$$p^{1-n} + q^{1-n} = c^{1-n}.$$

COR. If $n = 0$, $p + q = c$, and therefore the curve is an ellipse.

2. Having given the following simultaneous differential equations,

$$\frac{d^2x}{dt^2} = \frac{dR}{dx}, \quad \frac{d^2y}{dt^2} = \frac{dR}{dy}, \ldots\ldots$$

where $$R = f(r), \quad r^2 = x^2 + y^2 + \ldots\ldots;$$

prove that $$t = \int \frac{rdr}{\sqrt{\{r^2(2R+B) - A^2\}}},$$

A, B being arbitrary constants.

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} + \ldots = 2\left(\frac{dR}{dx}\frac{dx}{dt} + \frac{dR}{dy}\frac{dy}{dt} + \ldots\ldots\right),$$

and $$r^2 = x^2 + y^2 + \ldots\ldots$$

$$\therefore\ r\frac{dr}{dx} = x.$$

Also $$\frac{dR}{dx} = f'(r)\frac{dr}{dx};$$

$$\therefore\ \frac{dR}{dx}\frac{dx}{dt} + \frac{dR}{dy}\frac{dy}{dt} + \ldots\ldots$$

$$= f'(r)\left(\frac{dr}{dx}\frac{dx}{dt} + \frac{dr}{dy}\frac{dy}{dt} + \ldots\ldots\right)$$

$$= f'(r)\frac{dr}{dt};$$

$$\therefore\ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \ldots\ldots = 2f(r) + B$$

$$= 2R + B.$$

Also $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = x\frac{dR}{dy} - y\frac{dR}{dx}$$
$$= f'(r)\left(x\frac{dr}{dy} - y\frac{dr}{dx}\right) = 0;$$
$$\therefore\ x\frac{dy}{dt} - y\frac{dx}{dt} = \text{constant};$$
$$\therefore\ \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)^2 + \left(x\frac{dz}{dt} - z\frac{dx}{dt}\right)^2 + \ldots\ldots$$
or $$(x^2+y^2+\ldots)\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \ldots\right\} - \left(x\frac{dx}{dt} + y\frac{dy}{dt} + \ldots\right)^2 = A^2;$$
$$\therefore\ r^2(2R+B) - A^2 = \left(x\frac{dx}{dt} + y\frac{dy}{dt} + \ldots\ldots\right)^2 = \left(r\frac{dr}{dt}\right)^2;$$
$$\therefore\ t = \int\frac{rdr}{\sqrt{\{r^2(2R+B) - A^2\}}}.$$

2. Integrate the partial differential equation
$$q(1+q)\,r - (p+q+2pq)\,s + p(1+p)\,t = 0.$$

Employing Monge's method of solving such equations, we arrive at the equations
$$(q\,dy + p\,dx)\,\{(1+q)\,dy + (1+p)\,dx\} = 0,$$
and $$q(1+q)\,dp\,dy + p(1+p)\,dq\,dx = 0.$$

If we use the equation $(1+q)\,dy + (1+p)\,dx = 0$,
$$q\,dp - p\,dq = 0,\quad \therefore\ p = \alpha q,$$
and $$dy + dx + p\,dx + q\,dy = 0,\quad \therefore\ x + y + z = \beta;$$
$\therefore$ a first integral is $p - q\,\phi(x+y+z) = 0$;
$\therefore$ from the equations $dz = 0$ and $dy = dx\,\phi(x+y+z)$,
$z = \gamma$ and $dy + dx = dx\{1 + \phi(x+y+\gamma)\}$, $\therefore\ x = \delta + f(x+y+\gamma)$;
$\therefore$ the complete integral is $x = F(z) + f(x+y+z)$.

3. An annular surface is generated by the revolution of a circle about an axis in its own plane; prove that one of the principal radii of curvature, at any point of the surface, varies as the ratio of the distance of this point from the axis to its distance from the cylindrical surface described about the axis and passing through the centre of the circle.

Let AB (fig. 91) be the axis of revolution, C the centre of the revolving circle in any position, P any point in the circumference of this circle. Join CP; draw CN parallel to BA, and NPM at right angles to AB.

One principal radius of curvature of the surface at P is in the plane of the paper. The other principal radius at P, in the plane through PC, at right angles to the plane of the paper, is, by Meusnier's theorem, equal to

$$\frac{PM}{\cos\phi} = \frac{PM.CP}{PN} \propto \frac{PM}{PN}.$$

4. Give sufficient equations for calculating the motion of a right cone placed upon a perfectly rough inclined plane; and find the moment of the couple exerted by friction on the cone.

Shew that the length of the simple isochronous pendulum, when the cone oscillates about the lowest position, is

$$\frac{4k^2}{3r \sin\alpha \sin\beta},$$

2α being the angle of the cone, r the radius of its base, β the inclination of the plane, and k the radius of gyration round a generating line.

Let A be the vertex of the cone, (fig. 92),

Az perpendicular to the inclined plane,

Ay horizontal,

AGx_1 the axis of the cone,

GH perpendicular to AL the generating line in contact at the time t. GM perpendicular to Az.

And let the friction on the generating line be resolved into forces F, G in AL, and perpendicular to it, and the couple whose moment is N in plane xAy,

$AG = h$, $\angle LAx = \phi$,

ω = perpendicular velocity round AL,

$\omega \cos\beta$ = perpendicular velocity round Ax_1,

A the moment of inertia round AGx_1.

By the principle of *vis viva*, since the slant side is an instantaneous axis,

$$Mk^2\omega^2 = \text{constant} + 2Mg \sin\alpha \, GM \cos\phi \ldots\ldots\ldots(1),$$

and
$$A \frac{d(\omega \cos\beta)}{dt} = N \sin\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, equating the two expressions for the velocity of G,

$$GM \frac{d\phi}{dt} = -\omega h \sin\beta;$$

$$\therefore \frac{d\phi}{dt} \cot\beta = -\omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

$$\therefore k^2 \cot\beta \frac{d^2\phi}{dt^2} + g \sin\alpha \, h \sin\beta \sin\phi = 0;$$

therefore, for a small oscillation, since $h \tan\beta = \dfrac{3r}{4}$,

$$\frac{d^2\phi}{dt^2} + g \frac{3r \sin\alpha \sin\beta}{4k^2} \phi = 0;$$

therefore the length of the simple pendulum

$$= \frac{4k^2}{3r \sin\alpha \sin\beta},$$

and
$$N = \frac{g \sin\alpha \dfrac{3r}{4} \cot\beta \cos\beta}{k^2} A \sin\phi.$$

In order to illustrate a difficulty in forming the equations for determining the angular velocities about principal axes moveable in the body, we will proceed to determine all the forces, without using the principle of *vis viva*, by the general equations.

The equations for determining all the forces may be formed as follows:

R being the reaction of the plane through G, and L the moment of the couple, to which the whole reactions on the generating line can be reduced:

$$M \frac{d^2x}{dt^2} = F \frac{x}{h \cos\beta} - G \frac{y}{h \cos\beta} + Mg \sin\alpha \ldots\ldots(1),$$

$$M \frac{d^2y}{dt^2} = F \frac{y}{h \cos\beta} + G \frac{x}{h \cos\beta} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$0 = R - Mg \cos\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let Gy_1, Gz_1 be axes perpendicular to LAx_1, and in the plane Gy_1', Gz_1' fixed in the body, coinciding with Gy_1, Gz_1, at times t, and having this position at times $t + \Delta t$, when the body has turned through $\omega_1 \Delta t$ round Gx_1, $\omega_2 \omega_3$, $\omega'_2 \omega'_3$, angular velocities round these axes at time t;

$$\therefore \ \omega'_2 + \Delta\omega'_2 = \omega_3 \sin(\omega_1 \Delta t) + \left(\omega_2 + \frac{d\omega_2}{dt}\Delta t\right)\cos(\omega_1 \Delta t);$$

$$\therefore \ \omega'_2 - \omega_2 = \omega_3 \omega_1 \Delta t + \frac{d\omega_2}{dt}\Delta t;$$

$$\therefore \ \frac{d\omega'_2}{dt} = \frac{d\omega_2}{dt} + \omega_3 \omega_1.$$

Similarly, $$\frac{d\omega'_3}{dt} = \frac{d\omega_3}{dt} - \omega_2 \omega_1;$$

$$\therefore \ A\frac{d\omega_1}{dt} = N \sin\beta \quad \text{.................} \quad (4),$$

$$B\left(\frac{d\omega_2}{dt} + \omega_1 \omega_3\right) + (A - B)\,\omega_1 \omega_3 = L - Fh \sin\beta \ldots (5),$$

$$B\left(\frac{d\omega_3}{dt} - \omega_1 \omega_2\right) + (B - A)\,\omega_2 \omega_1 = N \cos\beta - Gh \ldots (6);$$

and by the geometry of the motion,

$$\omega_2 = 0 \quad \text{..} \quad (7),$$

$$\omega_1 = \omega \cos\beta \quad \text{................................} \quad (8),$$

$$\omega_3 = -\,\omega \sin\beta \quad \text{..............................} \quad (9),$$

$$x = h \cos\beta \cos\phi \quad \text{.......................} \quad (10),$$

$$y = h \cos\beta \sin\phi \quad \text{.......................} \quad (11),$$

$$\frac{d\phi}{dt} = -\,\omega \tan\beta \quad \text{..........................} \quad (12),$$

12 equations between F, G, R, L, N, x, y, ω_1, ω_2, ω_3, ω, ϕ, t.

By (1) and (2),

$$M\left(x\frac{d^2x}{dt^2} - y\frac{d^2x}{dt^2}\right) = Gh \cos\beta - Mg\,y \sin\alpha;$$

$$\therefore \ Mh^2 \cos^2\beta \frac{d^2\phi}{dt^2} = Gh \cos\beta - Mgh \cos\beta \sin\alpha \sin\phi.$$

By (5), (6), (7),

$$A\frac{d\omega_1}{dt}\cos\beta - B\frac{d\omega_3}{dt}\sin\beta = Gh\sin\beta,$$

or

$$(A\cos^2\beta + B\sin^2\beta)\frac{d\omega}{dt} = Gh\sin\beta;$$

$$\therefore\ (Mk^2 - Mh^2\sin^2\beta)\frac{d^2\phi}{dt^2} = -Gh\tan\beta\sin\beta,$$

and $Mh^2\sin^2\beta\dfrac{d^2\phi}{dt^2} = -Gh\tan\beta\sin\beta - Mgh\tan\beta\sin\beta\sin\alpha\sin\phi;$

$$\therefore\ Mk^2\frac{d^2\phi}{dt^2} = - - Mgh\tan\beta\sin\beta\sin\alpha\sin\phi,$$

$$\frac{d^2\phi}{dt^2} + \frac{3r}{4}\frac{g\sin\beta\sin\alpha\sin\phi}{k^2} = 0.$$

And, by (1) and (2),

$$M\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2}\right) = Fh\cos\beta + Mgx\sin\alpha,$$

$$x^2 + y^2 = h^2\cos^2\beta;$$

therefore

$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} = -\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}$$

$$= -h^2\cos^2\beta\left(\frac{d\phi}{dt}\right)^2;$$

therefore

$$F = -Mh\cos\beta\left(\frac{d\phi}{dt}\right)^2 - Mg\sin\alpha\cos\phi,$$

i.e. the centrifugal force from Oz and resolved part of weight $= -F$;

$$G = Mh\cos\beta\frac{d^2\phi}{dt^2} + Mg\sin\alpha\cos\phi,$$

or G is the resultant effective force and the component of the weight perpendicular to the plane LAx,

$$N = -A\frac{d^2\phi}{dt^2}\cos^2\beta,$$

$$R = \quad Mg\cos\alpha;$$

by (5),

$$L = Fh\sin\beta - A\left(\frac{d\phi}{dt}\right)^2\cot\beta\cos^2\beta,$$

whence all the forces are known.

5. The form of a homogeneous solid of revolution, of given superficial area, and described upon an axis of given length, is such that its moment of inertia about the axis is a maximum: prove that the normal at any point of the generating curve is three times as long as the radius of curvature.

If c be some constant quantity, then, the axis of x being that of revolution,

$$u = \int (y^4 dx + c^3 y ds) = \int \{y^4 + c^3 y (1 + p^2)^{\frac{1}{2}}\} \, dx.$$

Then, adopting the ordinary notation of the Calculus of Variations,

$$V = Pp + C,$$

C being a constant.

Hence

$$y^4 + c^3 y (1 + p^2)^{\frac{1}{2}} = \frac{c^3 y p^2}{(1 + p^2)^{\frac{1}{2}}} + C, \quad y^4 + \frac{c^3 y}{(1 + p^2)^{\frac{1}{2}}} = C \ldots\ldots(1),$$

Again, the formula for the limits gives us the relation

$$\frac{y_{,,} p_{,,}}{(1 + p_{,,}^2)^{\frac{1}{2}}} \cdot \delta y_{,,} - \frac{y_{,} p_{,}}{(1 + p_{,}^2)^{\frac{1}{2}}} \cdot \delta y_{,} = 0,$$

and therefore, $\delta y_{,}$ and $\delta y_{,,}$ being independent of each other,

$$\frac{y_{,} p_{,}}{(1 + p_{,}^2)^{\frac{1}{2}}} = 0, \qquad \frac{y_{,,} p_{,,}}{(1 + p_{,,}^2)^{\frac{1}{2}}} = 0.$$

Now, by the equation

$$\frac{y_{,} p_{,}}{(1 + p_{,}^2)^{\frac{1}{2}}} = 0,$$

either $p_{,} = 0$ or $y_{,} = 0$: on the former hypothesis $y_{,}$ would be arbitrary, which is evidently impossible. Hence $y_{,} = 0$; similarly $y_{,,} = 0$. Consequently the value of C in the equation (1) must be zero: hence

$$1 + p^2 = \frac{c^6}{y^6}, \quad \frac{dy^2}{dx^2} = \frac{c^6 - y^6}{y^6}, \quad \frac{dx}{dy} = \frac{y^3}{(c^6 - y^6)^{\frac{1}{2}}},$$

$$\frac{d^2 x}{dy^2} = 3y^2 \cdot \frac{c^6}{(c^6 - y^6)^{\frac{3}{2}}}.$$

Hence the radius of curvature

$$= \frac{\left(1+\frac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} = \frac{c^3}{3y^2}.$$

Also the normal $\quad = y\,(1+p^2)^{\frac{1}{2}} = \frac{c^3}{y^2}.$

Hence, C being the centre of curvature and PG the normal at any point P of the generating curve,

$$PG = 3PC.$$

6. Distinguish between secular and periodic variations. Are secular variations ever periodic?

The equations which connect the inclination and the longitude of the nodes of the orbits, in the case of Jupiter and Saturn, are of the form

$$\tan i \sin\Omega = G \sin(\alpha t+\gamma) + H \sin\delta,$$
$$\tan i \cos\Omega = G \cos(\alpha t+\gamma) + H \cos\delta.$$

For both orbits,

$\alpha = -25''.5756,\ \gamma = 125^\circ\,15'\,40'',\ \delta = 103^\circ\,38'\,40''$, and $H = .02905$,

$G = -.00661$ for Jupiter, and $= .01537$ for Saturn,

t being reckoned from A.D. 1700.

