SOLUTIONS

OF

THE PROBLEMS AND RIDERS

PROPOSED

IN THE SENATE-HOUSE EXAMINATION

FOR 1857.

BY

THE MODERATORS AND EXAMINERS.

Cambridge (Eng.) – University.

WITH AN APPENDIX

CONTAINING THE EXAMINATION PAPERS IN FULL.

"It is good to vary and intermingle asking of questions with telling of opinions."
BACON.

Cambridge:
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1857.

PREFACE.

ONE of the purposes for which the Board of Mathematical Studies was established, was to communicate to Students "correct views of the nature and objects of the Mathematical Examination;" and, in furtherance of this purpose, the Board has, from time to time, defined more strictly the limits of the subjects of examination, and informed the Students of the nature of the work which was required from them.

The Moderators and Examiners conceive that they are further promoting this object by giving to future Candidates for Mathematical Honors an opportunity of examining the Solutions of the Problems and Riders proposed at the last Examination as prepared by the Proposers themselves, and thus of ascertaining still more clearly the nature of the examination to which our Mathematical Students have been ordinarily subjected.

Cambridge, May 6th, 1857.

SOLUTIONS OF SENATE-HOUSE PROBLEMS AND RIDERS

FOR THE YEAR EIGHTEEN HUNDRED AND FIFTY-SEVEN.

THURSDAY, *Jan.* 8, 1857. 1 *to* 4.

1. THREE circles, A, B, C, (fig. 1) intersect in a common point, the other intersections of (B, C), (C, A), (A, B), being α, β, γ, respectively. If b, c, be points in B, C, respectively, such that b, α, c, lie in a straight line, prove that a, the intersection of $b\gamma$, $c\beta$, produced, lies in the circle A.

Since the sum of the angles of a triangle is equal to two right angles, we have

$$a + b + c = \pi;$$

and, since two opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles, we have also, I being the common point of intersection of the three circles,

$$b + \gamma I\alpha = \pi, \quad c + \alpha I\beta = \pi.$$

From the above three equations we see that

$$\gamma I\alpha + \alpha I\beta = \pi + a:$$

but $$\beta I\gamma + \gamma I\alpha + \alpha I\beta = 2\pi:$$

hence $$\beta I\gamma + a = \pi,$$

and therefore the point a must lie in the circle A.

This theorem was communicated to Liouville's *Journal de Mathématiques* by M. A. Miquel; Tome Troisième, année 1838.

2. Shew that the sum of all the harmonic means, which can be inserted between all the pairs of numbers the sum of which is n, is

$$\tfrac{1}{3}(n^2-1).$$

Let x be one number: then $n-x$ will be the other number of the pair; and, if H be the harmonic mean between them,

$$H=\frac{2x(n-x)}{n};$$

therefore,
$$\Sigma H=\frac{2}{n}\Sigma(nx-x^2)$$
$$=\frac{2}{n}\left\{n[1+2+\ldots+(n-1)]-[1^2+2^2+\ldots+(n-1)^2]\right\}$$
$$=\frac{2}{n}\left\{n\cdot\frac{n(n-1)}{2}-\frac{(n-1)n(2n-1)}{1.2.3}\right\}$$
$$=\frac{n-1}{3}\{3n-2n+1\}$$
$$=\tfrac{1}{3}(n^2-1).$$

3. Eliminate θ between the equations

$$\frac{x}{a}=\cos\theta+\cos 2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{y}{b}=\sin\theta+\sin 2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Squaring and adding the equations we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2+2\cos\theta;$$

and, from (1),
$$\frac{x}{a}+1=\cos\theta(1+2\cos\theta)$$
$$=\frac{1}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

Hence
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-3\right)-2\frac{x}{a}=0.$$

4. From a point on the side of a hill of constant inclination the angle of elevation of the top of an obelisk on its summit is observed to be α, (fig. 2), and, a feet nearer to the top of the hill, to be β; shew that, if h be the height of the obelisk, the inclination of the hill to the horizon will be

$$\cos^{-1}\left\{\frac{a}{h}\cdot\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}\right\}.$$

Let x be the distance from the second place of observation to the top of the obelisk:

then $$a\frac{\sin\alpha}{\sin(\beta-\alpha)}=x=h\,.\,\frac{\cos\theta}{\sin\beta}:$$

therefore $$\cos\theta=\frac{a}{h}\cdot\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)},$$

and $$\theta=\cos^{-1}\left\{\frac{a}{h}\cdot\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}\right\}.$$

5. Each of three circles, within the area of a triangle, touches the other two, touching also two sides of the triangle: if a be the distance between the points of contact of one of the sides, and b, c, be like distances on the other two sides, prove that the area of the triangle, of which the centres of the circles are the angular points, is equal to

$$\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}.$$

Let a', b', c', be the radii of the three circles, their respective centres being A, B, C.

Then, (fig. 3), $b'+c'$ being the hypotenuse of a right-angled triangle, one side of which is $b'-c'$ and the other is equal to a,

$$a^2=(b'+c')^2-(b'-c')^2=4b'c':$$

similarly $$b^2=4c'a',\quad c^2=4a'b'.$$

Hence $$\frac{bc}{a}=2a',\quad \frac{ca}{b}=2b',\quad \frac{ab}{c}=2c'.$$

Let $$BC=a_1,\quad CA=b_1,\quad AB=c_1.$$

Then $$a_1=b'+c',\quad b_1=c'+a',\quad c_1=a'+b',$$

and consequently,

$$a_1 + b_1 + c_1 = 2(a' + b' + c'),$$
$$b_1 + c_1 - a_1 = 2a',$$
$$c_1 + a_1 - b_1 = 2b',$$
$$a_1 + b_1 - c_1 = 2c'.$$

Hence the area of the triangle ABC is equal to

$$\{a'b'c'(a' + b' + c')\}^{\frac{1}{2}},$$
$$= \tfrac{1}{4}\left\{abc\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right)\right\}^{\frac{1}{2}},$$
$$= \tfrac{1}{4}(b^2c^2 + c^2a^2 + a^2b^2)^{\frac{1}{2}}.$$

6. The acute angles, which the distances of two points of an ellipse from the same focus make with the respective tangents at the points, are complementary to each other: prove that the square on the semi-axis minor is a mean proportional between the areas of the two triangles, of which the two points are the respective vertices, and the distance between the foci the common base.

Shew that the problem is impossible unless the axis minor is less than the distance between the foci.

Let P, P', (fig. 4) be the two points, and ϕ, ϕ', the inclinations of the focal distances of P, P', respectively, to the tangents at P, P'. Let A, A', be the areas of SPH, $SP'H$, respectively.

Then $$A = \tfrac{1}{2}SP.HP.\sin 2\phi$$
$$= \frac{SY.HZ.\sin 2\phi}{2\sin^2\phi} = BC^2.\cot\phi.$$

Similarly, putting A' for A and $\frac{1}{2}\pi - \phi$ for ϕ, we have

$$A' = BC^2.\tan\phi.$$

Hence $$A : BC^2 :: BC^2 : A'.$$

The greatest value of either A or A' is $\frac{1}{2}SH.BC$: hence, that the problem may be possible,

$$BC^4 < \tfrac{1}{4}SH^2.BC^2$$

or $$2BC < SH.$$

7. CP, CD, (fig. 5), are two conjugate semi-diameters of an ellipse: Rr is a tangent parallel to PD: a straight line CIJ cuts at a given angle PD, Rr, in I, J, respectively: prove that the loci of I, J, are similar curves.

Let the tangents at P, D, meet in a point T: join CT, cutting Rr and the ellipse in Q and the chord PD in N. Then, by a property of the ellipse, $PCDT$ is a parallelogram.

By similar triangles,

$$CI : CJ :: CN : CQ \text{.....................(1)}:$$

also, by a property of the ellipse,

$$CN : CQ :: CQ : CT,$$

and therefore, since $CT = 2\,CN$,

$$2\,CN^2 = CQ^2 \text{....................... (2)}.$$

By (1) and (2), we see that

$$CI^2 : CJ^2 :: CN^2 : CQ^2$$
$$:: \;1\; : \;2\;;$$

hence CI is to CJ in a constant ratio, and therefore the loci of I, J, are similar curves.

8. A fine string $ACBP$, (fig. 6), tied to the end A of a uniform rod AB of weight W, passes through a fixed ring at C, and also through a ring at the end B of the rod, the free end of the string supporting a weight P; if the system be in equilibrium, prove that

$$AC : BC :: 2P + W : W.$$

The equilibrium will not be affected if we suppose the weight P to be placed at B. Let G, on this supposition, be the centre of gravity of W and P: then, $2a$ being the length of AB,

$$AG = a.\frac{2P+W}{P+W}, \quad BG = a.\frac{W}{P+W}.$$

The system being in equilibrium, CG must be vertical. Since the forces along AC, BC, are each equal to P, they must make equal angles with the vertical line CG, or their resultant would not, as is necessary for equilibrium, act in the direction GC. Hence

$$AC : BC :: AG : BG :: 2P + W : W.$$

9. A picture is hung up against a rough vertical wall by a string fastened to a point in its back, so that the picture inclines forwards: apply the principle of the triangle of forces to find the inclination of the string to the wall, when its tension is the least possible.

Let $W =$ the weight of the picture, (fig. 7), $R =$ the horizontal reaction of the wall, $F =$ the friction, which acts vertically upwards, $T =$ the tension of the string. Also, let θ be the angle at which the string is inclined to the wall, and α be the angle which S, the resultant of F and R, makes with the wall.

Since there is equilibrium, S, T, and W, pass through the same point; and, by the principle of the triangle of forces,

$$\frac{T}{W} = \frac{\sin\alpha}{\sin(\alpha+\theta)},$$

α being such an angle that

$$\cot\alpha = \frac{F}{R}.$$

Hence T will be least when $\sin(\alpha+\theta)$ is greatest and $\sin\alpha$ least. But $\sin(\alpha+\theta)$ is greatest when $\alpha+\theta = \frac{1}{2}\pi$; and $\sin\alpha$ is least when the ratio $\frac{F}{R}$ is greatest, or when the picture is on the point of sliding. Hence, if $\tan\varepsilon$ be the coefficient of friction between the wall and the picture, we shall have

$$\alpha = \tfrac{1}{2}\pi - \varepsilon,\ \ \theta = \varepsilon, \text{ and } T = \tfrac{1}{2}W.\sec\varepsilon,$$

when T has its least value.

10. A lamina, cut into the form of an equilateral triangle, is hung up against a smooth vertical wall by means of a string attached to the middle point of one side, so as to have a corner in contact with the wall; shew that, when there is equilibrium,

the reaction of the wall and the tension of the string are independent of the length of the string, and that, if the string exceed a certain limit, equilibrium in such a position is impossible.

Let ABC (fig. 8) be the lamina, having the corner B in contact with the wall, AD the perpendicular from A on BC, ED the sustaining string, G the centre of gravity of ABC. Then, if W be the weight of the lamina, T the tension of the string, and R the reaction of the wall, the directions of T, W, R must pass through a point.

Let F be the point through which they pass.

Since, in the quadrilateral $BDGF$, each of the angles BDG, BFG, is a right angle, a circle may be described about it. Therefore the angle $BFD =$ the angle $BGD = 60°$, and the string DE is inclined at an angle of $30°$ to the wall, whatever be its length. Hence, from the triangle EBF, which has its sides parallel to the directions of the forces T, W, R,

$$\frac{T}{W} = \sec 30° = \frac{2}{\sqrt{3}}, \quad \frac{R}{W} = \tan 30° = \frac{1}{\sqrt{3}};$$

also $\dfrac{ED}{BD} = \dfrac{\sin EBD}{\sin 30°}$; and therefore $ED = 2BD \sin EBD$.

Now the greatest value of $\sin EBD$ is 1; and therefore the greatest length of string, which is consistent with equilibrium, is the length of a side of the triangle.

11. A ball is projected from the middle point of one side of a billiard table, so as to strike in succession one of the sides adjacent to it, the side opposite to it, and a ball placed in the centre of the table; shew that, if a and b be the lengths of the sides of the table, and e the elasticity of the ball, the inclination of the direction of projection to the side a of the table from which it is projected must be

$$\tan^{-1}\left\{\frac{b}{a} \cdot \frac{1+2e}{1+e}\right\}.$$

Let V be the velocity of projection, and θ the required inclination. Then the

velocity perpendicular to the side from which the ball starts $= V\sin\theta$;

.... parallel .... $= V\cos\theta$.

Now, since $V\sin\theta$ is not affected by the first impact, and is altered in the ratio of e to 1 by the second, we have, if T be the time till the ball is struck,

$$T = \frac{b}{V\sin\theta} + \frac{\frac{1}{2}b}{eV\sin\theta} = \frac{b}{V\sin\theta}\left(1 + \frac{1}{2e}\right) \ldots\ldots (1).$$

Also, since $V\cos\theta$ is not affected by the second impact, and is altered in the ratio of e to 1 by the first, we have

$$T = \frac{\frac{1}{2}a}{V\cos\theta} + \frac{\frac{1}{2}a}{eV\cos\theta} = \frac{a}{V\cos\theta}\left(\frac{1}{2} + \frac{1}{2e}\right) \ldots\ldots (2).$$

Hence, equating (1) and (2),

$$\tan\theta = \frac{b}{a}\,\frac{1+2e}{1+e},$$

and

$$\theta = \tan^{-1}\left\{\frac{b}{a}\,\frac{1+2e}{1+e}\right\}.$$

12. A perfectly elastic ball is projected at an inclination β to a plane inclined to the horizon at an angle α, so as to ascend it by bounds; find the inclination to the plane at which the ball rises at the n^{th} rebound, and shew that it will rise vertically if $\cot\beta = (2n+1)\tan\alpha$.

By the second law of motion we may consider separately the motion of the ball parallel and perpendicular to the plane.

Hence, if T be the time till the n^{th} rebound,

$$T = \frac{2V\sin\beta}{g\cos\alpha} + \frac{2V\sin\beta}{g\cos\alpha} + \ldots\ldots \text{ to } n \text{ terms}$$

$$= n.\frac{2V\sin\beta}{g\cos\alpha};$$

and the velocity along the plane at the n^{th} rebound

$$= V\cos\beta - g\sin\alpha . T$$

$$= V\cos\beta - \tan\alpha . 2nV\sin\beta.$$

Hence, if θ be the inclination of the ball's path to the plane at the n^{th} rebound,

$$\tan\theta = \frac{V\sin\beta}{V\cos\beta - 2nV\sin\beta.\tan\alpha}$$

$$= \frac{\tan\beta}{1-2n\tan\alpha.\tan\beta}.$$

If the ball rises vertically, $\theta = \frac{1}{2}\pi - \alpha$, and

$$\cot\alpha = \frac{\tan\beta}{1-2n\tan\alpha.\tan\beta}.$$

Hence $$\cot\alpha - 2n\tan\beta = \tan\beta,$$

and $$\cot\beta = (2n+1)\tan\alpha.$$

13. A string, charged with $n+m+1$ equal weights fixed at equal intervals along it, and which would rest on a smooth inclined plane, with m of the weights hanging over the top, is placed on the plane with the $(m+1)^{\text{th}}$ weight just over the top; shew that, if a be the distance between each two adjacent weights, the velocity which the string will have acquired, at the instant the last weight slips off the plane, will be

$$\{nag\}^{\frac{1}{2}}.$$

Since there is equilibrium, when m weights hang over the top of the plane, we have, α being the inclination of the plane to the horizon,

$$(n+1)g\sin\alpha = mg,$$

and therefore $$\sin\alpha = \frac{m}{n+1}.$$

Suppose the $(m+r)^{\text{th}}$ weight has just passed over the top of the plane with the velocity v_r: the acceleration of the string will be

$$\frac{(m+r)-(n-r+1)\sin\alpha}{m+n+1}g$$

$$= \frac{(m+r)(n+1)-(n+1-r)m}{(m+n+1)(n+1)}g$$

$$= \frac{r}{n+1}g.$$

Hence, if v_{r+1} be the velocity of the string when the $(m+r+1)^{\text{th}}$ weight has just passed the top,

$$v^2_{r+1} = v^2_r + 2ag\,\frac{r}{n+1}.$$

And, giving to r the values 1, 2, 3 ...... n, successively, and observing that $v_1 = 0$, we have the equations

$$v_2^2 = 2ag\,\frac{1}{n+1},$$

$$v_3^2 - v_2^2 = 2ag\,\frac{2}{n+1},$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$v^2_{n+1} - v_n^2 = 2ag\,\frac{n}{n+1};$$

adding
$$v^2_{n+1} = 2ag\,\frac{1+2+\ldots+n}{(n+1)}$$
$$= nag;$$

and therefore
$$v_{n+1} = (nag)^{\frac{1}{2}}.$$

Rem. The preceding solution requires that the velocity of each weight should not be altered in passing over the top of the plane, and that the weight should not shoot off the plane. These conditions will be satisfied, if we suppose the string to pass through an indefinitely short circular tube at the top of the plane, the curvature of which is such that the tangents at the extremities of its axis are in the directions of the ascending and descending portions of the string.