Prove the following circumstances of motion, that Jupiter's node regresses and Saturn's progresses from a longitude $\delta+\varepsilon-\frac{\pi}{2}$ through the angle $2\varepsilon \sim \pi$ in the time $\frac{2\varepsilon}{-\alpha}$, where ε is for each planet the least positive angle which satisfies the equation $G = H\cos\varepsilon$; that they arrive simultaneously at their mean position; and that in this position Jupiter's orbit has its maximum and Saturn's its minimum inclination.

Let Ox, Oy be rectangular axes, $GOx = \delta$, $OG = H$; and let circles be described with centre G and radii .00661 and .01537 (see fig. 93),

$$\left.\begin{matrix} OA,\ Oa \\ OB,\ Ob \end{matrix}\right\} \text{ tangents to the three circles respectively.}$$

Let OG meet the circles in T and S;

therefore for Saturn, $\varepsilon = BGO$,

for Jupiter, $\varepsilon = AGS$;

$$\delta - \left(\frac{\pi}{2} - \varepsilon\right) = GOx - GOB = BOx, \text{ for Saturn,}$$

$$\delta + \left(\varepsilon - \frac{\pi}{2}\right) = GOx + GOA = AOS, \text{ for Jupiter.}$$

Let Gx' be parallel to Gx, and let a line revolving in the positive direction from Ox' through $at + \gamma$ arrive at the position GQ produced backwards to P. Join OQ, OP; therefore, at time t,

$$OQ = \tan i, \quad QOx = \Omega, \quad \text{for Saturn,}$$

$$OP = \tan i, \quad POx = \Omega, \quad \text{for Jupiter.}$$

As t increases $at + \gamma$ diminishes;

therefore QGP revolves backwards;

therefore Saturn's node advances from OB to Ob, the longitude increases from $\delta + \varepsilon - \frac{\pi}{2}$ through BOb or $2\left(\frac{\pi}{2} - \varepsilon\right) = \pi - 2\varepsilon$, in the time $\frac{BGb}{-a} = \frac{2\varepsilon}{-a}$; and Jupiter's node recedes from AOx to aOx, the longitude decreasing from $\delta + \varepsilon - \frac{\pi}{2}$ through AOa or $2\left(\varepsilon - \frac{\pi}{2}\right) = 2\varepsilon - \pi$, in the time $\frac{1}{-a} \cdot \frac{APa}{GP} = \frac{2\varepsilon}{-a}$.

The nodes arrive simultaneously at the mean positions whose longitudes are $GOx = \delta$; in which position Jupiter's orbit's inclination is OT, a maximum, and Saturn's OS, a minimum.

Also, it is easily seen that the nodes in t regress together during the times that GQ revolves from aG to BG, produced backwards, while $at + \gamma$ changes by the angle BGa, or the difference between

$$\frac{\pi}{2} - \varepsilon \text{ and } \varepsilon' - \frac{\pi}{2}$$

for Saturn and Jupiter; *i.e.* for the time

$$\frac{\pi - \varepsilon - \varepsilon'}{-a},$$

during one revolution of Q during the time

$$\frac{2\pi - 2(\varepsilon + \varepsilon')}{-a}.$$

Otherwise.

$$\tan i \cos\Omega \frac{d\Omega}{dt} + \sec^2 i \sin\Omega \frac{di}{dt} = \alpha G \cos(at + \gamma),$$

$$-\tan i \sin\Omega \frac{d\Omega}{dt} + \sec^2 i \cos\Omega \frac{di}{dt} = -\alpha G \sin(at + \gamma);$$

therefore $\quad \tan i \dfrac{d\Omega}{dt} = \alpha G \cos(at + \gamma - \Omega) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$

or $\quad \tan^2 i \dfrac{d\Omega}{dt} = \alpha G\{G + H \cos(at + \gamma - \delta)\}$

$$= \alpha GH\{\cos\varepsilon + \cos(at + \gamma - \delta)\} \ \ldots\ldots\ldots (2).$$

Also, $\quad \tan^2 i = G^2 + H^2 + 2GH \cos(at + \gamma - \delta) \ldots\ldots (3).$

In the case of Jupiter, αGH is positive;

when $\quad at + \gamma - \delta = \pi, \quad \dfrac{d\Omega}{dt}$ is negative;

therefore Ω is decreasing most rapidly, and $\tan i$ is a maximum, G being negative, and $\Omega = \delta$; therefore this decrease takes place while

by (2), $at + \gamma - \delta$ changes from $\pi + \varepsilon$ to $\pi - \varepsilon$,

and, by (1), $at + \gamma - \delta$ changes from $\dfrac{3\pi}{2}$ to $\dfrac{\pi}{2}$;

or, ε being $> \dfrac{\pi}{2}$,

Ω decreases from $\delta + \left(\varepsilon - \dfrac{\pi}{2}\right)$ to $\delta - \left(\varepsilon - \dfrac{\pi}{2}\right)$;

therefore the node regresses from $\delta + \varepsilon - \dfrac{\pi}{2}$ through $2\varepsilon - \pi$, arriving at the mean position δ, where i is a maximum.

In the case of Saturn, αGH is negative;

therefore $\quad at + \gamma - \delta = \pi, \quad \dfrac{d\Omega}{dt}$ is positive,

Ω is increasing, and, G being positive, $\tan i$ is a minimum; the increase takes place while

$$at + \gamma - \delta \text{ changes from } \pi + \varepsilon \text{ to } \pi - \varepsilon,$$

$$at + \gamma - \Omega \text{ changes from } \frac{3\pi}{2} \text{ to } \frac{\pi}{2};$$

ε being $< \frac{\pi}{2}$, Ω increases from $\delta - \left(\frac{\pi}{2} - \varepsilon\right)$ to $\delta + \left(\frac{\pi}{2} - \varepsilon\right)$;

therefore the node progresses from $\delta + \varepsilon - \frac{\pi}{2}$ through $\pi - 2\varepsilon$,

arriving at the mean position δ, where i is a minimum.

The motion of the nodes is exhibited geometrically in fig. (93).

8. Draw the course of a small pencil of parallel rays, passing at such an angle through a biaxal crystal cut with parallel faces, that external cylindrical refraction takes place.

How may the constants a, b, c, corresponding to the axes of elasticity be obtained experimentally?

If the two faces of a prism, formed of a biaxal crystal, be perpendicular to each other, and one contain the two axes of elasticity a, c, and the other b, c; and if μ_a, μ_b be two refractive indices for the ordinary ray when the planes of refraction are perpendicular to the axes of a and b respectively; shew that D, the minimum deviation of the extraordinary ray, is given by the equation

$$\sin^2 D = (\mu_a^2 - 1)(\mu_b^2 - 1).$$

Let OA, OB (fig. 94) be the projections of the faces containing (a, c) and (b, c) respectively,

QR, RS, ST, directions of normals to the extraordinary wave front at incidence, 1st and 2nd refraction,

$\left.\begin{matrix}\phi, \phi' \\ \psi', \psi\end{matrix}\right\}$ the angles of $\left\{\begin{matrix}1^{st} \\ 2^{nd}\end{matrix}\right\}$ incidence and refraction,

$TDq = D$, the deviation,

u = velocity of wave in air,

v = velocity of extraordinary wave in the crystal;

therefore $$\phi' + \psi' = \frac{\pi}{2} \quad\text{..........................} (1),$$

$$D = \phi + \psi - \frac{\pi}{2} \quad\text{......................} (2),$$

$$\frac{\sin\phi}{u} = \frac{\sin\phi'}{v} \quad\text{..........................} (3),$$

$$\frac{\sin\psi}{u} = \frac{\sin\psi'}{v} \quad\text{..........................} (4),$$

$$v^2 = a^2\cos^2\phi' + b^2\cos^2\psi' \quad\text{................} (5),$$

$$D = \text{minimum} \quad\text{......................} (6),$$

By (5) and (1),

$$v^2 = a^2\sin^2\psi' + b^2\sin^2\phi';$$

therefore, by (3) and (4),

$$u^2 = a^2\sin^2\psi + b^2\sin^2\phi \quad\text{..................} (7);$$

therefore $\quad 0 = a^2\sin\psi\cos\psi\, d\psi + b^2\sin\phi\cos\phi\, d\phi,$

and by (2) and (6), $\quad 0 = d\psi + d\phi;$

therefore $\quad 0 = a^2\sin 2\psi - b^2\sin 2\phi,$

and by (7), $\quad 2u^2 = a^2 + b^2 - a^2\cos 2\psi - b^2\cos 2\phi;$

$$\therefore\ a^2 + b^2 - 2u^2 = a^2\cos 2\psi + b^2\cos 2\phi,$$

$$0 = a^2\sin 2\psi - b^2\sin 2\phi;$$

$$\therefore\ (a^2+b^2-2u^2)^2 = a^4 + b^4 + 2a^2b^2\cos 2(\phi+\psi);$$

$$\therefore\ 2a^2b^2\{1-\cos 2(\psi+\phi)\} = 4(a^2+b^2)u^2 - 4u^4,$$

$$a^2b^2\sin^2(\psi+\phi) = (a^2+b^2)u^2 - u^4;$$

$\therefore$ by (2), $$\sin^2 D = \cos^2(\psi+\phi)$$

$$= 1 - \frac{u^2}{a^2} - \frac{u^2}{b^2} + \frac{u^4}{a^2b^2}$$

$$= \left(\frac{u^2}{a^2} - 1\right)\left(\frac{u^2}{b^2} - 1\right)$$

$$= (\mu_a^2 - 1)(\mu_b^2 - 1).$$

EXAMINATION PAPERS FOR THE MATHEMATICAL TRIPOS 1854.

TUESDAY, *Jan.* 3. 9...12.

1. THE complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

If K be the common angular point of these parallelograms, and BD the other diameter, the difference of the parallelograms is equal to twice the triangle BKD.

2. Divide a given straight line into two parts so that the rectangle contained by the whole line and one of the parts shall be equal to the square of the other part.

Produce a given straight line to a point such that the rectangle contained by the whole line thus produced and the part produced shall be equal to the square of the given straight line.

3. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

If the opposite sides of the quadrilateral be produced to meet in P, Q, and about the triangles so formed without the quadrilateral circles be described meeting again in R; P, R, Q will be in one straight line.

4. Describe an isosceles triangle having each of the angles at the base double of the third angle.

Upon a given straight line, as base, describe an isosceles triangle having the third angle treble of each of the angles at the base.

5. If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.

EA, EA' are diameters of two circles touching each other externally at E; a chord AB of the former circle when produced touches the latter at C', while a chord $A'B'$ of the latter touches the former at C: prove that the rectangle contained by AB, $A'B'$ is four times as great as that contained by BC', $B'C$.

6. If a solid angle be contained by three plane angles, any two of them are greater than the third.

Within the area of a given triangle is described a triangle, the sides of which are parallel to those of the given one. Prove that the sum of the angles subtended by the sides of the interior triangle at any point not in the plane of the triangles is less than the sum of the angles subtended at the same point by the sides of the exterior triangle.

7. Prove that the rectangle contained by the latus rectum of a parabola and the abscissa of any point in the curve is equal to the square on the ordinate drawn to the axis.

If N be the foot of the ordinate, SY the perpendicular from the focus on the tangent, and T the point where the tangent meets the axis produced, NY is equal to TY.

8. Define the tangent to an ellipse, and shew that it makes equal angles with the focal distances of the point of contact.

If NP be the ordinate of P, Y, and Z, the points where the tangent at P meets the perpendiculars from the foci, $NY : NZ :: PY : PZ$.

9. The tangent at a point P of an ellipse cuts CA, CB produced in T, t respectively, and PN, Pn are the respective perpendiculars from P upon CA, CB; prove that $CT \cdot CN = AC^2$, and that $Ct \cdot Cn = BC^2$.

Shew that the subnormal is a third proportional to CT and BC.

10. The rectangle contained by the abscissæ of the major axis of an hyperbola is to the square on the ordinate as the square on the major axis is to the square on the minor axis.

If A, M be the extremities of the major axis of an ellipse, PP' a double ordinate, and AP, $P'M$ be produced to meet in Q; Q will lie in an hyperbola having the same axes as the ellipse.

11. Parallelograms, whose sides touch an hyperbola and its conjugate, and are parallel to conjugate diameters, have the same area.

If CP, CD be conjugate semi-diameters, and through C a straight line be drawn parallel to either focal distance of P, the perpendicular let fall from D on this straight line will be equal to half the minor axis.

12. If two spheres exterior to each other be inscribed in a right cone touching it in two circles on the same side of the vertex, and a plane be drawn touching the spheres and cutting the cone; shew that the section is an ellipse, that the points of contact of the spheres with the plane are the foci, and that the planes of the two circles contain the directrices.