14. A perfectly elastic ball is projected with a given velocity from a point between two parallel walls, and returns to the point of projection, after being once reflected at each wall; prove that its angle of projection is either of two complementary angles.

Let V be the velocity and θ the angle of projection, t the time of flight, a the distance between the walls.

Then, the vertical movement not being affected by the horizontal impacts,

$$2V\sin\theta = gt.$$

Also, the elasticity being perfect, the magnitude of the whole horizontal motion is the same as if the walls had not existed: hence

$$V\cos\theta . t = 2a.$$

From the two equations we see that

$$\sin 2\theta = \frac{2ag}{V^2},$$

a result which proves the proposition.

15. A particle is attracted towards one centre of force and repelled from another, both forces varying as the distance: prove that, if the absolute intensities of the forces are equal, the path of the particle is a parabola.

Let P be the particle, A and B the centres of force: then the two forces acting on P are represented by AP, PB: the resultant of these two forces is represented by AB. Hence the resultant force is constant in magnitude and invariable in direction. Hence, if the particle be projected at any inclination to AB, it will describe a parabola.

16. When a body arrives at a point P of an elliptic orbit, which it is describing about one focus S, the centre of force is suddenly transferred to the other focus H: supposing the orbit to remain the same as before, prove that, μ denoting the absolute force in the former, and μ' in the latter case,

$$\mu : \mu' :: SP^2 : HP^2.$$

Let v be the velocity at P. Then

$$AC.v^2 = \mu \frac{HP}{SP}:$$

also $$AC.v^2 = \mu' \frac{SP}{HP}.$$

Hence $$\mu : \mu' :: SP^2 : HP^2.$$

Aliter. The velocity at P being the same before and after the transference of the centre of forces and the orbit being the same, the deflection must be the same for both centres of force: hence the normal component of the force must be the same in both cases: hence, ϕ being the inclination of either focal distance to the normal, and F, F' the two central forces at P,

$$F\cos\phi = F'\cos\phi,$$

$$F = F',$$

$$\frac{\mu}{SP^2} = \frac{\mu'}{HP^2},$$

or

$$\mu : \mu' :: SP^2 : HP^2.$$

17. A solid triangular prism, the faces of which include angles α, β, γ, is placed in any position entirely within an inelastic gravitating fluid: if P, Q, R, be the pressures on the three faces, which are respectively opposite to the angles α, β, γ, prove that

$$P\operatorname{cosec}\alpha + Q\operatorname{cosec}\beta + R\operatorname{cosec}\gamma$$

is invariable so long as the depth of the centre of gravity of the prism is unchanged.

The centre of gravity of each face of the prism is in the transverse section of the prism which bisects its length. Let a, b, c, be the sides of this section, and a', b', c', the depths of its angular points. Let l be the length of the prism.

Let λ represent each of the equal fractions

$$\frac{a}{\sin\alpha}, \quad \frac{b}{\sin\beta}, \quad \frac{c}{\sin\gamma}:$$

then

$$P = g\rho al.\frac{b'+c'}{2}$$

$$= \tfrac{1}{2}g\rho l\lambda \sin\alpha\,(b'+c'),$$

$$P\operatorname{cosec}\alpha = \tfrac{1}{2}g\rho l\lambda\,(b'+c').$$

Similarly

$$Q\operatorname{cosec}\beta = \tfrac{1}{2}g\rho l\lambda\,(c'+a'),$$

$$R\operatorname{cosec}\gamma = \tfrac{1}{2}g\rho l\lambda\,(a'+b').$$

Hence

$$P\operatorname{cosec}\alpha + Q\operatorname{cosec}\beta + R\operatorname{cosec}\gamma = g\rho l\lambda\,(a'+b'+c')$$

$$= 3g\rho l\lambda h,$$

where h is the depth of the centre of gravity of the transverse section, that is, of the prism.

Hence, if h be constant,

$$P \operatorname{cosec}\alpha + Q \operatorname{cosec}\beta + R \operatorname{cosec}\gamma$$

is constant.

18. A heavy sphere is placed in a vertical cylinder, filled with atmospheric air, which it exactly fits. Find the density of the air in the cylinder when the sphere is in a position of permanent rest.

Let Π' be the pressure of the air at any point of the lower hemisphere, Π be the pressure of the air at any point of the upper hemisphere, W the weight of the sphere, and a its radius.

Then, since the pressure is uniform over each hemisphere, the resultant vertical pressure upward is equal to $\Pi' \times$ area of projection of the hemisphere on the horizontal plane

$$= \Pi' \pi a^2;$$

and the resultant vertical pressure downwards $= \Pi \pi a^2$.

Hence, since there is equilibrium,

$$W = \pi a^2 (\Pi' - \Pi);$$

but, if ρ, ρ', be the densities of the internal and external air

$$\frac{\rho'}{\rho} = \frac{\Pi'}{\Pi};$$

therefore $$W = \Pi \pi a^2 \left(\frac{\rho'}{\rho} - 1\right),$$

whence ρ' is known.

If δ be the density of the sphere, σ the density of mercury, and h the height of the barometer for the pressure Π,

we have $$\frac{4}{3} \pi \delta g a^3 = g \sigma h \pi a^2 \left(\frac{\rho'}{\rho} - 1\right),$$

and therefore $$\frac{\rho'}{\rho} = 1 + \frac{4}{3} \cdot \frac{\delta}{\sigma} \cdot \frac{a}{h}.$$

19. A solid formed of two co-axial right cones, of the same vertical angle, connected at their vertices, is placed with one end in contact with the horizontal base of a vessel; water is then poured into the vessel: shew that, if the altitude of the upper cone be treble that of the lower, and the common density of the spindle four-sevenths that of the water, it will be upon the point of rising when the water reaches to the level of its upper end.

Let h, $3h$, be the altitudes of the lower and upper cones, a, $3a$, the radii of their bases, ρ, σ, the densities of the fluid and spindle, P, P', the downward and upward pressures of the fluid upon the spindle.

Since P is the vertical pressure of the fluid upon the lower cone

$$P = \pi g\rho a^2 (4h - \tfrac{1}{3}h);$$

and, since P' must be equal to the weight of the fluid displaced by the upper cone,

$$P' = \pi g\rho \, . \, 9a^2 h:$$

also, if W be the weight of the spindle,

$$W = \pi g\sigma (9a^2 h + \tfrac{1}{3}a^2 h).$$

Since the spindle is on the point of rising,

$$P' = P + W;$$

hence
$$9\rho = \tfrac{11}{3}\rho + \tfrac{28}{3}\sigma,$$
$$16\rho = 28\sigma,$$
and
$$\sigma = \tfrac{4}{7}\rho.$$

20. A fish is floating in a cubical glass tank filled with water, with its head in one corner and its tail towards the one diagonally opposite; describe the appearance which will be presented to an eye looking towards the corner in the direction of the length of the fish, and in the same horizontal plane with it.

Since the distance of the image of a point in the diagonal from either face of the cube is to the distance of the point in the ratio of 1 to μ, where μ is the index of refraction from air into water, the effect of the glass on the position of the image not being taken into account, the image of the diagonal will be inclined to the face at an angle $\tan^{-1}\frac{1}{\mu}$ or $29°.18'$, taking the value of μ to be 1·335.

Hence each side of the fish will appear inclined at an angle $29°.18'$ to the adjacent face of the tank, and the appearance will be presented of two fishes joined at the head, and inclined to each other at an angle $31°.24'$.

21. Two rays emanate from a point in the circumference of a reflecting circle, in the plane of the circle: supposing that their n^{th} points of incidence are coincident, prove that the angle between their original directions is any one of a series of $n-1$ angles in arithmetical progression.

First suppose the two rays to emanate on opposite sides of the diameter through their starting point.

Let θ, ϕ, be the angles which their original directions make with this diameter. Then, λ being a positive integer, we must have

$$n(\pi - 2\theta) + n(\pi - 2\phi) = \lambda . 2\pi,$$

$$\theta + \phi = \frac{n-\lambda}{n}\pi.$$

Thus $\theta + \phi$ may have any one of the values

$$\frac{n-1}{n}\pi,\quad \frac{n-2}{n}\pi,\quad \frac{n-3}{n}\pi,\ \ldots\ldots\ldots\ \frac{\pi}{n}.$$

If the original directions are on the same side of the diameter,

$$n(\pi - 2\theta) - n(\pi - 2\phi) = \lambda . 2\pi,$$

whence $$\phi - \theta = \frac{\lambda}{n}\pi,$$

and therefore $\phi - \theta$ may have any one of the values

$$\frac{\pi}{n},\quad \frac{2\pi}{n},\quad \frac{3\pi}{n},\ \ldots\ldots\ \frac{n-1}{n}\pi,$$

the same series of values reversed.

22. A luminous globe falls from a point above the Earth's surface in a dark night: shew that it will look like a bright falling column, elongating as it descends.

If c_1, c_2, c_3, be the lengths of the apparent column at the ends of times t_1, t_2, t_3, from the commencement of the fall, prove that, gravity being considered constant, and the resistance of the air being neglected,

$$t_1(c_2 - c_3) + t_2(c_3 - c_1) + t_3(c_1 - c_2) = 0.$$

Let τ represent the duration of the impression of light on the eye. Then, if s be the space described by the globe at any epoch of the motion in the time τ, the globe will, at the end of the time τ, look like a luminous column of length s. Since s increases as the time of the fall increases, the apparent column will continually elongate.

Again, since the velocity acquired by the globe in the time $t_1 - \tau$ is $g(t_1 - \tau)$, we have

$$c_1 = g(t_1 - \tau)\tau + \tfrac{1}{2}g\tau^2,$$

whence $$\frac{c_1}{g\tau} = t_1 - \tfrac{1}{2}\tau:$$

similarly $$\frac{c_2}{g\tau} = t_2 - \tfrac{1}{2}\tau,$$

$$\frac{c_3}{g\tau} = t_3 - \tfrac{1}{2}\tau.$$

Hence $$t_1(c_2 - c_3) + t_2(c_3 - c_1) + t_3(c_1 - c_2) = 0.$$

TUESDAY, *Jan.* 20. 9 *to* 12.

1. ELIMINATE x, y, z, between the equations

$$\frac{y}{z}+\frac{z}{y}=a, \quad \frac{z}{x}+\frac{x}{z}=b, \quad \frac{x}{y}+\frac{y}{x}=c.$$

Squaring and adding the equations, we have

$$\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}+6=a^2+b^2+c^2.$$

Multiplying them together,

$$\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}+2=abc.$$

Hence, subtracting the latter of these equations from the former,

$$a^2+b^2+c^2-abc=4.$$

2. From a bag containing a counters, some of which are marked with numbers, b counters are to be drawn; and the drawer is to receive a number of shillings equal to the sum of the numbers on the counters which he draws; if the sum of the numbers on all the counters be n, what will be the value of his chance?

Let the value of any counter, as indicated by the number marked upon it, be denoted by C_r; then the whole number of combinations of b counters in which this counter occurs will be

$$\frac{(a-1)\ldots\ldots(a-b+1)}{1.2\ldots\ldots(b-1)}:$$

therefore its value in all the drawings will be

$$\frac{(a-1)......(a-b+1)}{1.2......(b-1)} C_r.$$

Hence the value of all the possible drawings will be

$$\frac{(a-1)......(a-b+1)}{1.2......(b-1)} \Sigma(C_r) \text{ or } \frac{(a-1)......(a-b+1)}{1.2......(b-1)} n \text{ shillings};$$

and the number of such possible drawings will be

$$\frac{a(a-1)......(a-b+1)}{1.2......b}:$$

hence the value of the chance $= \dfrac{\text{value of all possible drawings}}{\text{number ....}}$

$$= \frac{b}{a} n \text{ shillings}.$$

3. O is the middle point of a given straight line AA', (fig. 9): BOB' is a straight line perpendicular to AA': P, P', are two points in the plane of AA', BB': perpendiculars from P' upon AP, $A'P$, cut AA', in C, C', respectively: if OC, OC', be equal to each other and of given magnitude, prove that the distances of P, P', from BB' are in a constant ratio.

Let OA, OB, be taken as axes of x, y, respectively: let $OA = a$, $OC = c$. Then, x, y, being the coordinates of P, the equations to the perpendiculars through P' are

$$yy' = \quad (a-x)(x'-c),$$

and

$$yy' = -(a+x)(x'+c):$$

at the intersection P' of these two perpendiculars,

$$(a-x)(x'-c) = -(a+x)(x'+c),$$

and therefore

$$2ax' = -2cx,$$

or

$$x' : x :: -c : a,$$

a result which proves the proposition.

4. The foci of a given ellipse A lie in an ellipse B, the extremities of a diameter of A being the foci of B: prove that the eccentricity of B varies as the diameter of A.

Let PCP', (fig. 10), be a diameter of the ellipse (A), AA' being its axis major, and S either of its foci.

By a known property of the ellipse,

$$SP + SP' = AA' \qquad \text{.................... (1).}$$

Since P, P', are the foci of the ellipse (B), the relation (1) shews that the major axis of (B), which passes through S, is equal to AA', the major axis of (A). Hence the eccentricity of B is equal to the ratio of PP' to AA', and therefore varies as PP'.

5. C is the centre of an ellipse, (fig. 11), G the foot of a normal at any point P, and O the corresponding centre of curvature: find the distance of P from the axis minor, in order that the area of COG may be the greatest possible.

If x, y, be the coordinates of P, then, as may be seen in elementary treatises on the Differential Calculus, the distance of O from the major axis is equal to

$$\frac{a^2e^2}{b^4}y^3:$$

hence the area of COG is equal to

$$\frac{1}{2}\frac{a^2e^2}{b^4}y^3.e^2x,$$

and therefore varies as xy^3.

Put $$x = a\cos\frac{\phi}{2}, \quad y = b\sin\frac{\phi}{2}$$

then the area COG varies as

$$\cos\frac{\phi}{2}.\sin^3\frac{\phi}{2}$$

$$\propto \sin\phi.(1-\cos\phi)$$

$$\propto \sin\phi - \tfrac{1}{2}\sin 2\phi.$$

Hence, when the area is a maximum,

$$\cos\phi - \cos 2\phi = 0,$$

and therefore $$\phi = \pm \tfrac{2}{3}\pi :$$

hence $$x = a \cos\frac{\phi}{2} = \tfrac{1}{2}a\,;$$

that is, P's distance from the minor axis is equal to a quarter of the major axis.

6. The corners of a leaf of a book are turned down so as to meet and to make the length of one crease always n times that of the other; shew that each corner will describe a portion of the curve

$$x^2\{y^2 + (c-x)^2\}^3 = n^2(c-x)^2(x^2+y^2)^3,$$

the outer edge of the leaf, the length of which is c, being taken as the axis of x, and the lower edge as the axis of y.

Let AB be the outer edge of the leaf, (fig. 12), CD, $C'D'$, any pair of creases, and P the point of meeting of the corners:

let $$AC = a,\quad BC' = a',\quad AM = x,$$
$$AD = b,\quad BD' = b',\quad MP = y.$$

From the triangle PCM,

$$y^2 + (x-a)^2 = a^2,$$

and therefore $$a = \frac{x^2+y^2}{2x}.$$

Also, since $PD = AD$,

$$x^2 + (b-y)^2 = b^2,$$

and therefore $$b = \frac{x^2+y^2}{2y}.$$

Similarly $$a' = \frac{(c-x)^2+y^2}{2(c-x)},\quad b' = \frac{(c-x)^2+y^2}{2y}.$$

And since, by the question,

$$a'^2 + b'^2 = n^2(a^2+b^2),$$

we shall have, substituting,

$$\left\{\frac{(c-x)^2+y^2}{2(c-x)}\right\}^2+\left\{\frac{(c-x)^2+y^2}{2y}\right\}^2=n^2\left\{\left(\frac{x^2+y^2}{2x}\right)^2+\left(\frac{x^2+y^2}{2y}\right)^2\right\},$$

and therefore $\quad x^2\{y^2+(c-x)^2\}^3=n^2(c-x)^2(x^2+y^2)^3,$

the required locus.

7. A heavy ring is suspended from a point by any number of equal strings attached to it symmetrically; and another ring of the same weight but of smaller radius is in equilibrium when resting on the strings at their middle points; if R, r, be the radii of the rings and $2l$ the length of each string, shew that

$$4R^2-8Rr+3r^2-3l^2=0.$$

Let the strings be n in number, and let the tension of each string be T, the inclinations to the vertical of its upper and lower portions θ and ϕ, and W the weight of each ring.

Then, for the equilibrium of the lower ring, resolving the forces vertically, we have

$$nT\cos\phi-W=0;$$

and, for the equilibrium of the system,

$$nT\cos\theta-2W=0.$$

Hence $\quad \cos\theta=2\cos\phi \quad \text{.......................(1)}.$

Also, from the geometry, we have

$$r=l\sin\theta,\quad R-r=l\sin\phi;$$

whence, substituting in (1) and squaring,

$$1-\frac{r^2}{l^2}=4\left\{1-\frac{(R-r)^2}{l^2}\right\},$$

and therefore

$$4R^2-8Rr+3r^2-3l^2=0.$$

8. A thread without weight carrying a heavy bead has its extremities fastened to two points in the same vertical line; if the bead and thread be made to revolve uniformly about this line with an angular velocity ω; shew that, when the

bead is in equilibrium relatively to the thread, its distance below the horizontal plane midway between the points of attachment of the thread will be

$$\frac{g}{\omega^2} \cdot \frac{l^2}{l^2 - a^2},$$

$2l$ being the length of the thread, and $2a$ the distance between the points of attachment.

Let S, H, (fig. 13), be the points of attachment of the thread, C the point midway between them, and P the position of the bead, the plane of the paper being the plane in which S, H, P, lie at any instant.

The bead is held in equilibrium by its weight, the tensions of the string, and the centrifugal force.

Since the length of the string is invariable, P is a point in the ellipse of which the centre is C and the foci S and H; and the normal PG is the direction of the resultant of the tensions of the portions SP, HP, of the string. Let e be the eccentricity of this ellipse, x the abscissa of P. Hence the triangle PGM has its sides parallel to the directions of the equilibrating forces, and therefore, if m be the mass of the bead,

$$\frac{GM}{PM} = \frac{mg}{m\omega^2 PM},$$

and $$GM = \frac{g}{\omega^2};$$

hence $$x = e^2 x + \frac{g}{\omega^2},$$

and therefore $$x = \frac{g}{\omega^2} \cdot \frac{1}{1 - e^2}$$

$$= \frac{g}{\omega^2} \cdot \frac{l^2}{l^2 - a^2}.$$

9. A flexible chain, the ends of which are united, hangs over two pegs, in a horizontal line, in the form of two festoons; if P, P', be the tensions at the vertices of the festoons, and α, α',

the inclinations of the festoons to the horizon at either peg, prove that the weight of half the chain is equal to

$$P \tan\alpha + P' \tan\alpha'.$$

Prove also that the weight of a piece of the chain, equal in length to the distance between the vertices of the festoons, is equal to $P \sim P'$.

Let A (fig. 14) be either peg, C, C', the vertices of the two festoons: let T be the tension of the chain at A, W the weight of the whole chain.

Then, for the equilibrium of AC, we have, resolving horizontally,

$$T \cos\alpha = P.$$

Similarly, for the equilibrium of AC',

$$T \cos\alpha' = P.$$

Again, resolving vertically for the equilibrium of the whole chain, we have

$$W = 2T \sin\alpha + 2T \sin\alpha'.$$

From the above three equations, we see that

$$\tfrac{1}{2} W = P \tan\alpha + P' \tan\alpha'.$$

Again, by a property of the catenary, m being the mass of a unit of length of the chain, and h, h', the depths of C, C', below the horizontal line through the pegs,

$$T - P = mgh,$$

$$T - P' = mgh':$$

hence

$$P - P' = mg(h' - h)$$

$= $ the weight of a length CC' of the chain.

10. A triangular lamina has a small ring at each of its angular points, which slides on a smooth wire occupying the position of the circle circumscribing the triangle; determine the motion of the triangle when the wire is held in any position, and find the time of a small oscillation when the wire is so held that the triangle is nearly in its position of stable equilibrium.

Since the wire is smooth, the directions of the mutual actions between it and the rings will pass through the centre, O, of the circle; and the lamina will oscillate about an axis through O, perpendicular to its plane.

Hence, if $\alpha =$ the inclination of the plane of the lamina to the vertical,

$h =$ the distance of its centre of gravity, G, from O,

$k =$ its radius of gyration about an axis through G,

$\theta =$ the inclination of OG to its lowest position at the time t,

the equation of motion will be

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{h^2+k^2}\cos\alpha\sin\theta,$$

integrating, $$\frac{d\theta^2}{dt^2} = \frac{2gh}{h^2+k^2}\cos\alpha(\cos\theta-\cos\beta),$$

if $\theta = \beta$ initially, which determines its angular velocity in any position.

If the triangle be held nearly in its position of stable equilibrium, θ is small, and the equation of motion becomes

$$\frac{d^2\theta}{dt^2} + \frac{gh}{h^2+k^2}\cos\alpha\,.\,\theta = 0,$$

and the time of a small oscillation $= \pi\left(\dfrac{h^2+k^2}{gh}\sec\alpha\right)^{\frac{1}{2}}$.

11. Shew, by aid of the formulæ

$$2\cot 2x = \cot x - \tan x, \qquad 2\operatorname{cosec} 2x = \cot x + \tan x,$$

that if $$\tan x = a_1 x + a_3 x^3 + a_5 x^5 + \ldots\ldots,$$

then $$\cot x = \frac{1}{x} - \frac{a_1}{2^2-1}x - \frac{a_3}{2^4-1}x^3 - \frac{a_5}{2^6-1}x^5 - \ldots\ldots,$$

and $$\operatorname{cosec} x = \frac{1}{x} + \frac{2-1}{2^2-1}\frac{a_1}{2}x + \frac{2^3-1}{2^4-1}\frac{a_3}{2^3}x^3 + \frac{2^5-1}{2^6-1}\frac{a_5}{2^5}x^5 + \ldots$$

Since $x\cot x = 1$, when $x = 0$, and $\cot x = -\cot(-x)$, we may assume

$$\cot x = \frac{1}{x} + A_1 x + A_3 x^3 + \ldots\ldots + A_{2n+1}x^{2n+1} + \ldots\ldots;$$

and therefore, changing x into $2x$,

$$\cot 2x = \frac{1}{2x} + A_1 2x + A_3 2^3 x^3 + \ldots\ldots + A_{2n+1} 2^{2n+1} x^{2n+1} + \ldots\ldots;$$

but $$2\cot 2x = \cot x - \tan x,$$

and therefore, substituting,

$$\frac{1}{x} + 2A_1 2x + 2A_3 . 2^3 x^3 + \ldots\ldots + 2A_{2n+1} 2^{2n+1} x^{2n+1} + \ldots\ldots$$

$$= \frac{1}{x} + (A_1 - a_1)\, x + (A_3 - a_3)\, x^3 + \ldots\ldots + (A_{2n+1} - a_{2n+1})\, x^{2n+1} + \ldots\ldots$$

Equating the coefficients of x^{2n+1} in these two identical series, we get

$$2^{2n+2} . A_{2n+1} = A_{2n+1} - a_{2n+1};$$

whence $$A_{2n+1} = -\frac{a_{2n+1}}{2^{2n+2} - 1}.$$

And giving to n the values 0, 1, 2, 3,...... successively, we have

$$\cot x = \frac{1}{x} - \frac{a_1}{2^2 - 1}\, x - \frac{a_3}{2^4 - 1}\, x^3 - \ldots\ldots$$

Also, $2\operatorname{cosec} 2x = \tan x + \cot x$

$$= \frac{1}{x} + \left(a_1 - \frac{a_1}{2^2 - 1}\right) x + \left(a_3 - \frac{a_3}{2^4 - 1}\right) x^3 + \ldots\ldots$$

$$= \frac{1}{x} + \frac{2^2 - 2}{2^2 - 1}\, a_1 x + \frac{2^4 - 2}{2^4 - 1}\, a_3 x^3 + \ldots\ldots:$$

therefore $$\operatorname{cosec} 2x = \frac{1}{2x} + \frac{2 - 1}{2^2 - 1}\, a_1 x + \frac{2^3 - 1}{2^4 - 1}\, a_3 x^3 + \ldots\ldots;$$

and changing $2x$ into x,

$$\operatorname{cosec} x = \frac{1}{x} + \frac{2 - 1}{2^2 - 1}\, a_1 \frac{x}{2} + \frac{2^3 - 1}{2^4 - 1}\, a_3 \frac{x^3}{2^3} + \ldots\ldots$$

12. Circles are described upon the radii vectores of the loop of a lemniscate as diameters, passing through the pole; find the locus of their ultimate intersections, and shew that its area is double that of the loop.

Taking, as the equation of the lemniscate,

$$r^2 = a^2 \cos 2\theta,$$

the equation of one of the circles described as required, will be

$$\rho = a \cos(\phi - \theta) \cos^{\frac{1}{2}} 2\theta \dots\dots\dots\dots\dots\dots (1),$$

ρ and ϕ being the current coordinates.

Hence, in the consecutive circle, we must have, differentiating with respect to θ,

$$0 = \sin(\phi - \theta) \cos^{\frac{1}{2}} 2\theta - \cos(\phi - \theta) \cdot \frac{\sin 2\theta}{\cos^{\frac{1}{2}} 2\theta},$$

therefore $$\tan 2\theta = \tan(\phi - \theta),$$

or $$\theta = \frac{\phi}{3};$$

and the equation of the required locus is, substituting in (1) and squaring,

$$\rho^2 = a^2 \cos^3 \frac{2\phi}{3}.$$

Also the area required $= \frac{1}{2} \int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} \rho^2 d\phi$

$$= a^2 \int_0^{\frac{3}{4}\pi} \cos \frac{2\phi}{3} \left(1 - \sin^2 \frac{2\phi}{3}\right) d\phi$$

$$= a^2$$

$=$ double the area of the loop.

13. A semicircular tube of very small bore containing an elastic string fastened to one of its extremities is revolving with a uniform angular velocity ω about a vertical axis through that extremity perpendicular to its plane, and the string in its stretched state subtends an angle α at the centre of the circle the radius of which is a; shew that, if the modulus of elasticity be the weight of a length l of the unstretched string, and $lg = 4a^2\omega^2 \cos^2 \frac{\alpha}{2}$, the unstretched length of the string will be

$$2a \cos \frac{\alpha}{2} \log \tan \frac{\pi + \alpha}{4}.$$

Suppose the string, which in its stretched state extends over the arc Ap (fig. 15), to have extended in its unstretched state over AP; and let

$$\angle AOP = \theta,\quad POQ = \delta\theta,\quad T = \text{tension at } p,$$
$$\angle AOp = \phi,\quad pOq = \delta\phi,\quad T + \delta T = \ldots\ldots\ldots\ldots q,$$

PQ being an element of the string, which is stretched into pq.

Also, let m be the mass of a unit of length of the unstretched string and ϖ the modulus of elasticity.

For the equilibrium of the element pq, we have, resolving the forces, which act upon it, parallel to the tangent at p,

$$(T+\delta T)\cos\delta\phi - T + ma\delta\theta \cos\frac{\phi}{2}\,.\,\omega^2.\,2a\sin\frac{\phi}{2} = 0;$$

hence, proceeding to the limit,

$$\frac{dT}{d\theta} + ma^2\omega^2 \sin\phi = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

But, since the elementary arc $a\delta\theta$ is stretched into $a\delta\phi$ by the tension T,

$$\frac{\delta\phi - \delta\theta}{\delta\theta} = \frac{T}{\varpi},$$

and, in the limit,

$$\frac{d\phi}{d\theta} = 1 + \frac{T}{\varpi}.$$

Hence

$$\frac{d^2\phi}{d\theta^2} = \frac{1}{\varpi}\frac{dT}{d\theta}$$
$$= -\frac{ma^2\omega^2}{\varpi}\sin\phi, \text{ from (1)};$$

Multiplying by $2\dfrac{d\phi}{d\theta}$ and integrating,

$$\frac{d\phi^2}{d\theta^2} = C + \frac{2ma^2\omega^2}{\varpi}\cos\phi.$$

When $\phi = \alpha$, $T = 0$, and $\dfrac{d\phi}{d\theta} = 1$:

hence

$$\frac{d\phi^2}{d\theta^2} = 1 + \frac{2ma^2\omega^2}{\varpi}(\cos\phi - \cos\alpha).$$

But
$$\varpi = mlg$$
$$= 4ma^2\omega^2 \cos^2 \frac{\alpha}{2},$$

and therefore
$$\frac{d\phi^2}{d\theta^2} = \frac{\cos^2 \frac{\phi}{2}}{\cos^2 \frac{\alpha}{2}}.$$

Hence
$$\frac{d\theta}{d\phi} = \cos \frac{\alpha}{2} \cdot \frac{\sec^2 \frac{\phi}{4}}{1 - \tan^2 \frac{\phi}{4}}$$
$$= \tfrac{1}{2} \cos \frac{\alpha}{2} \left\{ \frac{\sec^2 \frac{\phi}{4}}{1 + \tan \frac{\phi}{4}} + \frac{\sec^2 \frac{\phi}{4}}{1 - \tan \frac{\phi}{4}} \right\},$$

and, integrating, $\theta = C + 2 \cos \frac{\alpha}{2} \log \dfrac{1 + \tan \frac{\phi}{4}}{1 - \tan \frac{\phi}{4}}$.

When $\theta = 0$, $\phi = 0$, and therefore $C = 0$; and, when $\phi = \alpha$, the corresponding value of $a\theta$ will be the unstretched length of the string; and therefore

$$\text{the unstretched length} = 2a \cos \frac{\alpha}{2} \log \tan \frac{\pi + \alpha}{4}.$$

14. Two spheres, the molecules of which attract according to the law of the inverse square, were originally in contact; if W, W', W'', be the labouring forces which have been expended in pushing them asunder in the line of their centres, when the distances between their centres are respectively a, a', a''; prove that

$$W\left(\frac{1}{a'} - \frac{1}{a''}\right) + W'\left(\frac{1}{a''} - \frac{1}{a}\right) + W''\left(\frac{1}{a} - \frac{1}{a'}\right) = 0.$$

Let x, x', be simultaneous distances of the centres of the spheres from their original point of contact: then, m, m', being the masses of the spheres, the corresponding labouring force is equal to

$$\int mm' \left\{\frac{dx}{(x+x')^2} + \frac{dx'}{(x+x')^2}\right\}$$

$$= mm' \int \frac{d(x+x')}{(x+x')^2}$$

$$= mm' \left(\frac{1}{c} - \frac{1}{x+x'}\right),$$

where c is a constant.