TUESDAY, *Jan.* 3. $1\frac{1}{2}$...4.

1. DIVIDE $\dfrac{48\frac{4}{7}}{1085\frac{7}{10}}$ by $\dfrac{7\frac{3}{11}}{174\frac{3}{17}}$, and reduce the quotient to the form $1.0\dot{7}1428\dot{5}$; and find what decimal of a guinea is equivalent to ·2835 of a pound sterling.

2. The capital of a firm consists of £713. 3*s.*, £964. 17*s.*, £2391. 3*s.*, subscribed by three partners; divide £2231. among them in proportion to their several capitals.

3. Find the interest on £10,000 for four years at 3 per cent., compound interest.

How many complete years will elapse before a sum of money has trebled itself at $3\frac{1}{2}$ per cent. compound interest?

Given $\log(10350) = 4{\cdot}0149400$, $\log 3 = {\cdot}4771213$.

4. Find the highest common measure of

$$x^4 - x^3 + 2x^2 + x + 3, \text{ and } x^4 + 2x^3 - x - 2.$$

5. Find the sum of n terms of a geometrical progression, whose first term and common ratio are given.

If S_n represent this sum, find the sum of S_1, S_2, $S_3 \ldots S_n$.

6. Shew that a quadratic equation cannot have more than two roots; and solve the following equations:

$$x + \frac{x}{2} + \frac{2x}{3} + \frac{3x}{4} = 1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4};$$

$$\frac{x^2 - a^2}{x^2 + a^2} + \frac{x^2 + a^2}{x^2 - a^2} = \frac{34}{15};$$

$$y + z = \frac{1}{x}, \quad z + x = \frac{1}{y}, \quad x + y = \frac{1}{z}.$$

7. If $a, b, c, \ldots$ be a series of quantities, and x be a quantity depending on them in such a manner that x varies as a when the rest are constant, and that x varies as b when the rest are constant, and so on; shew that, when they all vary, x varies as their product.

Apply this principle to the following case: assuming that the quantity of work done at a sitting varies as the cube root of the number of agents when the time is the same, and varies as the square root of the time when the number of agents is the same; find how long three men would take to do one-fifth of the work which twenty-four men can do in twenty-five hours.

8. Prove that $\log(m \times n \times r) = \log m + \log n + \log r$.

Why is $\log(1 + 2 + 3)$ equal to $\log 1 + \log 2 + \log 3$?

Given that $\log 2 = {\cdot}3010300$,
$\log 3 = {\cdot}4771213$,

find $\log(1080)$ and $\log(.0025)^{\frac{1}{5}}$.

9. Define the tangent of an angle, and shew from the definition that $\tan(180° + A) = \tan A$, for all values of A.

10. Find the value of $\sin 18°$.

In Euclid's construction for determining an isosceles triangle, the angles at whose base are double of the angle at the vertex, shew that the common chord of the two circles is equal to the base of the triangle.

11. Find A from the equation $\tan 2A = 8\cos^2 A - \cot A$.

If $\sin 3A = n \sin A$ be true for any values of A besides 0 or a multiple of $90°$, shew that n must be less than 3 and not less than -1. Solve the equation when $n = 2$.

If $\cos\theta \cos\phi = \sin(\alpha - \beta) \sin(\alpha + \beta)$,

and $\sin(\theta - \phi) \sin(\theta + \phi) = 4 \cos\alpha \cos\beta$; find $\cos\theta$, and $\cos\phi$.

12. In any triangle ABC, prove that

$$AB^2 = BC^2 + CA^2 - 2BC.CA \cos C.$$

AD is drawn to meet BC, or BC produced, in D, so that AD is equal to AC; shew that if the sum of AB and AC is n times BC, their difference is $\frac{1}{n}$th of BD.

13. Find the radius of the circle described about a triangle whose sides are given.

Shew that the radius of the circle inscribed in an isosceles triangle can never be greater than one-half of that of the circumscribed circle.

14. Two posts, AB and CD, are placed at the edge of a river at a distance AC equal to AB, the height of CD being such that AB and CD subtend equal angles at E, a point on the other bank exactly opposite to A; shew that the square of the breadth of the river is equal to $\frac{AB^4}{CD^2 - AB^2}$, and that AD and BC subtend equal angles at E.

WEDNESDAY, *Jan.* 4, 9...12.

1. Two unequal forces act in parallel lines and in opposite directions upon a rigid body moveable about a fixed point in their plane; shew that, if there be equilibrium, the moments of the forces with respect to the fixed point are equal.

Three straight tobacco-pipes rest upon a table, with their bowls, mouth downwards, in the angles of an equilateral triangle, the tubes being supported in the air by crossing symmetrically, each under one and over the other, so as to form another equilateral triangle; shew that the mutual pressure of the tubes varies inversely as the side of the last triangle.

2. If three forces acting upon a particle keep it at rest, shew that the forces are respectively in the ratio of the sines of the angles contained by the other two.

A smooth circular ring is fixed in a horizontal position, and a small ring sliding upon it is in equilibrium when acted on by two strings in the direction of the chords PA, PB; shew that, if PC be a diameter of the circle, the tensions of the strings are in the ratio of BC to AC.

If A and B be fixed points, is the equilibrium stable?

3. Define the centre of gravity of a system of heavy particles, and shew that in every case there exists one and only one such point.

From this fact deduce the property that the lines joining the middle points of opposite sides of any quadrilateral bisect each other.

4. Find the ratio of P to W in the single moveable pulley, when the strings are not parallel.

If a weight W be supported by a weight P hanging over a fixed pulley, the strings being parallel, shew that, in whatever position they hang, the position of their centre of gravity is the same.

5. Describe the construction and graduation of the common steelyard.

Shew that, if a steelyard be constructed with a given rod, whose weight is inconsiderable compared with that of the sliding weight, the sensibility varies inversely as the sum of the sliding weight and the greatest weight which can be weighed.

6. A rigid body, moveable round a fixed axis, is kept in equilibrium by two forces P and Q acting in a plane perpendicular to the axis; shew that, if the body be twisted slightly round the axis,

$$P \times P\text{'s velocity} = Q \times Q\text{'s velocity}.$$

Of what practical principle does this property furnish a proof in the particular case proposed?

7. Describe one of the simple experiments which involve the principle of the second law of motion, and shew how the probability of the law may be inferred from it.

8. What is meant by a unit, and what is usually taken as the unit of accelerating force?

If the force of gravity be taken as the unit of force, and a rate of ten miles an hour as the unit of velocity, what must be the units of time and space?

9. If a body be projected with the velocity u in the direction of a uniform force f, and v be the velocity, and s the space described, at the end of the time t, prove that

$$s = ut + \tfrac{1}{2}ft^2, \text{ and that } \frac{v-u}{f} = \frac{2s}{v+u} = t.$$

The velocity of a body increases from ten to sixteen feet per second in passing over thirteen feet under the action of a constant force; find the numerical value of the force.

10. A body is projected in a given direction from the top of a tower, determine its path, and find where it will strike the ground.

A plane is inclined at an angle of 45° to the horizon, and from the foot of it a body is projected upwards along the plane, and reaches the top with one fifth of its original velocity; where will it strike the ground?

11. Two balls of given masses and given elasticity are moving with given velocities in the same direction; determine their motion after impact.

Two balls are moving in the same straight line, one of them only being acted on by a force; if the force be constant and tend towards the other ball, shew that the times which elapse between consecutive impacts decrease in geometrical progression.

12. Prove that the time of falling in a straight line from the highest point of a vertical circle to any point in the circumference is less than to any point outside; and give a geometrical construction for the straight line of quickest descent to the circumference of a vertical circle from a given point within it.

Shew that the circumferences of two circles contain all points from which the time of quickest descent to a given vertical circle is the same.

WEDNESDAY, *Jan.* 4. $1\frac{1}{2}$–4.

1. FIND the pressure at any depth below the surface of a uniform heavy fluid.

If there be n fluids arranged in strata of equal thickness, and the density of the uppermost be ρ, of the next 2ρ, and so on, that of the last being $n\rho$; find the pressure at the lowest point of the n^{th} stratum, and thence prove that the pressure at any point within a fluid whose density varies as the depth is proportional to the square of the depth.

2. What must be the unit of weight in order that the equation $W = VS$ may hold, W being the weight of a homogeneous body whose volume is V cubic feet, and specific gravity S in tables in which the specific gravity of distilled water is 1.

Find the specific gravity of a mixture of given volumes of known fluids.

3. Prove that the pressure of a uniform heavy incompressible fluid on any surface is equal to the weight of a column of the fluid, the base of which is equal to the area of the surface, and altitude equal to the depth of the centre of gravity of the surface below the surface of the fluid.

A cylindrical vessel is filled with equal masses of two incompressible fluids which do not mix; supposing the whole pressures on the upper and lower portions of the concave surface of the vessel to be equal, compare their densities.

4. Prove that the resultant pressure of a fluid on the surface of a solid immersed in it is equal to the weight of the fluid displaced, and acts upwards in the vertical line through the centre of gravity of the fluid displaced.

A rod of length a and density ρ is moveable freely about one end, which is fixed at a depth c below the surface of a fluid of density σ; prove that the rod may remain at rest, when inclined to the vertical, provided that

$$\frac{\sigma}{\rho} > 1, \text{ and } < \frac{a^2}{c^2}.$$

Shew that such a position is one of stable equilibrium.

5. Describe the experiment by which it is shewn that the pressure of air at a given temperature varies inversely as the space it occupies. State the law connecting pressure, density, and temperature, when all vary; and if $p_1\rho_1t_1$, $p_2\rho_2t_2$, $p_3\rho_3t_3$, be three corresponding pressures, densities, and temperatures, shew that

$$t_1\left(\frac{p_2}{\rho_2} - \frac{p_3}{\rho_3}\right) + t_2\left(\frac{p_3}{\rho_3} - \frac{p_1}{\rho_1}\right) + t_3\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = 0.$$

6. Describe the construction and action of Smeaton's Air-Pump.

Supposing the upper valve of the barrel to open when the piston has gone half through one of its ascents, what was the density of the air in the receiver at the commencement of the ascent?

7. State the laws of refraction to which rays of light are subject. What is the greatest apparent zenith distance which a star can have, as seen by an eye under water.

8. A pencil of rays diverging from a point at a given distance from the centre, is incident directly on a concave spherical refracting surface, determine the distance of the geometrical focus of the refracted pencil from the centre.

An eye is placed close to the surface of a sphere of glass ($\mu = \frac{3}{2}$), which is silvered at the back; shew that the image which the eye sees of itself is $\frac{2}{5}$ of the natural size.

9. Find the position of the geometrical focus of a diverging pencil refracted through a plate of glass.

A rod, inclined at any angle to a plate of glass, is seen by an eye on the opposite side of the plate; shew that the length of the image of the rod formed by geometrical foci is equal to the length of the rod. Is the image, formed by the refraction at the first surface, of the same magnitude as either?

10. Find the deviation of a ray of light refracted through a prism in a plane perpendicular to the edge.

If rays in this plane are incident at one point of the prism in all directions, shew that, if the refracting angle be greater than $\sin^{-1}\frac{1}{\mu}$, rays incident from that side of the normal which is towards the edge of the prism will not pass through, and examine what rays will pass through.

11. Describe the construction of Newton's telescope, and find its magnifying power.

Two convex lenses have a common axis and equal focal lengths, and their distance is two-thirds of the focal length of either; find a point on the axis from which rays must diverge, in order that, after refraction through both lenses, the emergent pencil may consist of parallel rays. Trace the course of such a pencil.

12. Determine the angle subtended at the eye by the image of a short object seen through a concave lens, the axis of which passes through both the eye and the object.

A short-sighted person moves his eye-glass gradually from his eye towards a small object: shew that the linear magnitude of the image will keep increasing during the motion, and that the angle subtended by the image at the eye will be least when the eye-glass has advanced half way towards the object.

THURSDAY, *Jan.* 5. 9...12.

1. EXPLAIN what is meant by the limit of a varying quantity or ratio, and enunciate and prove Newton's first Lemma.

Two triangles, CAB, $C'AB'$, have a common angle A, and the sum of their sides about that angle the same in each; if CB, $C'B'$ intersect in D, and B' move up to B, then in the limit $DC : DB :: AB : AC$.

2. Define the circle of curvature at any point of a curve. If PQ be an arc, and QR a subtense, the chord of the circle of curvature at P parallel to QR is equal to the limit of the third proportional to QR and PQ. Find the chord of curvature through the focus of an ellipse.

EF is a chord of a given circle and S its middle point; construct the ellipse of which E is one point, S one focus, and the given circle the circle of curvature at E.

3. Shew that, in an orbit described under the action of a force tending to a fixed point, the velocity at any point is inversely proportional to the perpendicular from the centre of force on the tangent at that point.

A body is describing a parabola under the action of a force which always tends to the focus, and a straight line is drawn from the focus perpendicular to the tangent, and proportional to the velocity, at any point; shew that the extremity of this straight line will lie in a certain circle.

4. Given the velocities and the directions of motion at any three points of an orbit described under the action of a central force, find the centre of force.

If the velocities at the three points be respectively parallel and proportional to the opposite sides of the triangle of which they are the angular points, the centre of force is the centre of gravity of the triangle.

5. An ellipse is described under the action of a force tending to the focus; find the law of force and the velocity at any point.

If, without changing the velocity, the direction of motion of the body receive a very slight alteration, shew that the position of the major axis will be altered, unless the body be at one extremity of the latus rectum through the focus to which the force does not tend.

6. Enumerate the principal steps which led Newton to conclude that the Moon is retained in her orbit by the force of gravity.

Assuming that the Moon is retained in her orbit by the Earth's attraction alone, and that, approximately, her orbit is circular, her period about the Earth 27 days, the accelerating effect of gravity at the Earth's surface 32 feet per second, and the Earth's radius 4000 miles, find the distance of the Moon from the Earth's centre.