Hence
$$W = mm' \left(\frac{1}{c} - \frac{1}{a}\right),$$

$$W' = mm' \left(\frac{1}{c} - \frac{1}{a'}\right),$$

$$W'' = mm' \left(\frac{1}{c} - \frac{1}{a''}\right),$$

and therefore

$$W\left(\frac{1}{a'} - \frac{1}{a''}\right) + W'\left(\frac{1}{a''} - \frac{1}{a}\right) + W''\left(\frac{1}{a} - \frac{1}{a'}\right) = 0.$$

15. Normals to an ellipsoid through a curve traced on its surface intersect a principal plane in a circle of given radius; prove that the projection of the curve on the plane encloses an invariable area.

Let x, y, z, be the coordinates of any point of the curve. The equations to the normal are

$$(x'-x)\frac{a^2}{x} = (y'-y)\frac{b^2}{y} = (z'-z)\frac{c^2}{z}.$$

When $z'=0$, we have

$$x' = \frac{a^2-c^2}{a^2}x, \quad y' = \frac{b^2-c^2}{b^2}y.$$

Let r be the radius of a circle in the plane of xy, through which the normal passes, and h, k, the coordinates of its centre.

Then for the equation to the projection of the curve on the plane of xy, we have

$$r^2 = (x'-h)^2 + (y'-k)^2$$

$$= \left(\frac{a^2-c^2}{a^2}x - h\right)^2 + \left(\frac{b^2-c^2}{b^2}y - k\right)^2,$$

which represents an ellipse the area of which is equal to

$$\frac{\pi a^2 b^2 r^2}{(a^2-c^2)(b^2-c^2)}.$$

16. A curve is traced upon a terrestrial globe, of such a form that the longitude of any point is equal to its north polar distance; prove that the whole length of the curve between the north and south pole is equal to the meridian distance between the north and south poles of an oblate spheroid, the eccentricity of which is $\frac{1}{\sqrt{2}}$ and axis equal to the diameter of the globe.

Let a be the radius of the globe, and θ the longitude of any point of the curve which is traced upon it: then the whole length of the curve between the north and south pole is equal to

$$\int_0^\pi (a^2 d\theta^2 + a^2 \sin^2\theta . d\theta^2)^{\frac{1}{2}}$$

$$= a\int_0^\pi (1+\sin^2\theta)^{\frac{1}{2}}\, d\theta.$$

Again, the meridian distance between the north and south poles of the spheroid, a', b', being the semi-axes of the meridian, and ϕ the eccentric angle of any point, is equal to

$$\int_0^\pi [\{d(a'\cos\phi)\}^2 + \{d(b'\sin\phi)\}^2]^{\frac{1}{2}}$$

$$= \int_0^\pi (a'^2\sin^2\phi + b'^2\cos^2\phi)^{\frac{1}{2}}\, d\phi$$

$$= b'\int_0^\pi \left(1+\frac{a'^2-b'^2}{b'^2}\sin^2\phi\right)^{\frac{1}{2}} d\phi$$

$$= b'\int_0^\pi \left(1+\frac{e^2}{1-e^2}\sin^2\phi\right)^{\frac{1}{2}} d\phi.$$

The two distances are therefore equal if $2b' = 2a$, and $e = \frac{1}{\sqrt{2}}$.

17. A closed vessel in the form of a right cone is placed with its base on a horizontal plane: supposing it to be filled with fluid through a small orifice at its vertex, prove that the horizontal tension of the vessel at any point varies as the area of the circular section through the point.

The ordinary formula connecting the pressure and tensions at any point of a vessel filled with fluid is

$$p = \frac{t}{r} + \frac{t'}{r'},$$

where r, r', are the principal radii of curvature at the point.

In the present instance, one of the radii of curvature is infinite: hence, putting $r' = \infty$,

$$p = \frac{t}{r}.$$

Now p varies as the depth of the point below the vertex and therefore as the radius of the circular section. Also, by Meunier's theorem, the radius of the circular section is equal to $r \cos\alpha$, where α is the semi-angle of the cone; and therefore r varies as the radius of the circular section. Hence pr, which is equal to t, varies as the square of the radius, or as the area of the circular section.

18. A luminous point is placed at one of the foci of a semi-elliptic arc bounded by the axis major; prove that the whole illumination of the arc varies inversely as the latus rectum.

Let the equation to the ellipse be referred to the focus as pole, the prime radius vector coinciding with the major axis. Then the illumination of an arc ds is equal to

$$\frac{\lambda}{r^2} \,.\, ds \,.\, \text{cosine of angle of incidence}$$

$$= \frac{\lambda}{r^2} \,.\, ds \,.\, \frac{rd\theta}{ds}$$

$$= \frac{\lambda}{r}\, d\theta,$$

and, c denoting the semi-latus-rectum, the illumination of the whole arc is equal to

$$\frac{\lambda}{c}\int_0^\pi (1 + e\cos\theta)\, d\theta$$

$$= \frac{\lambda}{c}\Big|_0^\pi \left(\theta + e\sin\theta\right)$$

$$= \frac{\lambda}{c}\pi,$$

and therefore varies inversely as the latus rectum.

19. A homogeneous globe is placed upon a perfectly rough table, very near to a centre of force in the surface of the table, the law of attraction being that of the inverse square; prove that the square of the time of an oscillation varies as the volume of the sphere.

Let a represent the radius of the globe, m its mass; let r be the distance of its centre, and x of its point of contact with the table from the centre of force, at any time t; and let ω denote its angular velocity. Then, μ being a constant quantity,

$$m\left(\frac{dx^2}{dt^2} + \frac{2}{5}a^2\omega^2\right) = C - 2m\int \frac{\mu}{r^2}\,dr$$

$$= C + \frac{2m\mu}{r}:$$

but, the rolling being perfect,

$$\frac{dx^2}{dt^2} = a^2\omega^2,$$

hence $$\frac{7}{5}\frac{dx^2}{dt^2} = \text{constant} + \frac{2\mu}{r}$$

$$= \text{const.} + \frac{2\mu}{(a^2 + x^2)^{\frac{1}{2}}}$$

$$= \text{const.} - \frac{\mu x^2}{a^3}\text{ nearly;}$$

and therefore $$\frac{d^2x}{dt^2} + \frac{5\mu}{7a^3}x = 0,$$

which shews that t, the time of oscillation, is equal to

$$\frac{\pi}{\left(\frac{5\mu}{7a^3}\right)^{\frac{1}{2}}};$$

or that t^2 varies as a^3 and therefore as the volume of the globe.

20. An inelastic ball, of given radius, is dropped from the window of a carriage, travelling uniformly along a level road, upon the wheel, which it hits at the highest point: determine the subsequent motion of the ball relatively to the carriage, the rim of the wheel being perfectly rough.

Since the ball, at the instant of being dropped, has the same horizontal velocity as the carriage, the motion of the ball, relatively to the carriage, will be the same as if it were dropped upon the wheel revolving uniformly about the axle at rest.

Let ω be the angular velocity of the wheel so revolving, a, b, the radii of the wheel and ball, u', ω', the relative horizontal and angular velocities of the ball immediately after the impact, I the impulsive friction, M the mass of the ball and k its radius of gyration about a diameter.

Then
$$u' = \frac{I}{M},$$
$$\omega' = -\frac{Ib}{Mk^2},$$

the angular velocity of the ball being estimated in the same direction as that of the wheel.

And, since the point of contact is instantaneously at rest, relatively to the wheel,

$u' - b\omega' = a\omega = u$, if u be the velocity of the carriage,

therefore, substituting,

$$u' = \frac{k^2}{b^2+k^2}u = \frac{2}{7}u, \text{ since } k^2 = \frac{2}{5}b^2,$$

$$\omega' = -\frac{b}{b^2+k^2}u = -\frac{5}{7}\frac{u}{b}.$$

The ball will leave the wheel immediately after the impact, if the curvature of the parabola, which it would proceed to describe if free, be not greater than the curvature of a circle of radius $a+b$; and, since the radius of curvature of the parabola at its vertex is equal to

$$\tfrac{1}{2}\text{ lat. rect.}$$
$$=\frac{(\text{horizontal velocity})^2}{g}$$
$$=\frac{4}{49}\frac{u^2}{g},$$

the ball will leave the wheel if

$$\frac{4}{49}\frac{u^2}{g}\text{ be not less than } a+b,$$

$$\text{or if } u \text{ be not less than } \tfrac{7}{2}\{g(a+b)\}^{\frac{1}{2}}.$$

If the velocity of the carriage be less than $\frac{7}{2}\{g(a+b)\}^{\frac{1}{2}}$, the ball will proceed to roll upon the wheel.

At the time t after the impact, let F be the rolling friction, ϕ the angle through which the ball has rotated, P', fig. (16), the point in contact with the wheel, CAQ the angle through which the wheel has turned, $P'AQ=\theta$.

The equations of motion of the ball, relatively to the carriage, are

$$\frac{d^2\{(a+b)\theta\}}{dt^2}=-\frac{F}{M}+g\sin\theta,$$

$$\frac{d^2\phi}{dt^2}=\frac{Fb}{Mk^2};$$

and, since there is perfect rolling,

$$\frac{d.(a+b).\theta}{dt}-b\frac{d\phi}{dt}=a\omega,\quad\text{where } \omega=\frac{u}{a}.$$

Hence, differentiating and substituting, we have

$$\frac{F}{M}=\frac{k^2}{b^2+k^2}.g\sin\theta,$$

and, therefore, $$(a+b)\frac{d^2\theta}{dt^2}=\frac{b^2}{b^2+k^2}g\sin\theta.$$

Multiplying by $2\dfrac{d\theta}{dt}$ and integrating,

$$(a+b)\frac{d\theta^2}{dt^2}=C-\frac{2b^2}{b^2+k^2}g\cos\theta.$$

Now, at the commencement of motion, $\theta=0$, $\dfrac{d\phi}{dt}=\omega'$, and therefore

$$(a+b)\frac{d\theta}{dt}=a\omega+b\omega'$$

$$=u-\frac{5}{7}u.$$

Whence $$(a+b)\frac{d\theta^2}{dt^2}=\frac{4}{49}\frac{u^2}{a+b}+\frac{10}{7}g(1-\cos\theta).$$

When the ball flies off the wheel, the centrifugal force just balances the resolved part of its weight along the radius; therefore

$$(a+b)\frac{d\theta^2}{dt^2}=g\cos\theta,$$

and the angle θ is given by the equation,

$$g\cos\theta=\frac{4}{49}\frac{u^2}{a+b}+\frac{10}{7}g(1-\cos\theta);$$

whence $$\cos\theta=\frac{4}{119}\frac{u^2}{g(a+b)}+\frac{10}{17},$$

u^2, by the previous condition, being less than $\dfrac{49}{4}g(a+b)$.

WEDNESDAY, *Jan.* 21. 9 *to* 12.

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1. SHEW that the circle, which cuts orthogonally three given circles lying in a plane, has its centre at the radical centre of these circles.

Since the radical centre of the three circles is the intersection of the three radical axes of the circles, taken two and two; and since the radical axis of two circles possesses the property that from every point of it pairs of equal tangents may be drawn to the circles; therefore, from the radical centre, when it lies without the three circles, we may draw six equal tangents to the three circles. Hence the circle described with the radical centre as centre and with one of these tangents as radius will pass through the points of contact of the other five; and, since its radii drawn to the points of its intersection with the three circles are tangents to these circles, it will cut the circles orthogonally.

When the radical centre lies within any of the three circles, no circle can be drawn cutting them orthogonally.

2. If $\dfrac{a\sin^2\theta+b\sin^2\phi}{b\cos^2\theta+c\cos^2\phi}=\dfrac{b\sin^2\theta+c\sin^2\phi}{c\cos^2\theta+a\cos^2\phi}=\dfrac{c\sin^2\theta+a\sin^2\phi}{a\cos^2\theta+b\cos^2\phi}$;

then will
$$a^3+b^3+c^3=3abc.$$

Let each of the ratios be equal to r: then

$$r=\frac{(a+b+c)(\sin^2\theta+\sin^2\phi)}{(a+b+c)(\cos^2\theta+\cos^2\phi)}$$
$$=\frac{\sin^2\theta+\sin^2\phi}{\cos^2\theta+\cos^2\phi}.$$

Now $$a \sin^2\theta + b \sin^2\phi = r(b \cos^2\theta + c \cos^2\phi),$$
and therefore, $$a + b - (a + br) \cos^2\theta - (b + cr) \cos^2\phi = 0:$$
similarly, $$b + c - (b + cr) \cos^2\theta - (c + ar) \cos^2\phi = 0,$$
$$c + a - (c + ar) \cos^2\theta - (a + br) \cos^2\phi = 0.$$

Eliminating $\cos^2\theta$ and $\cos^2\phi$ by cross multiplication, we have

$$\begin{aligned}(a+b)\{(b+cr)(a+br) - (c+ar)^2\}\\ +(b+c)\{(c+ar)(b+cr) - (a+br)^2\}\\ +(c+a)\{(a+br)(c+ar) - (b+cr)^2\} = 0.\end{aligned}$$

Effecting the multiplications, the expression becomes

$$r(1-r)(a^3+b^3+c^3-3abc) = 0;$$

and since $r(1-r)$ involve θ and ϕ only, if the equations hold for given particular values of θ and ϕ, so that $r(1-r)$ is constant, we must have

$$a^3+b^3+c^3 = 3abc.$$

Since $$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
$$= \tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)\}^2,$$

we see further that, if the quantities a, b, c, be real, this relation leads to

$$a = b = c.$$

3. A parabola slides between two rectangular axes; find the curve traced out by any point in its axis; and hence shew that the focus and vertex will describe curves of which the equations are

$$x^2y^2 = a^2(x^2+y^2), \quad x^2y^2(x^2+y^2+3a^2) = a^6,$$

$4a$ being the latus rectum of the parabola.

Let ASP (fig. 17) be the position of the axis of the parabola at any instant, A being its vertex and S its focus, ϕ the angle which ASP makes with Ox, and x, y, the coordinates of P, a point in the axis: let $SP = \xi$, $OS = \rho$, $\angle SOx = \theta$.

Then $$x = \rho\cos\theta + \xi\cos\phi,$$
$$y = \rho\sin\theta + \xi\sin\phi.$$

But, since Ox, Oy, are tangents to the parabola at right angles to each other, O is a point in the directrix OM; and, since the tangent makes equal angles with OS and OM, we have

$$\angle SOM = 2\theta, \quad \phi + \theta = \tfrac{1}{2}\pi, \quad \text{and } 2a = \rho \sin 2\theta.$$

Hence
$$x = \frac{a}{\sin\theta} + \xi \sin\theta \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

$$y = \frac{a}{\cos\theta} + \xi \cos\theta \ \ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

Multiplying (1), (2), by $\sin\theta$, $\cos\theta$, respectively, squaring and subtracting, we have

$$x^2 \sin^2\theta - y^2 \cos^2\theta = 2a\xi\,(\sin^2\theta - \cos^2\theta) + \xi^2\,(\sin^4\theta - \cos^4\theta)$$
$$= (2a\xi + \xi^2)\,(\sin^2\theta - \cos^2\theta),$$

therefore
$$\sin^2\theta = \frac{y^2 - 2a\xi - \xi^2}{x^2 + y^2 - 2\,(2a\xi + \xi^2)}.$$

And since, from (1),

$$x^2 \sin^2\theta = (a + \xi \sin^2\theta)^2,$$

we have for the required locus, substituting for $\sin^2\theta$ its value,

$$x^2 \cdot \frac{y^2 - 2a\xi - \xi^2}{x^2 + y^2 - 2\,(2a\xi + \xi^2)} = \left\{a + \xi \cdot \frac{y^2 - 2a\xi - \xi^2}{x^2 + y^2 - 2\,(2a\xi + \xi^2)}\right\}^2,$$

or

$$x^2(y^2 - 2a\xi - \xi^2)(x^2 + y^2 - 4a\xi - 2\xi^2) = \{ax^2 + (a+\xi)y^2 - \xi(\xi + 2a)^2\}^2.$$

For the focus, $\xi = 0$, and therefore

$$x^2y^2\,(x^2 + y^2) = a^2\,(x^2 + y^2)^2,$$

or
$$x^2y^2 = a^2\,(x^2 + y^2).$$

For the vertex, $\xi = -a$, and therefore

$$x^2\,(y^2 + a^2)\,(x^2 + y^2 + 2a^2) = a^2\,(x^2 + a^2)^2,$$

or
$$x^2y^2\,(x^2 + y^2 + 3a^2) = a^6.$$

4. Shew that, if in the equation

$$ax^2 + by^2 + 2cxy - f = 0,$$

the parameter f alone vary, the focus of the conic represented will lie in either of two straight lines; if, either a or b vary, the other coefficients remaining constant, the focus will lie in a rectangular hyperbola; and, if c alone vary, the focus will lie in the curve

$$ab(x^4+y^4)-(a^2+b^2)x^2y^2+f(a-b)(x^2-y^2)=0.$$

Defining the focus as a point, such that the lines joining it to the two imaginary points on a circle at infinity shall both touch the curve, (Vide Salmon's *Higher Plane Curves*, p. 119), we have to determine p so that the line

$$x+y\sqrt{(-1)}=p,$$

may touch the conic

$$ax^2+by^2+2cxy=f.$$

Eliminating x between these equations, we get

$$\{b-a-2c\sqrt{(-1)}\}\,y^2+2p\,\{c-a\sqrt{(-1)}\}\,y+ap^2-f=0,$$

which must have equal roots in y; therefore

$$\{b-a-2c\sqrt{(-1)}\}\,(ap^2-f)=p^2\,\{c-a\sqrt{(-1)}\}^2,$$

and
$$p^2=f.\frac{a-b+2c\sqrt{(-1)}}{c^2-ab}.$$

Therefore, since x and y, the coordinates of the focus, are connected by the equation

$$\{x+y\sqrt{(-1)}\}^2=f.\frac{a-b+2c\sqrt{(-1)}}{c^2-ab},$$

we have
$$x^2-y^2=f.\frac{a-b}{c^2-ab}\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$xy=f.\frac{c}{c^2-ab}\ \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

for the determination of those coordinates.

If the parameter f alone vary, we have, eliminating f between (1) and (2), as the locus of the focus,

$$x^2-y^2=xy.\frac{a-b}{c};$$

which is the equation of two straight lines.

Also, since, from (2), $ab = c^2 - \dfrac{fc}{xy}$,

we have, from (1), $x^2 - y^2 = \dfrac{xy}{ac}(a^2 - ab)$.

Hence, eliminating b, if the parameter b vary, the locus of the focus is

$$x^2 - y^2 = \frac{xy}{ac}\left(a^2 - c^2 + \frac{fc}{xy}\right),$$

and therefore $ac(x^2 - y^2) + (c^2 - a^2)xy - fc = 0$,

which is the equation of a rectangular hyperbola.

Similarly, if the parameter a vary, the locus of the focus is

$$ab(x^2 - y^2) + (c^2 - b^2)xy - fc = 0.$$

Again, since, from (1), $c^2 = ab + f.\dfrac{a-b}{x^2 - y^2}$,

we have, eliminating c from (2), when c alone varies,

$$x^2y^2.f^2\left(\frac{a-b}{x^2-y^2}\right)^2 = f^2\left(ab + f.\frac{a-b}{x^2-y^2}\right),$$

or $ab(x^4 + y^4) - (a^2 + b^2)x^2y^2 + f(a-b)(x^2 - y^2) = 0$,

as the locus of the focus.

5. A right vertical cylinder with circular ends carries a hand upon its upper face, equal in length to a radius of the end, and moveable about an axis coincident with the axis of the cylinder: the extremity of the hand is attached by a fine elastic thread to a point in the circumference of the lower end of the cylinder; and, when the thread is vertical, it is stretched to its natural length: if the hand be made to revolve through any angle α, and then let go, find its angular velocity in any subsequent position; and shew that, if the angle of displacement, α, be very small, the time of an oscillation will be

$$n\int_0^\alpha \frac{d\theta}{(\alpha^4 - \theta^4)^{\frac{1}{2}}},$$

where n is constant.

Let OP, (fig. 18) be the position of the hand at the time t, AB the initial position of the string, $AB=l$, $OP=a$, $\angle BOP=\theta$. Now, if the cylinder be unwrapped, AP will be a straight line, and ABP a triangle, right-angled at B. Hence the length of the string at the time t is equal to

$$(l^2+a^2\theta^2)^{\frac{1}{2}};$$

and therefore, if T be the tension of the string at the time t, and ϖ its modulus of elasticity,

$$\frac{T}{\varpi}=\frac{(l^2+a^2\theta^2)^{\frac{1}{2}}-l}{l}.$$

Hence, if Mk^2 be the moment of inertia of the hand about the axis, the equation of motion will be

$$Mk^2\frac{d^2\theta}{dt^2}=-\,Ta\cos BPA$$

$$=-\varpi\frac{a}{l}\{(l^2+a^2\theta^2)^{\frac{1}{2}}-l\}\frac{a\theta}{(l^2+a^2\theta^2)^{\frac{1}{2}}}$$

$$=-\varpi\frac{a^2}{l}\left\{\theta-\frac{l\theta}{(l^2+a^2\theta^2)^{\frac{1}{2}}}\right\}.$$

Multiplying by $2\dfrac{d\theta}{dt}$, and integrating

$$Mk^2\frac{d\theta^2}{d\theta^2}=C-\varpi\frac{a^2}{l}\left\{\theta^2-2\frac{l}{a^2}(l^2+a^2\theta^2)^{\frac{1}{2}}\right\}.$$

When the motion commences, let $\theta=\alpha$; then

$$0=C-\varpi\frac{a^2}{l}\left\{\alpha^2-2\frac{l}{a^2}(l^2+a^2\alpha^2)^{\frac{1}{2}}\right\},$$

and $$Mk^2\frac{d\theta^2}{dt^2}=\varpi\frac{a^2}{l}(\alpha^2-\theta^2)-2\varpi\{(l^2+a^2\alpha^2)^{\frac{1}{2}}-(l^2+a^2\theta^2)^{\frac{1}{2}}\};$$

which gives the angular velocity of the hand in any position.

If α be very small, θ will be very small; hence, expanding and neglecting powers of α and θ above the fourth, we have, approximately,

$$Mk^2\frac{d\theta^2}{dt^2}=\varpi\frac{a^2}{l}(\alpha^2-\theta^2)-2\varpi\left\{l+\frac{1}{2}\frac{a^2}{l}\alpha^2-\frac{1}{8}\frac{a^4}{l^3}\alpha^4-l-\frac{1}{2}\frac{a^2}{l}\theta^2+\frac{1}{8}\frac{a^4}{l^3}\theta^4\right\}$$

$$=\varpi\frac{a^4}{4l^3}(\alpha^4-\theta^4),$$

and therefore the time of a small oscillation is equal to

$$n\int_0^{\alpha}\frac{d\theta}{(\alpha^4-\theta^4)^{\frac{1}{2}}}.$$

6. A narrow smooth semicircular tube is fixed in a vertical plane with its vertex upwards; and a heavy flexible string, passing through it, hangs at rest; shew that, if the string be cut at one of the ends of the tube, the velocity, which the longer portion of the string will have attained when it is just leaving the tube, will be

$$(ag)^{\frac{1}{2}}\left\{2\pi-\frac{a}{l}(\pi^2-4)\right\}^{\frac{1}{2}};$$

l being the length of the longer portion, and a the radius of the tube.

Let C, D, (fig. 19) be the ends of the tube, APB the position of the string at the time t; let $\angle COA=\theta$, $\angle COP=\phi$, Q a point in the pendent portion of the string, $DQ=\xi$. Then, by D'Alembert's principle, we have, m being the mass of a unit of length of the string,

$$\Sigma ma\,\delta\phi\left(a\frac{d^2\phi}{dt^2}+g\cos\phi\right)+\Sigma m\,\delta\xi\left(\frac{d^2\xi}{dt^2}-g\right)=0.$$

But, since the string is always stretched, we have

$$\frac{d\xi}{dt}=a\frac{d\phi}{dt}=a\frac{d\theta}{dt};$$

also the limits of ϕ are π and θ, and those of ξ are $l-a(\pi-\theta)$ and 0. Hence the equation of motion becomes

$$al\frac{d^2\theta}{dt^2}=g(l-a\pi+a\theta)-ag\int_\theta^\pi\cos\phi\,d\phi$$

$$=g(l-a\pi+a\theta)+ag\sin\theta.$$

Integrating, we have

$$al\frac{d\theta^2}{dt^2}=C+2g(l-a\pi)\theta+ag\theta^2-2ag\cos\theta.$$

When $\theta = 0$, $\frac{d\theta}{dt} = 0$; and, when $\theta = \pi$, $a\frac{d\theta}{dt}$ is the required velocity; hence, if v be that velocity,

$$0 = C - 2ag,$$

$$\frac{l}{a}v^2 = C + 2g(l - a\pi)\pi + ag\pi^2 + 2ag,$$

and therefore $$v = (ag)^{\frac{1}{2}}\left\{2\pi - \frac{a}{l}(\pi^2 - 4)\right\}^{\frac{1}{2}}.$$

Aliter. We may use the principle of vis-viva to obtain the value of v.

Let $\bar{y}$ be the height of the centre of gravity of the string above CD at the commencement of the motion, $\bar{y}'$ the value of $\bar{y}$ when the string has attained the velocity v: then

$$mlv^2 = 2mlg(\bar{y} - \bar{y}').$$

But $$ml\bar{y} = ma\pi . \frac{2a}{\pi} - m(l - a\pi) . \tfrac{1}{2}(l - a\pi),$$

and $$ml\bar{y}' = -ml . \tfrac{1}{2}l.$$

Hence $$lv^2 = g\{4a^2 - (l - a\pi)^2 + l^2\};$$

and therefore $$v = (ag)^{\frac{1}{2}}\left\{2\pi - \frac{a}{l}(\pi^2 - 4)\right\}^{\frac{1}{2}}.$$

7. If a, b, c, a', b', c', be the cosines of the inclinations of the faces of a tetrahedron, a and a', b and b', c and c', belonging respectively to the edges which do not meet; shew that

$$\begin{aligned} 1 + a^2a'^2 + b^2b'^2 + c^2c'^2 = a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2 \\ + 2a'bc + 2b'ca + 2c'ab + 2a'b'c' \\ + 2bcb'c' + 2cac'a' + 2aba'b'. \end{aligned}$$

Let F_1, F_2, F_3, F_4, be the areas of the four faces of the tetrahedron; then, projecting each set of three faces upon the fourth successively, we obtain the equations,

$$F_4 = aF_1 + bF_2 + cF_3,$$
$$F_3 = b'F_1 + a'F_2 + cF_4,$$
$$F_2 = c'F_1 + a'F_3 + bF_4,$$
$$F_1 = c'F_2 + b'F_3 + aF_4.$$

Putting $F_1 = xF_4$, $F_2 = yF_4$, $F_3 = zF_4$, and arranging, we have

$$ax + by + cz = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$b'x + a'y - z = -c \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$c'x - y + a'z = -b \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$x - c'y - b'z = a \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Hence, substituting for z its value from (2), we obtain

$$c^2 - 1 + (a + cb')\,x + (b + ca')\,y = 0,$$
$$b + ca' + (c' + a'b')\,x + (a'^2 - 1)\,y = 0,$$
$$a + cb' + (b'^2 - 1)\,x + (c' + a'b')\,y = 0.$$

Eliminating x, y, from these equations by cross multiplication, we get

$$\begin{aligned}&(c^2 - 1)\{(c' + a'b')^2 - (a'^2 - 1)(b'^2 - 1)\}\\ &+ (b + ca')\{(b'^2 - 1)(b + ca') - (c' + a'b')(a + cb')\}\\ &+ (a + cb')\{(a + cb')(a'^2 - 1) - (b + ca')(c' + a'b')\} = 0,\end{aligned}$$

or $$(c^2 - 1)(c' + a'b')^2 + (b'^2 - 1)(b + ca')^2 + (a'^2 - 1)(a + cb')^2$$
$$= (c^2 - 1)(a'^2 - 1)(b'^2 - 1) + 2(a + cb')(b + ca')(c' + a'b');$$

whence, effecting the multiplication, we obtain

$$\begin{aligned}1 + a^2a'^2 + b^2b'^2 + c^2c'^2 = a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2&\\ + 2a'bc + 2b'ca + 2c'ab + 2a'b'c'&\\ + 2bcb'c' + 2cac'a' + 2aba'b'.&\end{aligned}$$

8. Shew that the determination of the circular sections of the cone

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0,$$

may be made to depend upon the solution of the cubic equation

$$abc\mu^3 - (a^2 + b^2 + c^2)\,\mu^2 + 4 = 0;$$

and that the circular sections of the cone

$$\left(\frac{b}{c} + \frac{c}{b}\right) yz + \left(\frac{c}{a} + \frac{a}{c}\right) zx + \left(\frac{a}{b} + \frac{b}{a}\right) xy = 0$$

are parallel to the planes

$$ax + by + cz = 0, \qquad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0.$$

Let the equation of the sphere on which the circular sections lie be

$$x^2 + y^2 + z^2 + \alpha x + \beta y + \gamma z + \delta = 0 \dots\dots\dots\dots (1).$$

The equation of the cone, when combined with that of the sphere, must become the equation of two planes; and therefore, if l, m, n; l', m', n'; be proportional to the direction-cosines of these planes, we must have, conversely

$$(lx + my + nz + d)(l'x + m'y + n'z + d') - \mu(ayz + bzx + cxy) = 0 \dots (2),$$

as the equation of the sphere. Hence, comparing it with equation (1), we get

$$ll' = mm' = nn' = 1,$$

$$mn' + m'n = \mu a, \quad nl' + n'l = \mu b, \quad lm' + l'm = \mu c.$$

Substituting for l', m', n', the last three equations become

$$\frac{m}{n} + \frac{n}{m} = \mu a \dots\dots (3), \quad \frac{n}{l} + \frac{l}{n} = \mu b \dots\dots (4), \quad \frac{l}{m} + \frac{m}{l} = \mu c \dots\dots (5).$$

If we subtract the sum of the squares of these three equations from their product, as in page 17, we shall eliminate l, m, n, and obtain the equation

$$abc\mu^3 - (a^2 + b^2 + c^2)\mu^2 + 4 = 0,$$

for the determination of μ.