7. Define the terms Declination and Right Ascension. Account for the change of the Sun's declination in the course of a year, and discuss the consequent variations in the length of the day at a place between the pole and the arctic circle.

8. Account for the Moon's rising at different times on two successive nights; at what places is it possible for the Moon to continue above the horizon for more than twenty-four hours?

If an observer be stationed on the Moon's surface at the point nearest the Earth, describe the principal phenomena relating to the Sun and Earth which he would observe in one of his days. What circumstances would lead him to the conclusion that the Earth's apparent orbit was inclined to the Sun's?

9. Distinguish between a sidereal and a tropical year; and explain the Gregorian intercalation of a day in certain years, assuming the length of the tropical year to be 365.242218 days.

10. Describe the Transit Instrument, and give a method of detecting an error of deviation. Will this method apply at places near the Equator?

11. Explain the aberration of light, and shew in what direction the error of aberration takes place.

What limit is there to the position of a place in order that at some time in the day a star in the ecliptic may have its error of aberration in a vertical plane?

12. Explain the method of determining the longitude by Moon culminating stars.

What is the object of registering in the Nautical Almanac the time of passage of the Moon's semi-diameter across the Meridian?

THURSDAY, *Jan.* 5. 1...4.

PROBLEMS.

1. ABD, ACE are two straight lines touching a circle in B and C, and, if DE be joined, DE is equal to BD and CE together; shew that DE touches the circle.

2. O, A, B, C, are four points arranged in order in a straight line, so that OA, OB, OC, form an harmonic progression. Prove that, A and C being stationary, if O move towards A, B will also move towards A.

3. If a, b, c, be positive integers, and $a^{\frac{2}{b}}$, $b^{\frac{1}{ac}}$, $c^{\frac{2}{b}}$ be in geometrical progression, shew that $a^{\frac{2}{b^n}}$, $b^{\frac{1}{a^n c^n}}$, $c^{\frac{2}{b^n}}$, are also in geometrical progression.

4. If either of the two quantities $1 + 3^m$, $1 + 3^{m+4r}$, is a multiple of 10, prove that the other is also a multiple of 10, m and r being positive integers.

5. Find the value of $\tan a$ or $\tan\beta$ from the equations

$$\tan(a+\beta) = \tan a \cot\beta + \cot a \tan\beta,$$
$$\tan(a-\beta) = \tan a \cot\beta - \cot a \tan\beta.$$

6. If $A + B + C = 90°$, shew that the least value of

$$\tan^2 A + \tan^2 B + \tan^2 C$$

is 1.

7. Lines, drawn through Y, Z, at right angles to the major axis of an ellipse, cut the circles, of which SP, HP are diameters, in I, J respectively. Prove that IS, JH, BC, produced indefinitely, intersect each other in a single point.

8. From any point T, two tangents are drawn to a given ellipse, the points of contact being Q, Q': CQ, CQ', QQ', CT, are joined; V is the intersection of QQ', CT. Prove that the area of the rectilinear triangle QCQ' varies inversely as

$$\left(\frac{CV}{TV}\right)^{\frac{1}{2}} + \left(\frac{TV}{CV}\right)^{\frac{1}{2}}.$$

9. A piece of uniform wire is bent into three sides of a square $ABCD$, of which the side AD is wanting; shew that, if it be hung up by the two points A and B successively, the angle between the two positions of BC is $\tan^{-1} 18$.

10. A weight of given magnitude moves along the circumference of a circle, in which are fixed also two other weights: prove that the locus of the centre of gravity of the three weights is a circle. If the immoveable weights be varied in magnitude, their sum being constant, prove that the corresponding circular loci intercept equal portions of the chord joining the two immoveable weights.

11. A ball of elasticity e is projected from a point in an inclined

plane, and, after once impinging upon the inclined plane, rebounds to its point of projection: prove that, a being the inclination of the inclined plane to the horizon, and β that of the direction of projection to the inclined plane,

$$\cot a \,.\, \cot\beta = 1 + e.$$

12. Two heavy bodies are projected from the same point at the same instant in the same direction, with different velocities; find the direction of the line joining them at any subsequent time.

13. Three equal and perfectly elastic balls A, B, C move with equal velocities towards the same point, in directions equally inclined to each other; suppose first, that they impinge upon each other, at the same instant; secondly, that B and C impinge on each other, and immediately afterwards simultaneously on A; and thirdly, that B and C impinge simultaneously on A just before touching each other; and let $V_1 V_2 V_3$ be the velocities of A after impact on these suppositions respectively: shew that

$$V_2 = \tfrac{1}{5} V_1, \text{ and that } V_3 = \tfrac{7}{5} V_1.$$

14. CP, CD, are two conjugate semidiameters of an ellipse described by a body about a centre of force in the focus S: PP', DD', chords of the ellipse parallel to the major axis. Prove that, a, a', β, β', being the angular velocities of the body about S at P, P', D, D', respectively,

$$\frac{1}{(aa')^{\frac{1}{3}}} + \frac{1}{(\beta\beta')^{\frac{1}{3}}} = \text{a constant quantity.}$$

15. Supposing the velocity of a body in a given elliptic orbit to be the same at a certain point, whether it describe the orbit in a time t about one focus, or in a time t' about the other, prove that, $2a$ being the major axis, the focal distances of the point are equal to

$$\frac{2at'}{t+t'},\ \frac{2at}{t+t'}.$$

16. Three candles are placed in a room, and the two shorter being lighted throw shadows of the third upon the ceiling; if the directions of these shadows be produced, where will they meet?

17. Within a reflecting circle on the same side of the centre are two parallel rays, one dividing the circumference into arcs which are as 3 to 1, the other dividing it into arcs which are as 8 to 1; find the least value of n such that, after each ray has suffered n reflections, they may be again parallel.

18. One asymptote of an hyperbola lies in the surface of a fluid; find the depth of the centre of pressure of the area included between the immersed asymptote, the curve, and two given horizontal lines in the plane of the hyperbola.

19. A cone is totally immersed in a fluid, the depth of the centre of its base being given. Prove that, P, P', P'', being the resultant pressures

on its convex surface, when the sines of the inclination of its axis to the horizon are s, s', s'', respectively,

$$P^2(s'-s'')+P'^2(s''-s)+P''^2(s-s')=0.$$

20. Light emanating from a luminous circular disk, placed horizontally on the ceiling of a room, passes through a rectangular aperture in the floor: ascertain the form and area of the luminous patch on the floor of the room below.

Shew that neither the shape nor the area of the patch will be affected by any movement of the disk along the ceiling.

21. If c_1, c_2, c_3 be the lengths of the meridian shadows of three equal vertical gnomons, on the same day, at three different places on the same meridian, prove that the latitudes λ_1, λ_2, λ_3, of the places are connected together by the equation

$$c_1 \cdot \frac{(c_2-c_3)^2}{\tan(\lambda_2-\lambda_3)}+c_2 \cdot \frac{(c_3-c_1)^2}{\tan(\lambda_3-\lambda_1)}+c_3 \cdot \frac{(c_1-c_2)^2}{\tan(\lambda_1-\lambda_2)}=0.$$

MONDAY, *Jan.* 16. 9...12.

1. A SYSTEM of rigid bodies is under the action of no forces but their weights, mutual reactions, tensions of inextensible strings, and pressures on smooth fixed surfaces; prove that if the height of the centre of gravity above a fixed horizontal plane be a maximum or a minimum, the system will be in equilibrium.

Apply this principle to determine the position of equilibrium of two equal uniform rods, connected by a smooth hinge at one extremity, and resting symmetrically on two smooth pegs in the same horizontal line.

2. Determine the necessary and sufficient conditions that a system of forces acting on a rigid body may have a single resultant.

A portion of a curve surface of continuous curvature is cut off by a plane, and, at a point in each element of that portion, a force proportional to the element is applied in the direction of the normal; shew that, if all the forces act inwards or all outwards, they will in the limit have a single resultant.

3. A particle under the action of any forces rests on a surface whose equation is given; determine the conditions of equilibrium, (1) when the surface is smooth, (2) when it is rough.

Find the least coefficient of friction between a given elliptic cylinder and a particle, in order that, for all positions of the cylinder in which the axis is horizontal, the particle may be capable of resting at any point vertically over the axis.

4. A heavy elastic string is suspended from one extremity, and stretched by its own weight; determine its length when it is at rest.

If a heavy elastic string rest upon the convex side of a smooth curve in a vertical plane, shew how to determine the tension at any point.

5. If a particle be moving in any path, straight or curved, and, at the time t, s be its distance measured along its path from a fixed point; shew that $\frac{d^2s}{dt^2}$ is a measure of the accelerating force in the direction of motion.

If the position of a particle moving in a plane be determined by the coordinates ρ and ϕ, ρ being measured from a fixed circle along a tangent which has revolved through an angle ϕ from a fixed tangent, investigate the following expressions for the components of the accelerating force along and perpendicular to ρ respectively, (the latter being considered positive when it tends to increase ϕ):

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2 + a\frac{d^2\phi}{dt^2}, \quad \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right) + a\left(\frac{d\phi}{dt}\right)^2.$$

6. State the laws which regulate the magnitude and the direction of statical and of sliding friction.

Two equal bodies lie on a rough horizontal table, and are connected by a string which passes through a fine ring on the table; if the string be stretched, find the greatest velocity with which one of the bodies can be projected in a direction perpendicular to its portion of the string without moving the other body.

7. Find the differential equation to the path of a particle subject to a force, which tends to a fixed centre, and is a function of the distance from that point.

If there be several centres, the force towards each varying as the distance, and a number of particles be projected in different directions from the same point and with the same velocities, determine the curve which passes through the position of each particle at the instant when it has a given velocity.

8. A heavy particle is suspended from a fixed point by a fine string; find the time of a small oscillation in a vertical plane.

9. Having given the index of refraction between the two media A and B, and also between the two A and C, shew how to find that between B and C.

The index of refraction (μ) in a medium varies from point to point, being a function of the distances x and y from two planes at right angles to each other; a ray traverses the medium in a plane perpendicular to these two planes; if $\log \mu = f(x, y)$, prove that the curvature of the path of the ray varies as

$$f'(x)\frac{dy}{ds} - f'(y)\frac{dx}{ds}.$$

10. State the law determining the elastic force of a mixture of given quantities of air and vapour. Define the Dew Point, and shew the im-

portance of its determination. Why is a cloudy night unfavourable to the deposition of dew?

The barometer stands at 29·88 inches, and the thermometer is at the Dew Point; a barometer and a cup of water are placed under a receiver, from which the air is removed, and the barometer then stands at ·36 of an inch: find the space which would be occupied by a given volume of the atmosphere, if it were deprived of its vapour without changing its pressure or temperature.

11. Determine the condition that a curve surface, immersed in fluid, may have a centre of pressure; and shew how to find it, if this condition be satisfied.

12. Describe the reading microscope of the mural circle. What are "Runs"? Shew that the effects of the eccentricity and irregular form of the pivot are eliminated by taking the sum of opposite Microscope-readings corrected for Runs.

13. Determine the effect of precession on the declination of a given star: explain the advantage of using the constants A, B, C, D in applying the correction for aberration, precession, and nutation.

14. What is the greatest value of the inclination of the Moon's orbit to the ecliptic, for which there would have been a lunar eclipse at every opposition?

Find the lunar ecliptic limits; and determine whether there was or was not an eclipse of the Moon on the 31st of March 1847, from the following data, selected from the Nautical Almanac:

1847.	The Sun's		The Moon's			
	Semidiam.	Longitude.	Semidiam.	Parallax.	Longitude.	Latitude.
Mar. 31. Noon	16′1″.3	10°9′18″.3	14′44″.3	54′ 5″.0	185°56′16″.2	1°10′27″.1
Midnight	......	....	14′45″.8	54′10″.5	191°53′11″.3	0°37′55″.6
Apr. 1. Noon	16′1″.0	11°8′26″.1	....	....	....	....

Apr. 1. Sun's parallax 8″.58, longitude of Moon's ascending node 199° 26′.2.

MONDAY, *Jan.* 16, 1½...4.

1. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\ldots$ be the successive convergents of a continued fraction greater than unity, prove that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n.$$

Shew that the difference between the 1st and nth convergents is equal to

$$\frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \frac{1}{q_3 q_4} - \ldots\ldots + \frac{(-1)^n}{q_{n-1} q_n}.$$

2. Prove that impossible roots enter rational algebraical equations by pairs.

Shew that all the roots of the following equation are possible:

$$\frac{A_1^2}{x-a_1}+\frac{A_2^2}{x-a_2}+\frac{A_3^2}{x-a_3}+\ldots\ldots+\frac{A_n^2}{x-a_n}=1.$$

3. Give Cardan's method for the solution of a cubic equation. When is Cardan's method said to fail, and in what does the failure consist?

If $\alpha+\beta(\sqrt{-1})$ be a root of the equation $x^3+qx+r=0$, prove that α is a root of the equation $8x^3+2qx-r=0$.

4. Apply Horner's method to determine to four places of decimals the root of the following equation which lies between 1 and 2:

$$x^4-2x^3+21x-23=0.$$

5. Prove that the series $\tan\alpha-\frac{1}{3}\tan^3\alpha+\frac{1}{5}\tan^5\alpha-\ldots$*ad inf.* is equal to $n\pi+\alpha$, where n is zero or such a positive or negative integer as will make $n\pi+\alpha$ lie between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

Shew that, whatever positive integer m be, if $\phi=\frac{2}{(2m+1)\pi}$, $\frac{1}{\phi}-\phi-\frac{2}{3}\phi^3$ is a very approximate solution of the equation $\tan\theta=\theta$.

6. Investigate the condition of perpendicularity of two straight lines whose equations are

$$Ax+By+C=0,\quad A'x+B'y+C'=0.$$

Shew that, if the axes be inclined at an angle ω, the condition that the straight lines may be equally inclined to the axis of x in opposite directions, is

$$\frac{B}{A}+\frac{B'}{A'}=2\cos\omega.$$

If, besides being equally inclined to the axis of x, the straight lines pass through the origin and be perpendicular to one another, the equation of the straight lines is

$$x^2+2xy\cos\omega+y^2\cos2\omega=0.$$

7. Investigate the equations to the tangents at the extremities of two conjugate diameters of an ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the co-ordinates of the extremity of one of the diameters being given.

In an ellipse SQ and HQ, drawn perpendicularly to a pair of conjugate diameters, intersect in Q; prove that the locus of Q is a concentric ellipse.

8. Shew that the locus of the poles of all tangents to a given circle, with respect to another fixed circle, is a conic section, whose directrix is the polar of the centre of the first circle.

Employ the method of reciprocal polars to shew that, if three ellipses have one common focus, and pairs of common tangents be drawn to the ellipses taken two together, the three points of intersection of these pairs of tangents lie in a straight line.

9. Investigate the equation to a plane. Find the equation to a plane which passes through two parallel lines denoted by the equations

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \quad \frac{x-a'}{l} = \frac{y-b'}{m} = \frac{z-c'}{n}.$$

10. Investigate formulæ for the transformation of co-ordinates in passing from one system of three réctangular axes to another having the same origin.

Shew that the equation of a surface $yz + zx + xy = a^2$ may be reduced to the form

$$x^2 - \frac{y^2 + z^2}{2} = a^2.$$

11. If A, B, C, be extremities of the axes of an ellipsoid, and AC, BC be the principal sections containing the least axis, find the equations of the two cones whose vertices are A, B, and bases BC, AC respectively: shew that they have a common tangent plane, and a common parabolic section, the plane of the parabola and the tangent plane intersecting the ellipsoid in ellipses the area of one of which is double that of the other; and, if l be the latus rectum of the parabola, l_1, l_2 of the sections AC, BC, prove that

$$\frac{1}{l^2} = \frac{1}{l_1^2} + \frac{1}{l_2^2}.$$

12. Prove that, in a spherical triangle,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

where b and c are each less than 90°; and extend it to the case where one of these sides is greater than 90°.

Prove that, if p, q, r be the lengths of arcs of great circles drawn from A, B, C perpendicularly to the opposite sides,

$$\sin a \sin p = \sin b \sin q = \sin c \sin r = (1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c)^{\frac{1}{2}}.$$

TUESDAY, *Jan.* 17. 9...12

PROBLEMS.

1. If $\overset{m}{\underset{s}{C}}$ denote generally the number of combinations of m things s together and $\overset{m}{\underset{0}{C}}$ be taken to denote unity for all values of m; prove that, if

$$\underset{r}{S} = 1^r . \overset{n}{\underset{r}{C}} + 2^{r-1} . \overset{n-1}{\underset{r-1}{C}} + 3^{r-2} . \overset{n-2}{\underset{r-2}{C}} + 4^{r-3} . \overset{n-3}{\underset{r-3}{C}} + \dots\dots r^1 . \overset{n-r+1}{\underset{1}{C}} + \overset{n-r}{\underset{0}{C}},$$

then $$\underset{0}{S} + \underset{1}{S} + \underset{2}{S} + \underset{3}{S} + \dots + \underset{n}{S} = 1^{n+1} + 2^n + 3^{n-1} + \dots + (n-1)^3 + n^2 + (n+1)^1.$$

2. Straight lines Aa, $B\beta$, $C\gamma$, are drawn from the angular points A, B, C, of a triangle to bisect the opposite sides in a, β, γ, O being the point of intersection of the three lines. If the radii of the circles inscribed in the triangles BOa, COa; $CO\beta$, $AO\beta$; $AO\gamma$, $BO\gamma$; be represented by a_β, a_γ; b_γ, b_α; c_α, c_β; respectively; prove that

$$\frac{1}{a_\beta} - \frac{1}{a_\gamma} + \frac{1}{b_\gamma} - \frac{1}{b_\alpha} + \frac{1}{c_\alpha} - \frac{1}{c_\beta} = 0.$$

3. P is a point in a branch of an hyperbola, P' a point in a branch of its conjugate, CP, CP', being conjugate semi-diameters. If S, S', be the interior foci of the two branches, prove that

$$S'P' - SP = AC - BC.$$

4. On any chord of a parabola as diameter is described a circle cutting the parabola again in two points; if these points be joined, shew that the portion of the axis of the parabola included between the two chords is equal to its latus rectum.

5. If $r = f(\theta)$ and $y = f\left(\frac{x}{a}\right)$ be the equations to two curves, $f(\theta)$ being a function which vanishes for the values $\theta_1 \theta_2$ and is positive for all values between these limits, and if A be the area of the former between the limits

$$\theta = \theta_1, \quad \theta = \theta_2,$$

and M be the arithmetic mean of all transverse sections of the solid generated by the revolution, about the axis of x, of the portion of the latter curve between the limits $x = a\theta_1$, $x = a\theta_2$; shew that

$$M = \frac{2\pi}{\theta_2 \sim \theta_1} A.$$

6. A brick is divided by a plane, passing through one corner, and making an angle of 45° with the length of the brick; find the position of this plane in which the two parts are the most nearly equal.

7. If r, r', be the radii of curvature of an involute and evolute at corresponding points (x, y), (x', y'), prove that

$$rdx' \pm r'dy = 0, \quad rdy' \mp r'dx = 0;$$

and shew that, the involute being an ellipse of which the semi-axes are a, b, the greatest value of $\frac{r'}{r}$ is equal to

$$\frac{3}{2}\left(\frac{a}{b} - \frac{b}{a}\right).$$

8. Trace the curve whose equation is

$$y^2 = \frac{x^4 - c^4}{x(x-a)},$$

first supposing a to be less than c, then equal, then greater; and shew how the three forms of the curve pass into each other, when the value of a is supposed to increase gradually through the value c.

9. $SPHQ$ is a quadrilateral, P and Q being points in an ellipse of which S and H are the foci; if Q be fixed while P moves, find the locus of the centre of gravity of the perimeter of the quadrilateral.

10. From an external point P two tangents are drawn to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Supposing the locus of the centre of gaavity of the triangle, included between the two tangents and the chord of contact, to be an ellipse $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$, find the equation to the locus of P.

What must be the relation between a, b, a_1, b_1, in order that the locus of P may be an ellipse?

11. The radii vectores of any series of points in the path of a particle, moving about a centre of force, being in arithmetical progression, the times of arriving at these points, reckoned from a given epoch, form another arithmetical progression. Find the equation to the path.

12. In any machine in which two weights P and W are suspended by strings and balance each other in all positions, let P be replaced by a weight Q equal to pP; if in the ensuing motion W and Q move vertically, find the tensions of these strings, neglecting the friction of the machine and the masses of its several parts.

13. There are generally two directions in which a projectile may be projected with given velocity from a point A, so as to pass through another point B; shew that one of these directions is inclined to the vertical at the same angle that the other is inclined to the line AB. Hence shew that the locus of points, for which a given sight must be used in firing with a given charge of powder, is the surface generated by the revolution, about the vertical, of the path of the bullet obtained by aiming at the zenith with the given sight, and with the given charge of powder.

14. A prism whose base is a given regular polygon is surmounted by a regular pyramid whose base coincides with the head of the prism; find the inclination of the faces of the pyramid to its axis in order that the whole solid may contain a given volume with the least possible surface.

15. An ellipsoid is intersected in the same curve by a variable sphere, and a variable cylinder: the cylinder is always parallel to the least axis of the ellipsoid, and the centre of the sphere is always at one focus of a principal section containing this axis. Prove that the axis of the cylinder is invariable in position, and that the area of its transverse section varies as the surface of the sphere.

16. An elastic tube of circular bore is placed within a rigid tube of square bore which it exactly fits in its unstretched state, the tubes being

of indefinite length; if there be no air between the tubes and air of any pressure be forced into the elastic tube, shew that this pressure is proportional to the ratio of the part of the elastic tube that is in contact with the rigid tube, to the part that is curved.

17. OA, OB, are any equal arcs of two given great circles of a sphere, intersecting in O. A and B are joined by an arc of a great circle, and also by an arc of a small one described about O. Find the area of the lune included between the two joining arcs.

If $OA = \lambda$ and $AOB = 2\omega$, prove that the lune is greatest when

$$\cos^2\lambda = \frac{\tan\omega - \omega}{\omega\tan^2\omega}.$$

18. The ridges of two roofs are at right angles to each other, and the inclination of each roof to the horizon is θ; the shadow of a chimney falling upon them makes angles α and β with their ridges; shew that

$$\cos^2\theta = \cot\alpha\cot\beta.$$

19. The hour angles of two stars being ϵ, ϵ', and the azimuths a and a', when $a \sim a'$ has for a moment a stationary value; prove that the latitude λ of the place of observation is given by the formula

$$\sin\lambda = \frac{\sin 2a \,.\, \cot\epsilon - \sin 2a' \,.\, \cot\epsilon'}{\cos 2a - \cos 2a'}.$$

20. A thin hollow ring, of which the plane is vertical, and which contains a bead, is placed upon a smooth horizontal plane: prove that the bead, having been placed near the lowest point of the ring, will oscillate isochronously with a perfect pendulum the length of which is equal to

$$\frac{\mu a}{m+\mu},$$

a being the radius of the ring, μ its mass, and m the mass of the bead.

21. A uniform rod, not acted on by any forces, is in motion, its ends being constrained to slide along two fixed rods at right angles to each other in one plane. Prove that, during the whole motion, the wrenching force at any point of the moving rod varies as the product of the distances of the point from the two fixed rods.

TUESDAY, *Jan.* 17. 1½...4.

1. EXPLAIN the formation of focal lines in the reflection or refraction of a small oblique pencil.

A small pencil of diverging rays is incident on a prism at a given distance from the edge, the axis of the pencil being perpendicular to the edge; find the positions of the primary and secondary foci.

If the given distance be small, and the axis be incident at such an angle as to pass through the prism with minimum deviation, shew that the primary and secondary foci nearly coincide, and thence explain the necessity of certain precautions in order to obtain a pure spectrum in the decomposition of light by a prism.

2. What is meant by a secondary spectrum? A compound object-glass is to be formed of two lenses in contact; shew that, if, when the lenses are ground, achromatism is nearly but not quite secured, the defect may be remedied by slightly separating the lenses.

The refractive indices, corresponding to the letters D and F in the orange and blue, for certain kinds of crown and flint glass, are

Crown glass......1.5279, 1.5344,
Flint glass 1.6351, 1.6481;

twenty inches is to be the focal length of the proposed object-glass; find the focal lengths of the two lenses which, placed in contact, unite these lines.

3. Investigate a formula for calculating the first two tables in the Nautical Almanac by which the latitude is determined from observations of the Pole Star out of the Meridian.

What is the nature of the correction contained in the third table?

4. Determine the motion of a planet in geocentric longitude, and shew that all planets will sometimes appear stationary to an observer on the Earth.

If m be the ratio of the radius of the Earth's orbit to that of an inferior planet, n the ratio of their motions in longitude considered uniform, shew that the elongation of the planet as seen from the Earth, when the planet appears stationary, is equal to

$$\tan^{-1}\sqrt{\left(\frac{1-m^2n^2}{m^2-1}\right)}.$$

5. Determine the motion of a particle acted on by given forces and constrained to remain on a given surface.

A particle is in motion on the surface whose equation is $z=\phi(x, y)$, and is acted on by a constant accelerating force f parallel to the axis of z; if v be the velocity of the particle and its path be always perpendicular to the direction of the force, shew that, at any point of its path,

$$\frac{v^2}{f}=\frac{\left\{\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}^2}{\frac{d^2z}{dx^2}\left(\frac{dz}{dy}\right)^2-2\frac{d^2z}{dx\,dy}\frac{dz}{dx}\frac{dz}{dy}+\frac{d^2z}{dy^2}\left(\frac{dz}{dx}\right)^2}.$$

6. Investigate the general equations of fluid motion; and deduce from them the differential equation of the surfaces of equal pressure, when

a heavy elastic fluid is contained in a closed vessel, rotating with unform angular velocity about a vertical axis, and is at rest relatively to the vessel.

How is the constant to be determined in integrating for the pressure at any given point?

7. Explain the effect of the Sun's disturbing force upon the position of the line of nodes of the Moon's orbit, when the line of nodes is in quadratures; and shew that the horary motion of the line of nodes is to that of the Moon as

$$-3m^2 \cos(\theta - m\theta) \sin(\theta - N) \sin(m\theta - N) : 1,$$

N being the longitude of the node, θ that of the Moon, and $m\theta$ that of the Sun.

8. Define the principal axes of a rigid body, and shew that for every point in space there exists a system of such axes.

Shew that in general there is only one point for which the principal axes are parallel to those drawn through a given point; but that, if the given point be in one of the principal planes through the centre of gravity, there is an infinite number of such points lying in an hyperbola which passes through the given point.

9. The equation for the projection of the Moon's radius vector on the ecliptic is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} - \frac{T}{h^2u^3}\frac{du}{d\theta} - 2\left(\frac{d^2u}{d\theta^2} + u\right)\int \frac{Td\theta}{h^2u^3}$$

and $$\frac{P}{h^2u^2} = \frac{\mu}{h^2}\left(1 - \frac{3s^2}{2}\right) - m^2a\left(\frac{u'}{a'}\right)^3\left(\frac{a}{u}\right)^3 \frac{1 + 3\cos 2(\theta - \theta')}{2};$$

calculate that part of evection in the value of θ which is due to the radial force only.

Explain this term in connexion with the elliptic inequality,

$$2e \sin(cpt - a) + \frac{5e^2}{4} \sin 2(cpt - a).$$

WEDNESDAY, *Jan.* 18. $9\frac{1}{2}...12\frac{1}{2}$.

PROBLEMS.

1. THERE are n points in space, of which p are in one plane, and there is no other plane which contains more than three of them; how many planes are there, each of which contains three of the points?

2. A bag contains nine coins, five are sovereigns, the other four are equal to each other in value; find what this value must be, in order that the expectation of receiving two coins at random out of the bag may be worth twenty-four shillings.