We have also, from the equations (3) and (4),

$$\frac{l}{n} = \frac{1}{2}\{\mu b \pm \sqrt{(\mu^2 b^2 - 4)}\}, \qquad \frac{m}{n} = \frac{1}{2}\{\mu a \pm \sqrt{(\mu^2 a^2 - 4)}\};$$

and therefore, since

$$\frac{l'}{n'} = \frac{n}{l}, \qquad \frac{m'}{n'} = \frac{n}{m},$$

the circular sections will be parallel to the planes

$$\tfrac{1}{2}\{\mu b + \sqrt{(\mu^2 b^2 - 4)}\}x + \tfrac{1}{2}\{\mu a + \sqrt{(\mu^2 a^2 - 4)}\}y + z = 0 \dots (6),$$

$$\tfrac{1}{2}\{\mu b - \sqrt{(\mu^2 b^2 - 4)}\}x + \tfrac{1}{2}\{\mu a - \sqrt{(\mu^2 a^2 - 4)}\}y + z = 0 \dots (7).$$

If the equations of the cone be

$$\left(\frac{b}{c} + \frac{c}{b}\right)yz + \left(\frac{c}{a} + \frac{a}{c}\right)zx + \left(\frac{a}{b} + \frac{b}{a}\right)xy = 0,$$

the equations (3), (4), (5), become

$$\frac{m}{n}+\frac{n}{m}=\mu\left(\frac{b}{c}+\frac{c}{b}\right),\ \frac{n}{l}+\frac{l}{n}=\mu\left(\frac{c}{a}+\frac{a}{c}\right),\ \frac{l}{m}+\frac{m}{l}=\mu\left(\frac{a}{b}+\frac{b}{a}\right);$$

whence $\mu=1$ and l, m, n, are proportional to a, b, c, and therefore l', m', n', to $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, and the planes (6), (7), take the forms

$$ax+by+cz=0,\qquad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0.$$

9. Shew that, if

$$\alpha=0,\quad \beta=0,\quad \gamma=0,$$

be the equations of three planes which form a trihedral angle, the equation of a cone of the second order, which has its vertex at the angular point and touches two of the planes at their intersections with the third, is

$$\gamma^2-k\alpha\beta=0;$$

and that the equation of a surface of the second order enveloped by the cone is

$$\delta^2+\mu(\gamma^2-k\alpha\beta)=0,$$

$\delta=0$ being the equation of the plane of contact, and μ being constant.

Shew that if the enveloping cone of a series of ellipsoids be the asymptotic cone of a series of hyperboloids of two sheets, the curves of intersection of any ellipsoid with the series of hyperboloids will lie in planes parallel to the plane of contact of the cone and ellipsoid.

Since the equation

$$\gamma^2-k\alpha\beta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

involves the variables to the second order, and results from the elimination of the arbitrary parameter l between the equations

$$\gamma-l\alpha=0,\qquad \gamma-\frac{k}{l}\beta=0;$$

and since these equations represent a straight line passing through the angular point $\alpha\beta\gamma$; we see that (1) is the locus of a straight line passing through the angular point, and therefore is the equation of a cone of the second order.

Also, if we combine the equation (1) with either of the equations $\alpha = 0$, $\beta = 0$, it becomes $\gamma^2 = 0$. Hence the planes $\alpha = 0$, $\beta = 0$, meet the cone in two coincident lines; therefore they touch the cone, and $\gamma = 0$ passes through their lines of contact with it.

Again, since the equation

$$\delta^2 + \mu(\gamma^2 - k\alpha\beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

is of the second order, and since, if we combine it with the equation (1), it becomes $\delta^2 = 0$, we see that the cone meets this surface in two coincident plane curves, the equations of which are

$$\gamma^2 - k\alpha\beta = 0, \qquad \delta = 0;$$

and therefore, (2) is the equation of a surface of the second order enveloped by (1), $\delta = 0$ being the equation of the plane of contact.

If $\mu = -m^2$, (2) may be put under the form

$$(\delta + m\gamma)(\delta - m\gamma) + m^2 k\alpha\beta = 0,$$

which results from the elimination of the parameter l between the equations

$$\delta + m\gamma + ml\alpha = 0, \qquad \delta - m\gamma - m\frac{k}{l}\beta = 0,$$

and therefore represents a ruled surface of the second order. If μ be positive, (2) cannot be resolved into plane factors, and therefore cannot have a straight line coincident with it.

Since the plane of contact of an hyperboloid with its asymptotic cone is at infinity; and since this condition is analytically expressed by making the equation, which results from combining the surface with its asymptote, become

$$\text{constant} = 0,$$

we have for the equation of the hyperboloids of which (1) is the asymptotic cone,

$$\pm c^2 + \mu(\gamma^2 - k\alpha\beta) = 0,$$

where c is constant, the upper and lower signs belonging to the hyperboloids of two sheets and of one sheet respectively.

If we combine the equations

$$\delta^2 + \mu(\gamma^2 - k\alpha\beta) = 0\ldots\ldots(3), \qquad c^2 + \mu^2(\gamma^2 - k\alpha\beta) = 0\ldots\ldots(4),$$

we obtain $$\delta = \pm c\frac{\mu^{\frac{1}{2}}}{\mu'^{\frac{1}{2}}},$$

which, for all values of μ and μ', represents planes parallel to the plane $\delta = 0$; and hence any one of the surfaces represented by equation (3) will instersect all the surfaces represented by equation (4), in planes parallel to the plane $\delta = 0$.

10. A rigid spherical shell is filled with homogeneous inelastic fluid, every particle of which attracts every other with a force varying inversely as the square of the distance: shew that the difference between the pressures at the surface and at any point within the fluid varies as the area of the least section of the sphere through the point.

Let r be the distance of any particle P of the fluid from the centre of the shell; and through P describe a sphere concentric with the shell. The fluid exterior to this sphere exerts no resultant attraction on P, and the resultant attraction of the fluid within the sphere is the same as if it were condensed into the centre. Hence, if p be the pressure at P, a the radius of the shell, and p' the pressure at its surface,

$$dp = -\frac{4}{3}\pi r^3 \cdot \frac{\mu}{r^2}dr, \quad \mu \text{ being constant},$$

$$= -\frac{4}{3}\pi\mu r dr,$$

and hence $$p = C - \frac{2}{3}\pi\mu r^2.$$

When $r = a$, $p = p'$; and therefore

$$p - p' = \frac{2}{3}\pi\mu(a^2 - r^2)$$

$$= \frac{2}{3}\mu \times \text{area of the least section through } P.$$

11 A uniform beam is revolving uniformly in a vertical plane about a horizontal axis through its middle point; and,

at the instant it is passing through its horizontal position, a perfectly elastic ball, the mass of which is one-third that of the beam, is projected horizontally from a point vertically above the axis, so as to hit the beam at one extremity, then to rebound to the other, and so on for ever, bounding and rebounding along the same path; shew that if θ be the angle, on each side of its horizontal position, through which the beam revolves, θ will be given by the equation

$$\theta \tan \theta = 1.$$

Let m, $3m$, be the masses of the ball and beam, and $2a$ the length of the beam.

Since the ball always describes the same path, the direction of its motion, when it impinges upon the beam, must be perpendicular to the beam, and the motion of both ball and beam must be just reversed at each impact.

Let v, v', be the velocities of the ball immediately before impact and at the instant compression ceases, $-\omega$, ω', the angular velocities of the beam at the same epochs, and I the impulsive pressure during compression; we have

$$v' = v - \frac{I}{m},$$

$$\omega' = -\omega + \frac{Ia}{3mk^2}$$

$$= -\omega + \frac{I}{ma}, \text{ since } k^2 = \frac{a^2}{3};$$

and since the ball and the extremity of the beam are, at this instant, moving with the same velocity,

$$v' = a\omega';$$

therefore $$v - \frac{I}{m} = -a\omega + \frac{I}{m},$$

and $$2\frac{I}{m} = v + a\omega.$$

Hence, at the instant restitution ceases, the velocity of the ball

$$= v - 2\frac{I}{m}$$

$$= -a\omega,$$

and the angular velocity of the beam

$$= -\omega + 2\frac{I}{ma}$$

$$= \frac{v}{a},$$

and consequently, if $v = a\omega$, the motion of both ball and beam will be reversed after each impact.

But, if $2t$ be the time which elapses between any two successive impacts,

$$\theta = \omega \,.\, t,$$

$$a\cos\theta = v\sin\theta \,.\, t$$

$$= a\omega\sin\theta \,.\, t$$

$$= a\theta\sin\theta,$$

and therefore $\qquad \theta\tan\theta = 1.$

12. A homogeneous sphere, of elasticity e, rotating uniformly about a horizontal diameter, falls upon a perfectly rough inclined plane through such a height h that its angular velocity is not affected by the first impact, and then proceeds to descend the plane directly by bounds; if u_n be the velocity of the sphere along the plane after the n^{th} impact, shew that

$$u_n = (2gh)^{\frac{1}{2}}\sin\alpha\left(1 + \frac{10}{7}\cdot\frac{e - e^n}{1 - e}\right),$$

and that the range which the sphere describes upon the plane before it ceases to hop will be

$$4h\sin\alpha\frac{e}{(1-e)^2}\left(1 - \frac{4}{7}\cdot\frac{e^2}{1+e}\right),$$

α being the inclination of the plane to the horizon.

Since the angular velocity of the sphere is not affected by the first impact, if u, ω, be its vertical and angular velocities, at the instant when it impinges, and a its radius, we shall have

$$u^2 = 2gh, \qquad a\omega = u\sin\alpha.$$

Let F_{n+1} be the tangential impulse on the sphere during the $(n+1)^{\text{th}}$ impact; v_n, v_{n+1}, the velocities of the sphere along the plane immediately after the n^{th} and $(n+1)^{\text{th}}$ impacts; ω_n, ω_{n+1}, its angular velocities at the same epochs; then, since there is perfect rolling,

$$u_n = a\omega_n, \qquad u_{n+1} = a\omega_{n+1}.$$

Now, if t_n be the time of the n^{th} bound, the velocity of the sphere along the plane, at the instant before the n^{th} impact, will be equal to

$$u_n + g\sin\alpha \,.\, t_n$$

$$= u_n + g\sin\alpha \,.\, \frac{2e^n u\cos\alpha}{g\cos\alpha}$$

$$= u_n + 2e^n u \sin\alpha;$$

therefore
$$u_{n+1} = u_n + 2e^n u\sin\alpha - \frac{F_{n+1}}{M},$$

and
$$\omega_{n+1} = \omega_n + \frac{F_{n+1}a}{Mk^2},$$

Mk^2 being the moment of inertia of the sphere about a diameter. Hence, since there is perfect rolling,

$$a\omega_n + \frac{F_{n+1}a^2}{Mk^2} = u_n + 2e^n u\sin\alpha - \frac{F_{n+1}}{M};$$

therefore
$$\left(1 + \frac{a^2}{k^2}\right)\frac{F_{n+1}}{M} = u_n - a\omega_n + 2e^n u\sin\alpha,$$

and
$$\frac{F_{n+1}}{M} = \frac{2}{7}\,.\,2e^n u\sin\alpha.$$

Hence
$$u_{n+1} = u_n + 2e^n u\sin\alpha - \frac{4}{7}e^n u\sin\alpha,$$

and
$$u_{n+1} - u_n = \frac{10}{7}e^n u\sin\alpha.$$

Integrating,
$$u_n = C - \frac{10}{7}u\sin\alpha\,\frac{e^n}{1-e}.$$

When $n = 1$, $a_n = u\sin\alpha$;

therefore
$$u\sin\alpha = C - \frac{10}{7}u\sin\alpha\,\frac{e}{1-e},$$

and
$$u_n = u\sin\alpha + \frac{10}{7}u\sin\alpha\,\frac{e-e^n}{1-e}$$
$$= (2gh)^{\frac{1}{2}}\sin\alpha\left(1+\frac{10}{7}\,\frac{e-e^n}{1-e}\right).$$

Also, if R_n be the range described upon the plane between the n^{th} and $(n+1)^{\text{th}}$ impacts,

$$R_n = u_n t_n + \frac{1}{2}g\sin\alpha\, t_n^2,$$
$$= u\sin\alpha\left(1+\frac{10}{7}\,\frac{e-e^n}{1-e}\right)\frac{2e^n u}{g} + \frac{1}{2}g\sin\alpha\left(\frac{2e^n u}{g}\right)^2$$
$$= 2\,\frac{u^2\sin\alpha}{g}\left\{\left(1+\frac{10}{7}\,\frac{e}{1-e}\right)e^n + \left(1-\frac{10}{7}\,\frac{1}{1-e}\right)e^{2n}\right\};$$

and therefore, the whole range described, which is the sum of all the values of R_n, from $n=1$, to $n=\infty$, will be

$$4h\sin\alpha\left\{\left(1+\frac{10}{7}\,\frac{e}{1-e}\right)\Sigma_1^\infty e^n + \left(1-\frac{10}{7}\,\frac{1}{1-e}\right)\Sigma_1^\infty e^{2n}\right\}$$
$$= 4h\sin\alpha\left\{\left(1+\frac{10}{7}\,\frac{e}{1-e}\right)\frac{e}{1-e} + \left(1-\frac{10}{7}\,\frac{1}{1-e}\right)\frac{e^2}{1-e^2}\right\}$$
$$= 4h\sin\alpha\,\frac{e}{(1-e)^2}\left\{1-e+\frac{10}{7}e + \left(1-e-\frac{10}{7}\right)\frac{e}{1+e}\right\}$$
$$= 4h\sin\alpha\,\frac{e}{(1-e)^2}\left(1-\frac{4}{7}\,\frac{e^2}{1+e}\right).$$

THURSDAY, *Jan.* 22. 9 *to* 12.

1. IF p, p', be the reciprocals of the perpendiculars from the centre of an ellipse upon SP, HD, where S, H, are the foci respectively nearest to P, D, the ends of two conjugate semi-diameters, prove that, b being the reciprocal of the semi-axis minor,

$$\frac{(pp'-b^2)^2}{(p-b)^2+(p'-b)^2}$$

is a constant quantity.

Let, in the first place, b represent the semi-axis minor, and p, p', the perpendiculars from C upon SP, HP, (fig. 20). Let x, y, be the coordinates of P. Then, since twice the area of the triangle CSP is equal to either $(a-ex)\,p$ or to aey, we have

$$\left(1-e\frac{x}{a}\right)\frac{p}{b}=e\frac{y}{b}.$$

Similarly, $\frac{a}{b}y$, $\frac{b}{a}x$, being the magnitudes of the coordinates of D,

$$\left(1-e\frac{y}{b}\right)\frac{p'}{b}=e\frac{x}{a}.$$

Obtaining $\frac{x}{a}$ and $\frac{y}{b}$ from these two equations, we see that

$$be\frac{x}{a}\left(\frac{1}{pp'}-\frac{1}{b^2}\right)=\frac{1}{p}-\frac{1}{b},$$

and

$$be\frac{y}{b}\left(\frac{1}{pp'}-\frac{1}{b^2}\right)=\frac{1}{p'}-\frac{1}{b}.$$

Squaring the last two equations, adding, and attending to the equation to the ellipse, we have

$$\frac{1}{b^2e^2}=\frac{\left(\frac{1}{pp'}-\frac{1}{b^2}\right)^2}{\left(\frac{1}{p}-\frac{1}{b}\right)^2+\left(\frac{1}{p'}-\frac{1}{b}\right)^2},$$

a result which shews that, the symbols b, p, p', being now used to denote the reciprocals of b, p, p', the expression

$$\frac{(pp'-b^2)^2}{(p-b)^2+(p'-b)^2}$$

is invariable.