3. Having given that u, v, and z are functions of the independent variables x and y, and that one of the equations for determining them is $\frac{du}{dx} = v\frac{dz}{dx}$; transform this equation into one in which x and z shall be the independent variables.

4. Trace the curves whose equations are $\tan^2\frac{x}{a} + \tan^2\frac{y}{a} = 1$; and

$$xy(y-x)^2 - ay^3 = a^4.$$

5. Find the value of $\int_0^{\frac{1}{4}\pi} \tan^{-1}\{m\sqrt{(1-\tan^2 x)}\}\,dx$; and shew either from your result, or from the area of the former of the two curves proposed in the preceding question, that $\int_0^{\pi} \tan^{-1}\sqrt{(1-\tan^2 x)}\,dx$ is equal to $\cdot 17$ nearly.

6. Determine the form of the function $f(\theta)$ from the equation

$$f(2\theta) = \cos\theta f(\theta);$$

with the condition $f(0) = m$.

Apply the result to find the centre of gravity of a circular arc.

7. A rod is marked at random at two points, and then divided into three parts at those points; shew that the probability of its being possible to form a triangle with the pieces is $\frac{1}{4}$.

Again: a piece is cut off the end of a rod, and the remainder is cut into two pieces at random; shew that the probability of its being possible to form a triangle with the pieces is in this case $\log_\varepsilon 2 - \frac{1}{2}$.

8. One helix rolls upon another, (the inclination of the curve to the axis being the same in both,) in such a way that the osculating planes of the two curves at the point of contact coincide, find the curve traced out by a point in the rolling curve.

9. A, B, C are three fixed points, and P a point which moves first half way to A, then half way to B, then half way to C, then half way to A again, and so on for ever; shew that from whatever position P start, its path approximates to the perimeter of a certain triangle whose area is one-seventh of the area of the triangle ABC.

10. A string has a heavy particle at one end, and a small smooth ring at the other; a loop, formed by passing the particle through the ring, surrounds a fixed rough horizontal cylinder, the string being in one plane perpendicular to the axis: find the limiting positions of equilibrium; and shew that in every position of equilibrium the three angles at the ring will be all obtuse unless the coefficient of friction exceed $\frac{2\log_\varepsilon 2}{7\pi}$.

11. Two parallel vertical walls are one smooth and the other rough, and between them is supported a hemisphere with its curved surface in contact with the smooth wall, and a point in its rim in contact with the rough wall; find the pressures on the walls, and the least coefficient of friction consistent with equilibrium.

12. A body moves under the action of a force whose direction always touches a given plane curve, shew that, so long as the curvature is continuous, the areas, which it sweeps out about the moving point of contact, are not proportional to the times.

13. A body describes a cycloid under the action of a force, which in every position of the body is directed towards the centre of the corresponding generating circle; find the law of the force and of the motion of the centre of force.

14. A surface of the second order circumscribes a tetrahedron, and each face of the tetrahedron is parallel to the tangent plane at the opposite angular point; shew that the centre of the surface coincides with the centre of gravity of the tetrahedron.

15. A horizontal cylinder revolves with uniform velocity about its axis, and an endless chain, passing round it, revolves with it in such a manner that the form of the chain in space is always the same; shew that the form of the curve is independent of the velocity.

16. An inclined plane is fixed on a table, and from the foot of it a body is projected upwards along the plane with the velocity due to the height h; after passing over the top of the plane the body strikes the table at a distance x from the foot of the plane; shew that, if the length of the plane be l, and a its inclination to the horizon be less than $\frac{1}{4}\pi$, the greatest value of x for given values of h and a is $\frac{h}{\sin a \cos a}$, and corresponds to the value $l = 2h\,\frac{\cot 2a}{\cos a}$.

17. A slender ring, moveable in a vertical plane, has a fixed rough cylinder passing through it, the axis of the cylinder being perpendicular to the plane of the ring; the ring whirls round in its own plane so as always to be in contact with the cylinder, and to roll on it without sliding: if $V_1 V_2$ be the velocities of the centre of the ring when in its highest and lowest positions respectively, and if P be the point of contact, O the centre of the ring, when the tendency to slide is greatest, and OA a vertical drawn downwards through O, shew that

$$\cos POA = 2\,\frac{V_2^2 - V_1^2}{V_2^2 + V_1^2}.$$

Explain the result when $V_2^2 > 3V_1^2$.

18. A cylindrical vessel is moveable about a horizontal axis passing through its centre of gravity, and is placed so as to have its axis vertical; if water be poured in, shew that the equilibrium is at first unstable; and find the condition which must be satisfied, in order that it may be possible to make the equilibrium stable by pouring in enough water.

19. Given the directions of three plane mirrors in space, construct a straight line, such that, if light from it be reflected by the three mirrors in succession, the third image shall be parallel to the straight line.

20. Shew that, in latitude 60°, on the 21st of March, the setting sun is visible for about 69 seconds longer from the top than from the bottom of a tower 66 feet high, taking the earth's radius 4000 miles and neglecting the effect of refraction.

21. Shew how to determine graphically the path of the centre of graduation of a mural circle, by observing the differences between the readings of any three microscopes, (severally corrected for runs,) for various positions of the instrument.

WEDNESDAY, *Jan.* 18, $1\frac{1}{2}...4$.

1. PROVE Leibnitz' Theorem,

$$\frac{d^n(uv)}{dx^n} = u\frac{d^n v}{dx^n} + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{1.2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \dots\dots + \frac{d^n u}{dx^n}v.$$

If $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$, shew that

$$x^2\frac{d^{n+2}y}{dx^{n+2}} + (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (n^2+1)\frac{d^n y}{dx^n} = 0.$$

2. If y be a function of x, and x, y be given functions of r and θ, shew how to transform an expression involving x, y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ ...... into one involving r, θ, $\dfrac{dr}{d\theta}$, $\dfrac{d^2r}{d\theta^2}$ ......

If $x = r\cos\theta$, $y = r\sin\theta$, shew that $\dfrac{x + y\dfrac{dy}{dx}}{x\dfrac{dy}{dx} - y} = \dfrac{1}{r}\dfrac{dr}{d\theta}$.

3. If $f(x)$ be a continuous function of x, shew that, when x increases, $f(x)$ increases or diminishes according as $f'(x)$ is positive or negative; deduce tests which are sufficient for distinguishing between the maximum and minimum values of $f(x)$, supposing them to exist for certain values of x.

Find the least triangle which can be described about a given ellipse, having a side parallel to the major axis.

4. If $x^n f(y)$ contain all the terms involving the highest power of x in the rationalized equation of a curve, shew that $f(y) = 0$ is the equation of all the asymptotes parallel to the axis of x.

If the equation, arranged in the form of a series of homogeneous functions of descending order, be $x^n f\left(\frac{y}{x}\right) + x^{n-r}\phi\left(\frac{y}{x}\right) + \ldots = 0$, and $f(z) = 0$ have two equal roots different from zero, each equal to a; shew that if $r = 1$, there is a parabolic asymptote whose equation is

$$(y - ax)^2 = x\frac{-2\phi(a)}{f''(a)};$$

and, if $r = 2$, there are two parallel rectilinear asymptotes whose equations are

$$y = ax \pm \left\{\frac{-2\phi(a)}{f''(a)}\right\}^{\frac{1}{2}}.$$

5. If r, θ be co-ordinates of a point in a plane curve, and ϕ the angle between the radius-vector and tangent at that point, prove that

$$\cos\phi = \frac{dr}{ds}, \text{ and } \sin\phi = r\frac{d\theta}{ds}.$$

S and H are two fixed points, and a curve is described such that, if P be a point in it, the rectangle contained by SP and HP is constant; shew that the straight lines drawn from S at right angles to SP and from H at right angles to HP meet the tangent at P in points equidistant from P.

6. Trace the curve whose equation is

$$\frac{a}{r} = 1 - \tan\theta.$$

7. Find the values of the following integrals,

$$\int\frac{dx}{\sqrt{(x^2-6x+13)}}, \int\frac{dx}{(1-x^2)^{\frac{3}{2}}}, \int_0^{\frac{1}{2}\pi}\frac{d\theta}{1+e\cos\theta} \text{ (}e\text{ being} < 1\text{)}, \int_0^{2a} x^n\sqrt{(2ax-x^2)}\,dx.$$

8. State between what limits the summation of $dx\,dy\,dz$ should be performed, in order to obtain the volume contained between the conical surface whose equation is $x^2 + y^2 = (a-z)^2$ and the planes whose equations are $x = z$, and $x = 0$; and find the volume by this or any other method.

9. Give a geometrical interpretation of the singular solution of a differential equation.

Investigate the singular solution of the equation

$$8y^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + 9y^2 - x^2 = 0,$$

and shew that it is the envelope of a series of circles described on the subnormal of a rectangular hyperbola as diameter.

10. Shew that the differential equation of all surfaces which are generated by a circle, whose plane is parallel to the plane of yz, and which passes through the axis of x and through two curves respectively in the planes of xz and xy, is

$$(y^2+z^2)\,t + 2\,(z-yq)\,(1+q^2) = 0.$$

11. Find the general functional equation to surfaces generated by the motion of a straight line which always intersects and is perpendicular to a given straight line.

If the surface, whose equation referred to rectangular co-ordinates, is

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + 1 = 0,$$

be capable of generation in this manner, shew that

$$a + b + c = 0, \qquad aa'^2 + bb'^2 + cc'^2 = 2a'b'c' + abc.$$

THURSDAY, *Jan.* 19. 9...12.

PROBLEMS.

1. Two circles of radii r, r', touch a straight line at the same point on opposite sides: a circle, of which the radius is R and of which the straight line is a chord, touches both the former circles. Prove that the length of the chord is equal to

$$\frac{4R}{\left(\frac{r}{r'}\right)^{\frac{1}{2}} + \left(\frac{r'}{r}\right)^{\frac{1}{2}}}.$$

2. Prove that, n being any positive integer, and e the base of Napier's logarithms,

$$e^n > \frac{(n+1)^n}{1.2.3\ldots\ldots n}.$$

3. From a focus S of a conic section $ARQPA$ three radii vectores SR, SQ, SP, are drawn, the angles PSQ, QSR, being invariable. Prove that the tangent at P intersects the chord RQ in a point of which the locus is another conic section.

Supposing e to be the eccentricity of the original conic section and e' of the conical locus, shew that, if $\angle RSQ = 2\alpha$, and $\angle QSP = \beta$,

$$\frac{e'^2}{e^2} = \frac{\sin^4\frac{\alpha}{2}}{\sin^2\frac{\alpha+\beta}{2}} + \frac{\cos^4\frac{\alpha}{2}}{\cos^2\frac{\alpha+\beta}{2}}.$$

4. Tangents PP', PP'', are drawn from a point P to touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

at points P', P''. Supposing the harmonic mean between the abscissæ of the points P', P'', to be equal to that between their ordinates, shew that the locus of P consists of four arcs of a curve of the third order.

Trace the curve and shew that, when $a = b$, the curve degenerates into a straight line and an ellipse.

5. The distances of the successive angular points of a plane polygon from a given point O within its area are given. Supposing the polygonal area to be the greatest possible, prove that, C_{r-1}, C_r, C_{r+1}, being any three consecutive angular points, no two of which are in a line with O, the line $C_{r-1}C_{r+1}$ is perpendicular to the distance OC_r.

6. A rectangular column is formed by placing a number of smooth cubical blocks one above another, the base of the column resting upon a horizontal plane. All the blocks above the lowest are then twisted in the same direction about an edge of the column, first the highest, then the two highest, and so on, in each case as far as is consistent with equilibrium. Prove that the sum of the sines of the inclinations of a diagonal of the base of any block to the like diagonals of the bases of all the blocks above it is equal to the sum of the cosines.

7. A uniform chain of length l hangs over two fixed points, which are in a horizontal line: from its middle point is suspended by one end another chain of equal thickness and of length l'. Supposing each of the two tangents of the former chain at its middle point to make an angle θ with the vertical, find the distance between the two fixed points.

Shew that the value of θ can never exceed that given by the equation

$$\tan^2\frac{\theta}{2} = \frac{l - l'}{l + l'}.$$

8. If $\dfrac{a^2x^2}{(v^2 - a^2)^2} + \dfrac{b^2y^2}{(v^2 - b^2)^2} = 1$, and if, for any assigned values of x and y, the expression

$$v^4 \cdot \left\{\frac{x^2}{(v^2 - a^2)^2} + \frac{y^2}{(v^2 - b^2)^2}\right\}$$

has only one value, prove that

$$a^2x^2 + b^2y^2 = 4\,(a^2 - b^2)^2.$$

9. A great circle of a sphere intersects two given great circles, drawn through a point O, in points A, B, such that the product of $\tan OA$, $\tan OB$, is invariable. If P be the intersection of this circle with the consecutive one of the series of circles described according to the same law, prove that $\cot^2 OP \propto \sin POA \,.\, \sin POB$.

10. Investigate an equation for the form of the floats in the paddle wheels of a steam vessel in order that they may enter the water without splashing.

If $u = h\omega$, where u = the velocity of the vessel, ω = the angular velocity of the wheels, and h = the height of the centres of the wheels above the water, shew that the floats of each wheel must have the forms of arcs of involutes of a concentric circle touching the water level.

11. A hollow vertical polygonal prism, open at both ends, rests upon a horizontal plane. Every two contiguous faces are moveable about their common edge. Supposing the prism to be in equilibrium, when filled with fluid, prove that

$$\frac{c_1}{\sin a_1} = \frac{c_2}{\sin a_2} = \frac{c_3}{\sin a_3} = \dots,$$

$a_1, a_2, a_3, \dots$ being the angles of a transverse section $A_1A_2A_3\dots A_nA_1$, and $c_1, c_2, c_3, \dots$ denoting the lines $A_nA_2, A_1A_3, A_2A_4, \dots$

Hence shew that there will be equilibrium when the points $A_1, A_2, A_3, \dots$ lie all in the circumference of a circle.