2. If forces P, Q, R, acting at the centre O (fig. 21) of a circular lamina along the radii OA, OB, OC, be equivalent to forces P', Q', R', acting along the sides BC, CA, AB, of the inscribed triangle, prove that

$$\frac{P.P'}{BC}+\frac{Q.Q'}{CA}+\frac{R.R'}{AB}=0.$$

Since the sum of the moments of P', Q', R', about any point must be equal to the sum of the moments of P, Q, R, about the same point, we have, taking moments about A, B, C, successively,

$$P'.\frac{\Delta ABC}{BC}=R.\frac{\Delta COA}{OC}-Q.\frac{\Delta AOB}{OB},$$

$$Q'.\frac{\Delta BCA}{CA}=P.\frac{\Delta AOB}{OA}-R.\frac{\Delta BOC}{OC},$$

$$R'.\frac{\Delta CAB}{AB}=Q.\frac{\Delta BOC}{OB}-P.\frac{\Delta COA}{OA}:$$

multiplying these equations by P, Q, R, respectively, and adding, we have

$$\frac{P.P'}{BC}+\frac{Q.Q'}{CA}+\frac{R.R'}{AB}=0.$$

3. A fine thread just encloses, without tension, the circumference of an ellipse: supposing a centre of force, attracting

inversely as the square of the distance, to be placed at one of the foci, prove that the sum of the tensions of the thread at the ends of any focal chord is invariable, and that the normal pressure on the ellipse at any point varies inversely as the cube of the conjugate diameter.

Let PSP' (fig. 22) be any focal chord: let t be the tension at P; let $SP=r$, $SP'=r'$: then, ds being an indefinitely small arc Pp, and ϕ the inclination of the tangent at P to SP, we have, resolving along the tangent the forces which act on ds,

$$t+dt=t+\frac{\mu}{r^2}.ds.\cos\phi,$$

$$dt=\frac{\mu}{r^2}\,dr,$$

$$t=C-\frac{\mu}{r}.$$

Now the attraction of the central force tends to draw every element of the thread towards the nearer apse; hence the tension of the string must be zero at the nearer apse, that is, $t=0$ when $r=a(1-e)$: hence

$$0=C-\frac{\mu}{a(1-e)},$$

and therefore $$t=\mu\left\{\frac{1}{a(1-e)}-\frac{1}{r}\right\}.$$

Let t' be the tension at P': then

$$t'=\mu\left\{\frac{1}{a(1-e)}-\frac{1}{r'}\right\}:$$

hence $$t+t'=\mu\left\{\frac{2}{a(1-e)}-\left(\frac{1}{r}+\frac{1}{r'}\right)\right\}:$$

but, by a property of the ellipse,

$$\frac{1}{r}+\frac{1}{r'}=\frac{2}{a(1-e^2)}:$$

hence $$t+t'=\mu\left\{\frac{2}{a(1-e)}-\frac{2}{a(1-e^2)}\right\}$$

$$=\frac{2\mu e}{a(1-e^2)}=\text{a constant}.$$

Again, N denoting the normal pressure between the ellipse and the thread at P, we have, resolving along the normal the forces which act on ds,

$$Nds = t\sin\psi + \frac{\mu}{r^2}\sin\phi . ds,$$

where ψ is the angle between the normals at P, p. Hence, ρ denoting the radius of curvature at P,

$$N = \frac{t}{\rho} + \frac{\mu}{r^2}\sin\phi,$$

$$N\rho = t + \frac{\mu}{2r^2} . 2\rho\sin\phi$$

$$= t + \frac{\mu}{2r^2} . PV,$$

$$N . \frac{CD^3}{ab} = t + \frac{\mu}{2r^2} . \frac{CD^2}{a}$$

$$= \frac{\mu}{a(1-e)} - \frac{\mu}{r} + \frac{\mu}{2r^2} . \frac{r(2a-r)}{a}$$

$$= \frac{\mu}{a(1-e)} - \frac{\mu}{r} + \frac{\mu}{2r}\left(2 - \frac{r}{a}\right)$$

$$= \frac{\mu}{a(1-e)} - \frac{\mu}{2a}$$

$$= \frac{\mu(1+e)}{2a(1-e)},$$

and therefore $N \propto (CD)^{-3}$.

4. Prove that the eccentricity of a section of an ellipsoid, made by a plane through its least axis, varies inversely as the distance, from this axis, of the point in which it cuts a centric circular section.

Let COC' (fig. 23) be the least axis, P the point in which the section CPC' cuts the circular section BPB'. Let Q be the intersection of the curve CPC' and the plane of OA, OB: join OQ and draw PN at right angles to OQ. Let $ON = r$, $PN = z$, and $e =$ the eccentricity of the section CPC'.

Then $$1 - e^2 = \frac{OC^2}{OQ^2} = \frac{OC^2}{ON^2} . \frac{ON^2}{OQ^2} = \frac{c^2}{r^2}\left(1 - \frac{z^2}{c^2}\right):$$

but, b being the radius of the circular section, $OP=b$: hence

$$1-e^2=\frac{c^2}{r^2}\cdot\left(1-\frac{b^2-r^2}{c^2}\right)$$
$$=\frac{c^2}{r^2}-\frac{b^2}{r^2}+1,$$
$$e^2=\frac{b^2-c^2}{r^2},$$

and therefore $$e\propto\frac{1}{r}.$$

5. OA', OB', are two quadrants on the surface of a sphere, at right angles to each other: a great circle cuts them in A, B, respectively: from A', B', through any point P of the great circle, are drawn arcs $B'PM$, $A'PN$, cutting OA', OB', in M, N, respectively: if $PN=\phi$, $PM=\psi$, $\angle OAB=\lambda$, $\angle OBA=\mu$, prove that

$$\sin^2\lambda.\cos^2\phi-2\cos\lambda\cos\mu\sin\phi\sin\psi+\sin^2\mu\cos^2\psi=1.$$

Let $AP=\alpha$, $BP=\beta$: then, AMP, BNP, AOB, being right-angled triangles, we have, by Napier's rules,

$$\sin\alpha.\sin\lambda=\sin\psi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$\sin\beta.\sin\mu=\sin\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$\cot\lambda.\cot\mu=\cos(\alpha+\beta) \ldots\ldots\ldots\ldots\ldots (3).$$

Multiplying the equations (1), (2), (3), together, and dividing by $\sin\alpha\,\sin\beta$, we have

$$\cos\lambda.\cos\mu=\sin\phi\,\sin\psi\,(\cot\alpha\,\cot\beta-1),$$
$$(\cos\lambda\,\cos\mu+\sin\phi\,\sin\psi)^2$$
$$=(\operatorname{cosec}^2\alpha-1)(\operatorname{cosec}^2\beta-1)\sin^2\phi\,\sin^2\psi$$
$$=(\sin^2\lambda-\sin^2\psi)(\sin^2\mu-\sin^2\phi),\text{ by (1) and (2)},$$

and therefore

$$\cos^2\lambda\,\cos^2\mu+2\cos\lambda\,\cos\mu\,\sin\phi\,\sin\psi$$
$$=\sin^2\lambda\,\sin^2\mu-\sin^2\lambda\,\sin^2\phi-\sin^2\mu\,\sin^2\psi,$$
$$(1-\sin^2\lambda)(1-\sin^2\mu)+2\cos\lambda\,\cos\mu\,\sin\phi\,\sin\psi$$
$$=\sin^2\lambda\,\sin^2\mu-\sin^2\lambda\,(1-\cos^2\phi)-\sin^2\mu\,(1-\cos^2\psi),$$
$$\sin^2\lambda\,\cos^2\phi-2\cos\lambda\,\cos\mu\,\sin\phi\,\sin\psi+\sin^2\mu\,\cos^2\psi=1.$$

6. If a polygon of a given number of sides be inscribed in the orbit of a planet, such that all its sides subtend equal angles at the Sun, prove that the sum of the angular velocities of the planet about the Sun, at the angular points of the polygon, is independent of the position of the polygon.

If r be the distance of the planet from the Sun and ω its angular velocity about the Sun at any moment, $r^2\omega = h$. Hence, the equation to the orbit being

$$\frac{c}{r} = 1 + e\cos\theta,$$

we have $$\frac{c^2\omega}{h} = \frac{c^2}{r^2} = 1 + 2e\cos\theta + \tfrac{1}{2}e^2 + \tfrac{1}{2}e^2\cos 2\theta.$$

Hence the sum of the angular velocities will be constant if, α being equal to $\frac{2\pi}{n}$, each of the series

$$s_1 = \cos\theta + \cos(\theta+\alpha) + \cos(\theta+2\alpha) + \ldots + \cos\{\theta + (n-1)\alpha\},$$

$$s_2 = \cos 2\theta + \cos(2\theta + 2\alpha) + \cos(2\theta + 4\alpha) + \ldots + \cos\{2\theta + (n-1)2\alpha\},$$

is independent of θ.

Now $$\sin\left(\theta + \frac{\alpha}{2}\right) - \sin\left(\theta - \frac{\alpha}{2}\right) = 2\sin\frac{\alpha}{2}\cos\theta,$$

$$\sin\left(\theta + \frac{3\alpha}{2}\right) - \sin\left(\theta + \frac{\alpha}{2}\right) = 2\sin\frac{\alpha}{2}\cos(\theta + \alpha),$$

$$\sin\left(\theta + \frac{5\alpha}{2}\right) - \sin\left(\theta + \frac{3\alpha}{2}\right) = 2\sin\frac{\alpha}{2}\cos(\theta + 2\alpha),$$

....

....

$$\sin\left\{\theta + (2n-1)\frac{\alpha}{2}\right\} - \sin\left\{\theta + (2n-3)\frac{\alpha}{2}\right\} = 2\sin\frac{\alpha}{2}\cos\left\{\theta + (n-1)\alpha\right\}.$$

Hence $$2.\sin\frac{\alpha}{2}.s_1 = \sin\left\{\theta + (2n-1)\frac{\alpha}{2}\right\} - \sin\left(\theta - \frac{\alpha}{2}\right)$$

$$= \sin\left(\theta - \frac{\pi}{n}\right) - \sin\left(\theta - \frac{\pi}{n}\right) = 0.$$

Thus $s_1 = 0$, and, similarly, $s_2 = 0$. Hence

$$\frac{c^2}{h}\Sigma(\omega) = n\left(1 + \tfrac{1}{2}e^2\right),$$

$\frac{nh}{c^2}\left(1 + \frac{1}{2}e^2\right)$, the value of $\Sigma(\omega)$, being independent of θ.

7. A uniform homogeneous wire PAP', of which A is the middle point, is bent into the form of an arc of a loop of the lemniscate of which A becomes the vertex: prove that the resultant attraction on the wire, arising from a centre of force at the node O, attracting according to the law of the inverse square, varies as

$$\left(\frac{1}{OP^2} - \frac{1}{OA^2}\right)^{\frac{1}{2}}.$$

If A denote the resultant attraction, then, the equation to the lemniscate being $r^2 = a^2\cos 2\theta$, we have

$$A = \mu\int\frac{ds}{r^2}\cos\theta$$

$$= \mu\int\frac{\cos\theta}{r}\cdot\left(\frac{dr^2}{r^2} + d\theta^2\right)^{\frac{1}{2}}.$$

But, from the equation to the curve,

$$2\log r = 2\log a + \log\cos 2\theta,$$

and therefore $$\frac{dr}{r} = -\frac{\sin 2\theta}{\cos 2\theta}.d\theta.$$

Hence $$A = \frac{\mu}{a}\int_0^\theta\frac{\cos\theta}{(\cos 2\theta)^{\frac{1}{2}}}\cdot\left(\frac{\sin^2 2\theta}{\cos^2 2\theta} + 1\right)^{\frac{1}{2}}d\theta$$

$$= \frac{\mu}{a}\int_0^\theta\frac{\cos\theta\, d\theta}{(\cos 2\theta)^{\frac{3}{2}}}$$

$$= \frac{\mu}{a}\cdot\frac{\sin\theta}{(1 - 2\sin^2\theta)^{\frac{1}{2}}}$$

$$= \frac{\mu}{a}\cdot\frac{\sin\theta}{\cos 2\theta} = \frac{\mu\sin\theta}{r}.$$

But, from the equation to the curve,

$$\sin^2\theta = \frac{a^2 - r^2}{2a^2}:$$

hence
$$A \propto \frac{(a^2 - r^2)^{\frac{1}{2}}}{r}$$

$$\propto \left(\frac{1}{OP^2} - \frac{1}{OA^2}\right)^{\frac{1}{2}}.$$

8. A small light is placed at the focus of a perfect reflector in the form of a paraboloid of revolution: prove that the brightness, due to reflection, at any point within the volume of the paraboloid, varies inversely as the square of the focal distance of the end of the diameter through the point.

Let P, P', (fig. 24) be any two points, in the generating parabola, indefinitely near to each other: let S be the focus: join SP, SP', and draw PQ, $P'Q'$, parallel to the axis, to meet any ordinate in Q, Q', respectively. Take R, R', in SP, SP', respectively, so that $SR = c = SR'$. Then, if $SP = r$ and $\angle PSP' = d\theta$, the volume generated by the revolution of RSR' about the axis of the parabola is equal to

$$\int_0^c r\,d\theta\,dr \,.\, 2\pi r \sin\theta$$

$$= 2\pi \sin\theta\,d\theta \int_0^c r^2\,dr$$

$$= \tfrac{2}{3}\pi c^3 \sin\theta\,d\theta.$$

Also, if y be the ordinate of P, the area of the annulus generated by the revolution of QQ' about the axis is equal to $2\pi y\,dy$.

Hence the brightness at Q varies as

$$\frac{\tfrac{2}{3}\pi c^3 \sin\theta\,d\theta}{2\pi y\,dy}$$

$$\propto \frac{\sin\theta\,d\theta}{dx}, \text{ where } x \text{ is the abscissa of } P,$$

$$\propto \frac{\sin\theta\,d\theta}{dr}:$$

but $$\frac{1}{r} = 1 + \cos\theta, \quad \frac{dr}{r^2} = \sin\theta\, d\theta:$$

hence the brightness at Q varies as $\frac{1}{r^2}$.

9. A hollow homogeneous cylinder, of given material, which is perfectly brittle and incompressible, is partially inserted into a fixed horizontal tube just wide enough to admit it: prove that the greatest length which the free portion of the cylinder can have, without snapping off, varies as the square root of the radius of its external surface.

Let μ denote the tenacity of the material of the cylinder, a the radius of its internal and a' of its external surface, c its free length when it is on the point of snapping off. Let r be the distance of any point in the mouth of the tube from its centre, θ the inclination of r to the line drawn vertically upwards from the centre of the mouth.

Then the moment of the tenacity, to prevent snapping off, about the lowest point of the mouth, is equal to

$$\begin{aligned} &\mu \int_0^{2\pi} \int_a^{a'} r\, d\theta\, dr\, (a' + r\cos\theta) \\ &= \mu \int_0^{2\pi} d\theta \left(a' . \frac{a'^2 - a^2}{2} + \frac{a'^3 - a^3}{2} \cos\theta\right) \\ &= \mu\pi a' (a'^2 - a^2). \end{aligned}$$

Again, ρ being the density of the material of the cylinder, the moment of the weight of the cylinder about the same point, is equal to

$$g\rho . \pi c (a'^2 - a^2) . \tfrac{1}{2} c.$$

Hence, when the cylinder is on the point of snapping off,

$$\tfrac{1}{2} g\rho . \pi c^2 (a'^2 - a^2) = \mu\pi a' (a'^2 - a^2),$$

and therefore $c \propto a'^{\frac{1}{2}}$.

10. A centre of force, repelling inversely as the square of the distance, lies below the surface of a homogeneous inelastic fluid, which is also acted on by gravity and is at rest: the

intensity of the force, at a point in the surface of the fluid vertically above its centre, is equal to that of gravity: prove that the external surface of the fluid has a horizontal asymptotic plane, and that the centre of force is environed by an internal cavity, the summit of which is at the external surface of the fluid.

Find the volume of the cavity in terms of its length.

Let P, (fig. 25), be any point in the fluid, O the centre of force: let $OP = r$, $p =$ the pressure at P, $\rho =$ the density of the fluid. Let xx' be a vertical line through O, and Oy be horizontal.