12. A filament of fluid oscillates in a thin cycloidal tube of uniform bore the axis of the cycloid being vertical and its vertex downwards. Supposing the filament to be placed initially with its lower end at the lowest point of the tube, find the pressure at any point of the filament at any time.

Shew that the pressure is a maximum, during the whole motion, at the middle point of the filament.

13. A ray experiences a series of reflections between two plane inclined mirrors. Prove that all the segments of the ray, produced indefinitely, are tangents to every one of an infinite series of spheres.

14. A narrow self-luminous rectangular lamina is placed with one end at the edge of a circular plate: the lamina is at right angles to the plate and its plane passes through the centre of the plate: find the whole illumination on the plate.

If the length of the lamina be equal to the diameter of the plate, its intrinsic brightness and breadth being given, prove that the illumination varies as the diameter of the plate.

15. Prove that an infinite number of plane centric sections of an hyperboloid of one sheet may be drawn, each possessing the following property, viz. that the normals to the surface at the curve of section all pass through two straight lines lying in the same plane with the two possible axes.

Shew that these centric planes envelope the asymptotic cone, while the two straight lines envelope an ellipse.

16. Prove that the envelope of a sphere, of which any one of one series of circular sections of an ellipsoid is a diametral plane, is a spheroid touching a sphere, described on the mean axis of the ellipsoid as diameter, in a plane perpendicular to any one of the same series of circular sections.

17. The Sun's centre, in proceeding from Aries to the Summer Solstice, passes, when at a distance ϕ from the Solstice, through the zenith of a certain place. Prove that, supposing the Earth's orbit circular and the plane of the equator invariable in position, it will not again pass exactly through the zenith of this place in moving from the Solstice to Libra, unless

$$\frac{\tan n\phi}{\tan\phi} = \sec\omega,$$

n denoting the ratio of the Earth's angular velocity about its axis to its angular velocity about the Sun.

18. Deternime $u_{x,t}$ from the equation

$$c^2 \frac{d^2}{dt^2} u_{x+2,t} = \Delta^2 u_{x,t},$$

where Δ affects x only; and, having given the expressions for $u_{x,0}$, $\frac{d}{d_0} u_{x,0}$, shew how to determine the values of the arbitrary functions which appear in the result.

If $u_{x,0} = ax + b$ and $\frac{d}{d_0} u_{x,0} = a'r^x$, shew from your formulæ that

$$\frac{d}{dt} u_{x,t} = \tfrac{1}{2}a' . r^x . (\mu^t + \mu^{-t}),$$

a', r, μ, being constant quantities.

19. Determine the differential equation to a family of curves which possess the following property: if we take in one of the curves any three points P, P', P'', so related that C', C'', the centres of curvature at P', P'', lie respectively in the ordinates PM, $P'M'$, produced if necessary, the ratio of $M'M''$ to MM' shall be invariable.

Shew from your result that the elastica, the equation to which is

$$dy = \frac{x^2 dx}{(c^4 - x^4)^{\frac{1}{2}}},$$

is an individual of the family.

20. A small heavy insect, placed at an end of the horizontal diameter of a thin heavy motionless ring, which is moveable about its centre in a vertical plane, starts off to crawl round the ring so as to describe in space equal angles in equal times about its centre. Determine its velocity relatively to the ring in any position.

21. A series of perfectly rough semicylinders are fixed, side by side, upon their flat faces directly across a straight road of constant inclination. Determine the inclination of the road in order that a rough circular inelastic hoop, just started downwards from the summit of one of the cylindrical ridges, may travel directly along the road with a uniform mean velocity.

22. A brittle rod AB, attached to smooth fixed hinges at A and B, is attracted towards a centre of force C acording to the law of nature.

Supposing the absolute force to be indefinitely augmented, prove that the rod will eventually snap at a point E, the position of which is defined by the equation

$$\cos \angle AEC = \frac{\sin \frac{\alpha - \beta}{2}}{\sin \frac{\alpha + \beta}{2}},$$

where α, β, denote the angles BAC, ABC, respectively.

23. A vessel, of given capacity, in the form of a surface of revolution with two circular ends, is just filled with inelastic fluid which revolves about the axis of the vessel, and is supposed to be free from the action of gravity: investigate the form of the vessel that the whole pressure which the fluid exerts upon it may be the least possible, the magnitudes of the circular ends being given.

Shew that, for a certain relation between the radii of the circular ends, the generating curve of the surface of revolution is the common catenary.

24. If α, β, γ, be the direction-cosines of one of the two lines of vibration of the plane front of a wave in a biaxal crystal, and α', β', γ', those of either of the two lines of vibration of a plane front intersecting the former plane front at right angles and passing through the line (α, β, γ), prove that

$$\frac{\alpha'}{\alpha}(b^2 - c^2) + \frac{\beta'}{\beta}(c^2 - a^2) + \frac{\gamma'}{\gamma}(a^2 - b^2) = 0,$$

and that $$\frac{(b^2 - c^2)^2}{\alpha\alpha'} + \frac{(c^2 - a^2)^2}{\beta\beta'} + \frac{(a^2 - b^2)^2}{\gamma\gamma'} = 0.$$

THURSDAY, *Jan.* 19. $1\frac{1}{2}$...4.

1. If a and b be two numbers prime to each other, shew that, when a, $2a$, $3a$, ... $(b-1)a$ are divided by b, the remainders are all different from each other; and shew that there is an infinite number of positive integral solutions of the equation $ax - by = c$, when a and b are prime to each other, and c is a whole number.

Shew that, if m and n are prime to each other, the equations $x^m - 1 = 0$ and $x^n - 1 = 0$ have no common root but unity.

2. Shew that the area of a spherical triangle varies as the excess of the sum of its angles above two right angles; and prove Lhuillier's theorem,

$$\tan \frac{E}{4} = \sqrt{\left(\tan \frac{s}{2} \tan \frac{s-a}{2} \tan \frac{s-b}{2} \tan \frac{s-c}{2}\right)}.$$

3. If straight lines, represented by $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$, taken in order, form a quadrilateral, and a, b, c, d, be such that $au_1 + bu_2 + cu_3 + du_4$ vanishes for all values of x and y, shew that the curve of the second order, represented by the equation $\lambda u_1 u_2 + \mu u_3 u_4 = 0$, circumscribes the quadrilateral, and that $\lambda bu_2 = \mu au_4$ represents a tangent to the curve.

4. The variable parameter in an equation $u = 0$ to a family of curves being represented by a, prove that, if there be a cusp in their envelope, its coordinates will satisfy the three equations

$$u = 0, \quad \frac{du}{da} = 0, \quad \frac{d^2u}{da^2} = 0.$$

Apply this theorem to find the cusps of the curve which envelopes the family of lines represented by the equation

$$\frac{x}{\cos a} + \frac{y}{\sin a} = c.$$

5. Integrate the differential equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{y}{a^2} = 0.$$

Obtain a general expression for $\psi(x)$ from the equation

$$\psi(x) + \psi(1 - x) = c.$$

6. A lamina, in the form of a semi-ellipse bounded by the axis minor, is moveable about the centre as a fixed point, and falls from the position in which its plane is horizontal; find the pressure on the fixed point for any position of the lamina, and determine the impulse which must be applied at the centre of gravity, when the lamina is vertical, in order to reduce it to rest.

If this force be applied perpendicularly to the lamina at the extremity of an ordinate through the centre of gravity, instead of being applied at the centre of gravity itself, about what axis will the lamina begin to revolve?

7. Enunciate and prove the principle of Vis Viva, shewing that it will not be true unless the expression

$$\Sigma m(Xdx + Ydy + Zdz)$$

is a perfect differential or zero.

Describe the nature of those forces which disappear from this expression, and of those which render it a perfect differential. What kind of forces would render it not a perfect differential?

A thin uniform smooth tube is balancing horizontally about its middle point, which is fixed; a uniform rod, such as just to fit the bore of the tube, is placed end to end in a line with the tube, and then shot into it with such a horizontal velocity that its middle point shall only just reach that of the tube: supposing the velocity of projection to be known, find the angular velocity of the tube and rod at the moment of the coincidence of their middle points.

8. Investigate the differential equation for the Moon's latitude.

What are the points which require particular attention in obtaining approximately the Moon's latitude in terms of her longitude?

9. Prove that, in the planetary theory,

$$\frac{dR}{d\theta} = \frac{dR}{d\varpi} + \frac{dR}{d\epsilon}.$$

For what purpose is this transformation made?

10. In the shutter of a dark chamber there is a small rectangular aperture, covered symmetrically by a convex lens; homogeneous light diverges upon the lens from such an external point in its axis that after refraction its geometrical focus lies in the opposite wall of the chamber; investigate the character of the bands formed on the wall in the neighbourhood of the geometrical focus.

State the dynamical principle in virtue of which you are at liberty to adopt the method of summation, as employed in this and similar problems.

FRIDAY, *Jan.* 20. 9...12.

1. The position of a point in space being determined by the polar coordinates $r'\theta'\phi'$, where θ' is the angle through which r' has revolved, from a fixed line Oz, in a plane which has revolved through an angle ϕ' from a fixed plane zOx; shew that the equation to the tangent plane at a point $r\theta\phi$ of a surface is

$$\frac{r^2}{r'} = \frac{d}{d\theta}\left[r\{\sin\theta\cos\theta' - \sin\theta'\cos\theta\cos(\phi-\phi')\}\right] + \frac{\sin\theta'\sin(\phi-\phi')}{\sin\theta}\frac{dr}{d\phi}.$$

2. If x be an integer, shew that

$$\Sigma_1^\infty\left(\frac{1}{x^{2n}}\right) \text{ is equal to } \frac{2^{2n-1}B_{2n-1}\pi^{2n}}{1.2.3\ldots 2n},$$

B_{2n-1} being the n^{th} of Bernoulli's numbers.

Shew, by Bernoulli's numbers or otherwise, that

$$\frac{1^2}{1^2+1}\cdot\frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\ldots\ldots \textit{ad inf.} = \frac{2\pi}{e^\pi - e^{-\pi}}.$$

3. Define the terms convergent and divergent when applied to a series of quantities real or imaginary.

Investigate a rule which is ordinarily sufficient to ascertain whether a series is or is not convergent.

Are the following series convergent?

$$\frac{3}{2}x + \frac{5}{5}x^2 + \frac{7}{10}x^3 + \frac{9}{17}x^4 + \ldots + \frac{2n+1}{n^2+1}x^n + \ldots, \text{ where } x \text{ is real;}$$

$$1 + x\cos a + x^2\cos 2a + \&c., \text{ where } x \text{ is real or imaginary.}$$

4. If $f(x)$ be finite for all values of x between l and $-l$, prove that, whatever be the form of the function, the following equation holds for all values of x included between these limits,

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(x')\,dx' + \frac{1}{l}\,\Sigma\left\{\int_{-l}^{l}\cos\frac{n\pi(x-x')}{l}f(x')\,dx'\right\},$$

and deduce the formula,

$$f(x) - \frac{1}{\pi}\int_0^\infty da \left\{\int_{-\infty}^\infty dx' \cos\{a(x-x')\} f(x')\right\}.$$

5. What is meant by the potential of an attracting mass with respect to any point? If V be the potential with respect to a point whose co-ordinates are a, b, c, shew that

$$\frac{d^2V}{da^2} + \frac{d^2V}{db^2} + \frac{d^2V}{dc^2} = 0, \quad \text{or} \quad -4\pi\rho,$$

according as the point is extraneous, or forms part of the attracting mass, ρ in the latter case being the density of the mass at the point a, b, c.

A uniform circular lamina attracts a point situated in a line drawn perpendicularly to it through its centre; shew that

$$V = 2\pi\{\sqrt{(x^2 + a^2)} - x\},$$

a being the radius of the lamina, and x the distance of the point from it; and deduce the resultant attraction exerted by the lamina upon the point.

6. In the Planetary Theory, when the disturbing function is developed preparatory to the determination of the perturbations in longitude and radius vector, shew that $p \sim q$ is the order of the principal term in which $pn - qn'$ is the coefficient of t; assuming that this law holds for u, u', and for powers and products of powers of u and u'.

What terms must be reserved for examination as likely to be of importance?

7. Solve the differential equation for the vibratory motion of the air contained in an indefinite cylindrical tube; and shew that when such motion is produced by a vibrating plate placed at one end of a finite tube, of which the other end is open, if the period of vibration have a certain relation to the length of the tube, it is possible for the character of the vibrations to remain permanently the same.

If such a tube be sounding its fundamental note, what would be the effect of making a small aperture in the side of the tube, first at its middle point, secondly a little nearer to the open end?

8. Find the difference of retardation of the two waves produced by a thin lamina cut from a uniaxal crystal perpendicular to its axis, when a ray of common light is incident nearly parallel to the axis: describe the rings produced by interposing such a lamina between a polarizing and an analyzing plate, the planes of incidence at the two plates being inclined at an angle of 45° to each other.

If two such laminæ, one cut from a positive and the other from a negative uniaxal crystal, be placed together and interposed, what must be the ratio of their thicknesses in order that neither rings nor brushes may be visible?

FRIDAY, *Jan.* 20, $1\frac{1}{2}...4$.

1. IF $f(p, q, r, s, ...) = 0$, where $p, q, r, s, ...$ are the distances of any point in a curve from fixed points in its plane, or of any point in a surface from fixed points, and if a set of forces proportional to $f'(p), f'(q)...$ act on the point, along the distances $p, q, r ...$, prove that their resultant acts along the normal at that point.

If $\sin\lambda : \sin\mu :: p^n : q^n$, where λ, μ, are the respective inclinations of p, q to the normal at any point of the curve $f(p, q) = 0$, prove that, c being a constant, $$p^{1-n} + q^{1-n} = c^{1-n}.$$

2. Having given the following simultaneous differential equations,

$$\frac{d^2x}{dt^2} = \frac{dR}{dx}, \quad \frac{d^2y}{dt^2} = \frac{dR}{dy}, ...$$

$$\text{where} \quad R = f(r), \quad r^2 = x^2 + y^2 + ...;$$

prove that

$$t = \int \frac{rdr}{\sqrt{\{r^2(2R + B) - A^2\}}}, \ A, B \text{ being arbitrary constants.}$$

Integrate the partial differential equation

$$q(1+q)r - (p+q+2pq)s + p(1+p)t = 0.$$

3. Prove that the radius of curvature of an oblique section, at any point of a surface, coincides with the projection, upon the plane of the section, of the radius of curvature of the normal section through the same tangent line.

An annular surface is generated by the revolution of a circle about an axis in its own plane; prove that one of the principal radii of curvature, at any point of the surface, varies as the ratio of the distance of this point from the axis to its distance from the cylindrical surface described about the axis and passing through the centre of the circle.

4. Give sufficient equations for calculating the motion of a right cone placed upon a perfectly rough inclined plane; and find the moment of the couple exerted by friction on the cone.

Shew that the length of the simple isochronous pendulum, when the cone oscillates about the lowest position, is

$$\frac{4k^2}{3r \sin a \sin\beta},$$

$2a$ being the angle of the cone, r the radius of its base, β the inclination of the plane, and k the radius of gyration round a generating line.

5. If $u = \int Vdx$ has a maximum or minimum value, prove that

$$N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - ... = 0.$$

How must this equation be modified when the result of some given operation performed upon the variables and their extreme values is given?

The form of a homogeneous solid of revolution, of given superficial area, and described upon an axis of given length, is such that its moment of inertia about the axis is a maximum: prove that the normal at any point of the generating curve is three times as long as the radius of curvature.

6. Distinguish between secular and periodic variations. Are secular variations ever periodic?

The equations which connect the inclination and the longitude of the nodes of the orbits, in the case of Jupiter and Saturn, are of the form

$$\tan i \sin\Omega = G\sin(at+\gamma) + H\sin\delta,$$
$$\tan i \cos\Omega = G\cos(at+\gamma) + H\cos\delta.$$

For both orbits,

$$a = -25''.5756,\quad \gamma = 125^\circ\ 15'\ 40'',\quad \delta = 103^\circ\ 38'\ 40'',\quad \text{and } H = .02905,$$
$$G = -.00661 \text{ for Jupiter, and } = .01537 \text{ for Saturn,}$$

t being reckoned from A. D. 1700.

Prove the following circumstances of motion, that Jupiter's node regresses and Saturn's progresses from a longitude $\delta + \epsilon - \frac{1}{2}\pi$ through the angle $2\epsilon \sim \pi$ in the time $\dfrac{2\epsilon}{-a}$, where ϵ is for each planet the least positive angle which satisfies the equation $G = H\cos\epsilon$; that they arrive simultaneously at their mean position; and that in this position Jupiter's orbit has its maximum and Saturn's its minimum inclination.

7. Assuming that the angular accelerating force, exerted by the Sun on the Earth, about a diameter of the Earth's equator at right angles to the line joining the centres of the Earth and Sun, varies as $\sin SP \cos SP$, where P is the Earth's pole, and S the Sun's centre; investigate the solar precession of the equinoxes.

8. Draw the course of a small pencil of parallel rays, passing at such an angle through a biaxal crystal cut with parallel faces, that external cylindrical refraction takes place.

How may the constants a, b, c corresponding to the axes of elasticity be obtained experimentally?

If the two faces of a prism, formed of a biaxal crystal, be perpendicular to each other, and one contain the two axes of elasticity a, c, and the other b, c; and if μ_a, μ_b be two refractive indices for the ordinary ray when the planes of refraction are perpendicular to the axes a and b respectively; shew that D, the minimum deviation of the extraordinary ray, is given by the equation

$$\sin^2 D = (\mu_a^2 - 1)(\mu_b^2 - 1).$$

THE END.

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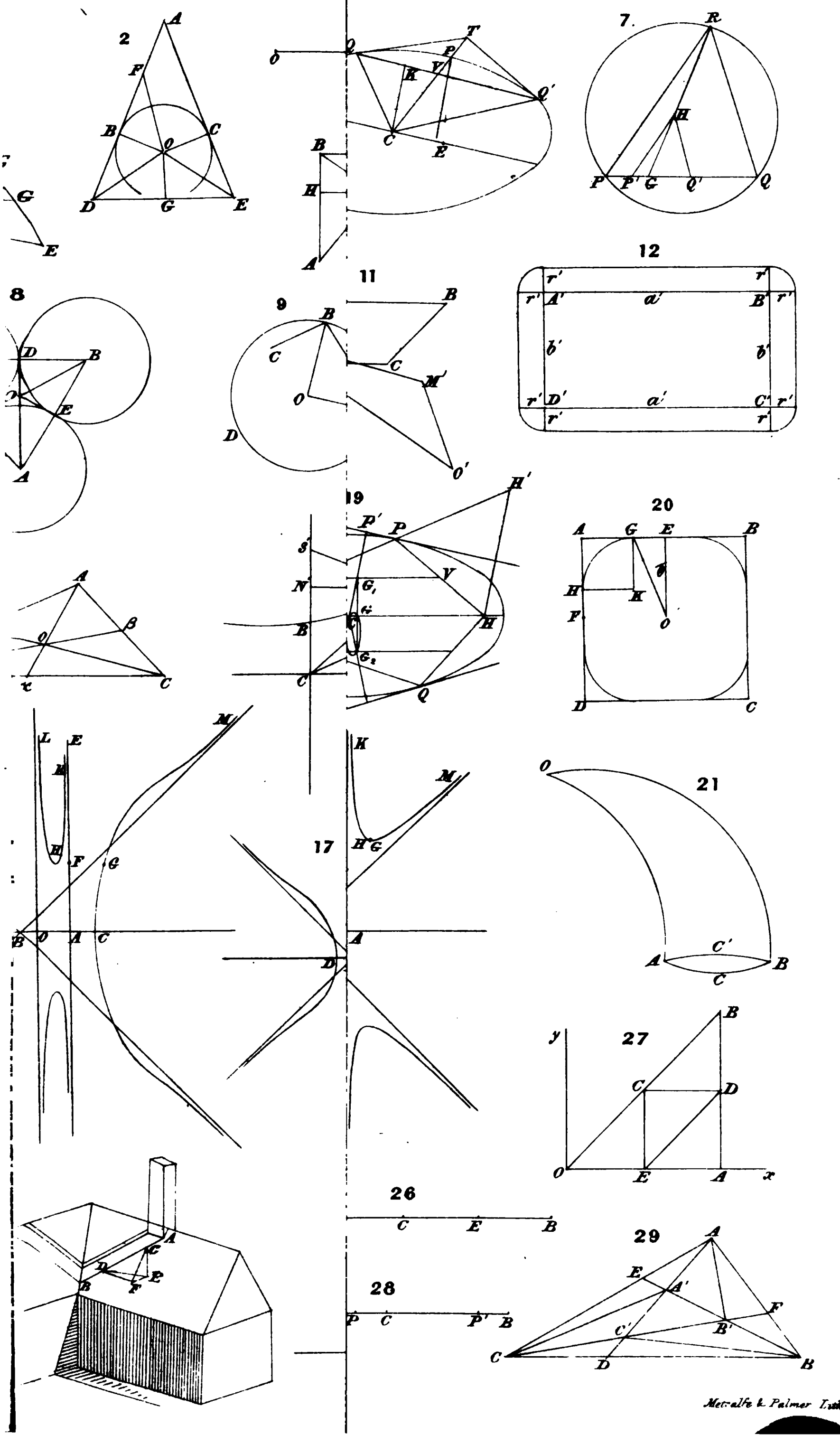

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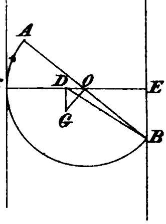

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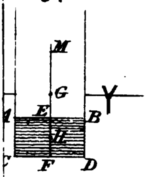

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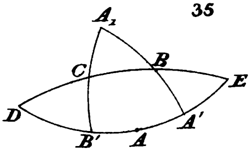

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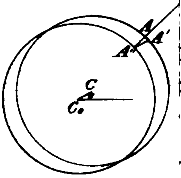

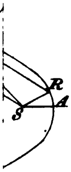

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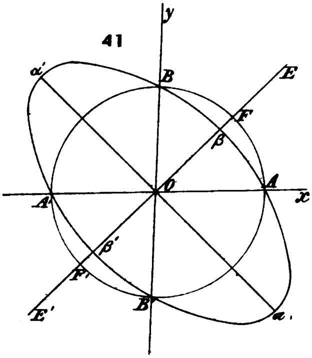

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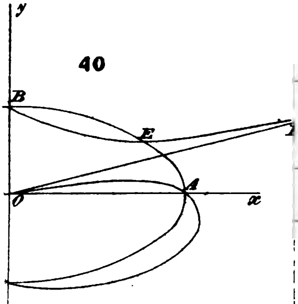

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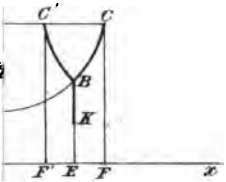

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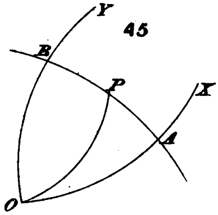

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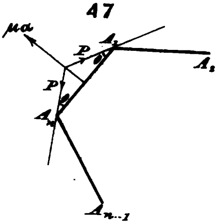

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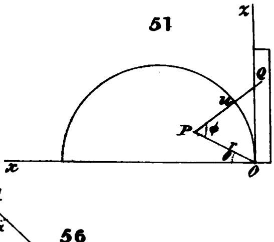

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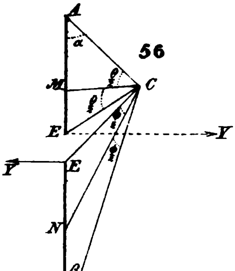

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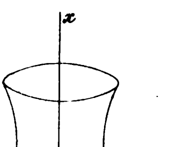

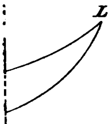

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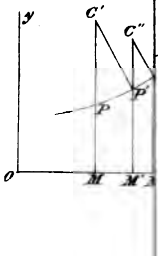

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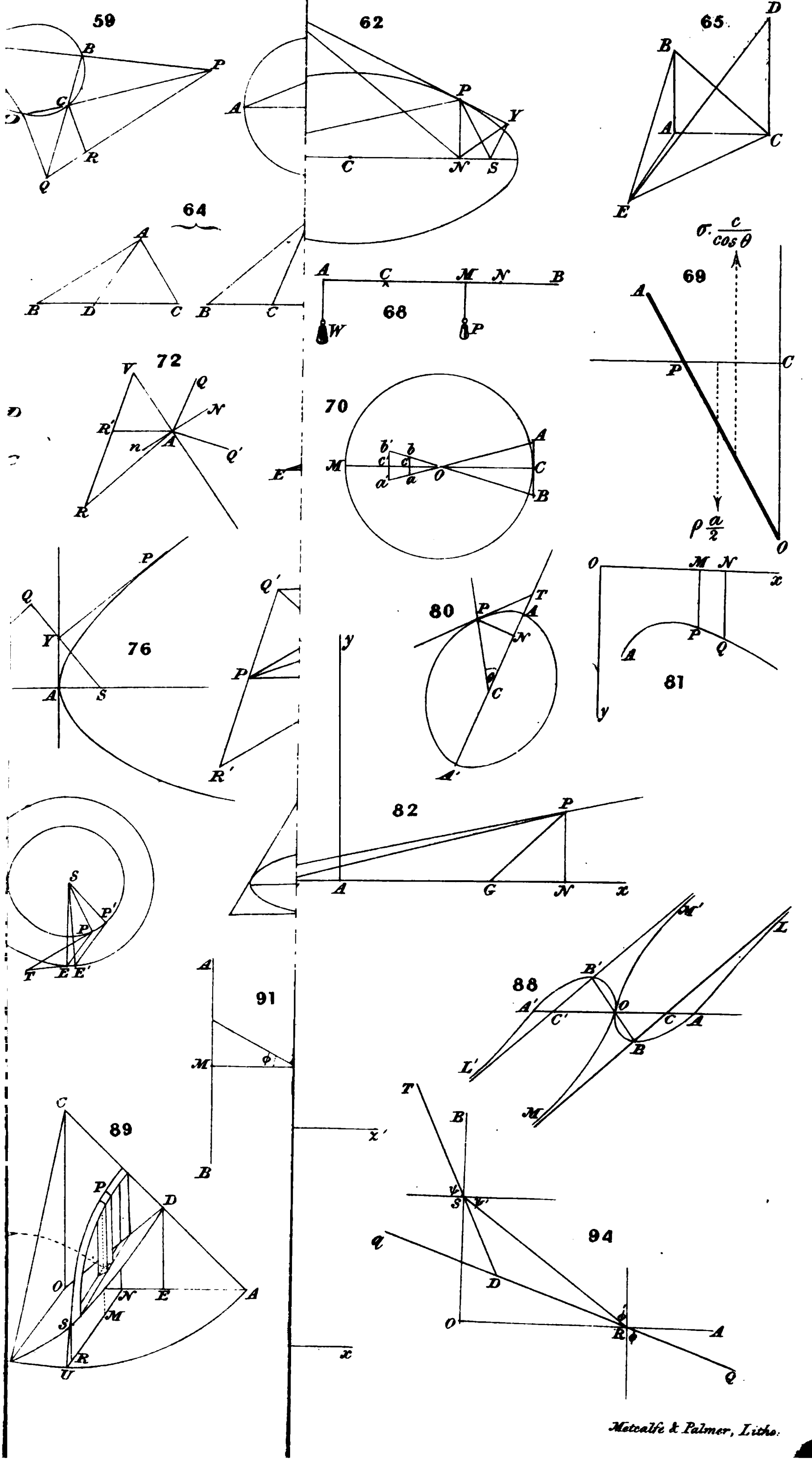

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