Then, for the equilibrium of the fluid, we have

$$\frac{dp}{\rho} = g\,dx + \frac{\mu}{r^2}\,dr,$$

$$\frac{p}{\rho} = gx - \frac{\mu}{r} + C.$$

Let A be a point, vertically above O, in the surface of the fluid, at which the intensity of the central force is equal to that of gravity: then, if $OA = a$, $\frac{\mu}{a^2}$ is equal to g: also $p = 0$ when $x = -a$: hence

$$\frac{p}{\rho} = gx - \frac{a^2 g}{r} + C,$$

$$0 = -ga - ag + C,$$

and therefore $$\frac{p}{g\rho} = x + 2a - \frac{a^2}{r} \quad\text{..........................(1)}.$$

The equation to the free surface is

$$r = \frac{a^2}{x + 2a} \quad\text{.......................... (2)}.$$

The equation (2) represents a surface generated by the revolution of a curve consisting of the two portions $HABAK$ and HLK, HK being a horizontal asymptote.

From (2) we readily see that $OB = a(\sqrt{2} - 1)$, and thus $AB = a\sqrt{2}$. Also $AV = 2a$, $AL = a(\sqrt{2} + 1)$.

The portion HLK of the curve must be rejected, because, as the equation (1) shews, p is negative when $x + 2a$ is negative.

In order that there may be fluid at any point, $\frac{p}{g\rho}$ must be positive: hence, by (1), we must have

$$r > \frac{a^2}{x+2a}:$$

this shews that AB and also $HAKV$ are free from fluid.

Thus we see that there is no fluid above the infinite arcs AH, AK, and that the point O is within a cavity. The volume of the cavity is equal to

$$\pi\int_{-a}^{a(\sqrt{2}-1)} y^2 dx$$

$$= \pi\int_{-a}^{a(\sqrt{2}-1)} \left\{\frac{a^4}{(x+2a)^2} - x^2\right\} dx, \text{ by (2)},$$

$$= \pi \left[-\frac{a^4}{x+2a} - \tfrac{1}{3}x^3\right]_{-a}^{a(\sqrt{2}-1)}$$

$$= \pi\left\{a^3 - \tfrac{1}{3}a^3 - \frac{a^3}{1+\sqrt{2}} - \tfrac{1}{3}a^3(\sqrt{2}-1)^3\right\}$$

$$= \pi a^3\left\{\tfrac{2}{3} - (\sqrt{2}-1) - \tfrac{1}{3}(5\sqrt{2}-7)\right\}$$

$$= \tfrac{4}{3}\pi a^3(3-2\sqrt{2}).$$

Let $AB=c$: then $a\sqrt{2}=c$: hence the volume of the cavity is equal to

$$\frac{2\pi c^3}{3\sqrt{2}}(3-2\sqrt{2}).$$

11. A carriage is travelling along any level road: prove that the sum of the squares of the shadows cast on the ground by any two spokes of a wheel, which are at right angles to each other, varies during the journey as the square of the secant of the Sun's zenith distance.

Prove also that, if the road run due east and west,

$$\sin\alpha = \frac{\tan 2\theta}{\tan 2z},$$

α being the azimuth and z the zenith distance of the Sun, and θ the corresponding inclination of a spoke to the horizon when its shadow is greatest or least.

The shadow of a spoke on the ground will be the same as on a horizontal plane through the centre of the wheel. We will suppose such a plane to receive the shadow.

Let OP (fig. 26) be a spoke, O the centre of the wheel: let PN, a vertical line, meet the horizontal plane through O in N: let OP be the shadow on this horizontal plane. Join ON, PP', $P'N$, and draw $P'O'$, at right angles to ON produced. Let $\angle NP'O' = \alpha$, and $\angle NPP' = z$, $\angle PON = \theta$. Then, a being the length of the spoke OP, and c of its shadow OP',

$$c^2 = (a\cos\theta)^2 + (a\sin\theta\tan z)^2 - 2a\cos\theta\,.\,a\sin\theta\tan z\cos\left(\frac{\pi}{2}+\alpha\right),$$

$$\frac{c^2}{a^2} = \sin^2\theta\tan^2 z + \cos^2\theta + \sin\alpha\,.\,\sin 2\theta\,.\,\tan z\,\dots\dots\dots(1).$$

Let c' be what c becomes when θ is replaced by $\frac{1}{2}\pi+\theta$: then

$$\frac{c'^2}{a^2} = \cos^2\theta\tan^2 z + \sin^2\theta - \sin\alpha\,.\,\sin 2\theta\,.\,\tan z\,\dots\dots\dots(2).$$

From (1) and (2), we see that

$$c^2 + c'^2 = a^2\sec^2 z,$$

which shews that, whether the road be winding or straight, $c^2 + c'^2$ varies as the square of $\sec z$.

Again, from (1),

$$\frac{2c^2}{a^2} - \sec^2 z = (1-\tan^2 z)\cos 2\theta + 2\sin\alpha\sin 2\theta\tan z:$$

when c is a maximum or minimum, for given values of α and z,

$$0 = (1-\tan^2 z)\sin 2\theta - 2\sin\alpha\cos 2\theta\tan z,$$

and therefore $$\sin\alpha = \frac{\tan 2\theta}{\tan 2z},$$

where, if the road run due east or west, α is the Sun's azimuth.

The following is a different solution of the problem:

Let O (fig. 27) be the centre of the wheel, OA, OB, vertical and horizontal radii, AC, CB', their projections upon the horizontal plane, OP another radius inclined at an angle θ to the horizon, CQ its projection, QN the projection of PM: then

$$CB' = OB, \qquad QN = \quad PM \quad = OP\cos\theta,$$
$$CA = OA\tan z, \qquad CN = OM\tan z = OP\sin\theta \, . \tan z.$$

Now if a line bisect a set of parallel lines, its orthogonal projection will bisect the projections of the parallel lines. Hence, since radii of a circle at right angles to each other bisect each the system of chords parallel to the other, they will be projected into conjugate diameters of the ellipse which is the projection of the circle. Therefore CA, CB', are conjugate semi-diameters inclined at an angle $\frac{1}{2}\pi - \alpha$; and, if a', b', be the lengths of the shadows of any other spokes at right angles to each other, we have

$$a'^2 + b'^2 = AC^2 + B'C^2, \text{ by a property of the ellipse,}$$
$$= r^2\tan^2 z + r^2, \ r \text{ being the radius of the wheel,}$$
$$= r^2\sec^2 z.$$

Again, $CQ^2 = CN^2 + QN^2 + 2CN.QN\cos ACB'$

$$= r^2\sin^2\theta\tan^2 z + r^2\cos^2\theta + 2r^2\sin\theta\cos\theta\tan z\sin\alpha;$$

and CQ is to be a maximum or a minimum by the variation of θ; therefore the value of θ will be given by the equation,

$$0 = 2(\cos^2\theta - \sin^2\theta)\tan z\sin\alpha - 2(1-\tan^2 z)\sin\theta\cos\theta:$$

whence
$$\frac{2\tan z}{1-\tan^2 z}\sin\alpha = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta},$$

and
$$\sin\alpha = \frac{\tan 2\theta}{\tan 2z}.$$

12. OA, OB, OC, are meridians on a surface of revolution, passing through three points A, B, C, which are connected together by the shortest arcs BC, CA, AB: BC cuts OB, OC, at angles λ_1, λ_2; CA cuts OC, OA, at angles λ_3, λ_4; and AB cuts OA, OB, at angles λ_5, λ_6: prove that

$$\sin\lambda_1 . \sin\lambda_3 . \sin\lambda_5 = \sin\lambda_2 . \sin\lambda_4 . \sin\lambda_6.$$

Let P, Q, (fig. 28), be two points, indefinitely near to each other, on the shortest arc between any two proposed points on the surface. In the arc OQ take q such that the arc Oq is equal to the arc OP. Let y = the distance of PQ from the axis of revolution, $d\phi$ = the angle between the planes of OP, OQ, and $Qq = ds$.

Then $$PQ^2 = ds^2 + y^2 d\phi^2,$$
and therefore, since s must be some function of y,

$$PQ = \left\{ f(y) \frac{dy^2}{d\phi^2} + y^2 \right\}^{\frac{1}{2}} d\phi$$
$$= \{p^2 f(y) + y^2\}^{\frac{1}{2}} d\phi, \quad \text{where } p = \frac{dy}{d\phi}.$$

Under the condition of the problem, we must make

$$\int \{p^2 f(y) + y^2\}^{\frac{1}{2}} d\phi$$

a minimum.

By the formula of the Calculus of Variations

$$V = Pp + \beta,$$

where β is a constant, we have

$$\{p^2 f(y) + y^2\}^{\frac{1}{2}} = \frac{p^2 f(y)}{\{p^2 f(y) + y^2\}^{\frac{1}{2}}} + \beta,$$
$$y^2 = \beta \{p^2 f(y) + y^2\}^{\frac{1}{2}},$$
$$y^4 = \beta^2 \left(y^2 + \frac{ds^2}{d\phi^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But, if θ denote the angle between PQ, Qq,

$$y d\phi = \tan\theta \, . \, ds:$$

hence, from (1), $$y^4 = \beta^2 y^2 \operatorname{cosec}^2\theta,$$
$$y \sin\theta = \beta.$$

By this result it appears that, if a, b, c, be the respective distances of the points A, B, C, from the axis of revolution,

$$b \sin\lambda_1 = c \sin\lambda_2,$$
$$c \sin\lambda_3 = a \sin\lambda_4,$$
$$a \sin\lambda_5 = b \sin\lambda_6,$$

and therefore

$$\sin\lambda_1 \, . \sin\lambda_3 \, . \sin\lambda_5 = \sin\lambda_2 \, . \sin\lambda_4 \, . \sin\lambda_6 .\text{*}$$

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* For this problem and also for problem (6) the Junior Moderator is indebted to Mr. R. L. Ellis of Trinity College.

13. A little animal, the mass of which is m, is resting on the middle point of a thin uniform quiescent bar, the mass of which is m' and the length $2a$, the ends of the bar being attached by small rings to two smooth fixed rods at right angles to each other in a horizontal plane: supposing the animal to start off along the bar with a velocity V, relatively to the bar, prove that, θ being the inclination of the bar to either rod, the angular velocity initially impressed upon the bar will be equal to

$$\frac{3m}{3m+4m'} \cdot \frac{V \sin 2\theta}{a}.$$

Let AB, (fig. 29), be the bar, OAx, OBy, the rods; θ the angle BAO, m the mass of the little animal, m' the mass of the bar, k the radius of gyration of the bar about its middle point G, ω the initial angular velocity of the bar.

Since the impulses on the system pass through the intersection C of the normals to the rods at A, B, the algebraic sum of the initial moments of the momenta of the system about this point must vanish. Now the sum of the initial moments of the momenta of the particles of the rod about C is equal to $m'\omega(a^2+k^2)$. Again, the velocity of G is perpendicular to OC and is equal to $a\omega$, and therefore the initial velocity of the little animal in this direction is equal to

$$a\omega - V \sin 2\theta:$$

hence the moment of the momentum of the little animal about C is equal to

$$m(a\omega - V \sin 2\theta)\, a.$$

Hence $$m'\omega(a^2+k^2) + m(a\omega - V \sin 2\theta)\, a = 0,$$

$$\omega = \frac{3m}{3m+4m'} \cdot \frac{V \sin 2\theta}{a}.^{*}$$

14. A narrow tube, in the form of a common helix, is wound round an upright cylinder, initially at rest, which is pierced

* A solution of this problem, based on the same physical conception, was given by one of the Candidates in the Senate-House Examination.

by two smooth fixed rods, parallel to each other and horizontal: supposing a molecule to be placed within the tube, at a point of which the distance from the axis of the cylinder is parallel to the rods, find the velocity of the cylinder when the molecule arrives at any proposed point of the tube.

Prove that, m, m', being the masses of the molecule and cylinder, the velocities which the cylinder has acquired, at the successive arrivals of the molecule at points most distant from the plane in which the axis of the cylinder moves, will have their greatest values when, α being the inclination of the helix to the horizon,

$$\tan^2\alpha = \frac{m'}{m+m'}.$$

Let Ox, (fig. 30), be the line of motion of the centre C of the base of the cylinder: let Oz be a vertical line. Let P be the place of the molecule at any time t. Draw the vertical line PN to meet the circumference of the base of the cylinder in N: draw NM at right angles to Ox and join NC.

Let $OM = x$, $MN = y$, $PN = z$, $OC = x'$, $\angle OCN = \theta$. Then, by the principle of Vis Viva, if c be the initial value of z,

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right) + m'\frac{dx'^2}{dt^2} = 2mg(c-z) \;\ldots\ldots\; (1).$$

By the Principle of the Conservation of the Motion of the Centre of Gravity,

$$m\frac{dx}{dt} + m'\frac{dx'}{dt} = 0 \;\ldots\ldots\ldots\ldots\ldots\; (2);$$

also, from the geometry, if a be the radius of the cylinder,

$$x' - x = a\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$y = a\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$c - z = a\theta\tan\alpha \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From (2) and (3), we have

$$\frac{dx}{dt} = \frac{m'a}{m+m'}\sin\theta\frac{d\theta}{dt} \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and

$$\frac{dx'}{dt} = -\frac{ma}{m+m'}\sin\theta\frac{d\theta}{dt} \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

From (1), (4), (5), (6), (7), there is

$$\{m\cos^2\theta + m' + (m+m')\tan^2\alpha\}\frac{d\theta^2}{dt^2} = \frac{2g}{a}(m+m')\,\theta\tan\alpha,$$

and therefore, by (7),

$$\frac{dx'^2}{dt^2} = \frac{m^2a}{m+m'}\cdot\frac{2g\theta\tan\alpha\sin^2\theta}{m\cos^2\theta + m' + (m+m')\tan^2\alpha}.$$

If $\theta = (2\lambda+1)\frac{\pi}{2}$, λ being an integer,

$$\frac{dx'^2}{dt^2} = \frac{m^2a}{m+m'}\cdot\frac{(2\lambda+1)\pi g\tan\alpha}{m' + (m+m')\tan^2\alpha},$$

and therefore $\frac{dx'}{dt}$ is a maximum by the variation of α when

$$m' + (m+m')\tan^2\alpha - 2(m+m')\tan^2\alpha = 0,$$

or when

$$\tan^2\alpha = \frac{m'}{m+m'}.$$

TUESDAY, *Jan.* 6. 9 *to* 12.

1. PARALLELOGRAMS upon the same base, and between the same parallels, are equal to one another.

ABC is an isosceles triangle, of which *A* is the vertex: *AB*, *AC*, are bisected in *D* and *E* respectively; *BE*, *CD*, intersect in *F*: shew that the triangle *ADE* is equal to three times the triangle *DEF*.

Join *AF*, (fig. 31), meeting *DE* in *G*. Then, since the triangle is isosceles, the triangle *AFD* is equal to the triangle *AFE*, and the triangle *GFD* to the triangle *GFE*.

Now $\Delta AED = \Delta BED$, since base $AD =$ base DB;

that is, $2\Delta AGD = \Delta BFD + 2\Delta DFG$.

But $\Delta AFD = \Delta BFD$, since the bases AD, BD, are equal,

or $\Delta AGD + \Delta DGF = \Delta BFD$.

Hence $2\Delta AGD = \Delta AGD + \Delta DGF + 2\Delta DFG$,

or $\Delta AGD = 3\Delta DGF$;

and therefore

$$\Delta AED = 3\Delta DEF.$$

2. In any triangle, the square on the side subtending either of the acute angles is less than the sum of the squares on the sides including this angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular drawn to this side, produced if necessary, from the opposite angular point.

The base of a triangle is given and is bisected by the centre of a given circle, the circumference of which is the locus of the vertex: prove that the sum of the squares on the two sides of the triangle is invariable.

Let AB, (fig. 32), be the base of the triangle, O the centre of the circle, C the vertex of the triangle. Join CA, CB, CO, and draw CM at right angles to AB.

Then

square on AC = square on OA + square on OC
+ twice rectangle OA, OM:

also square on BC + twice rectangle OB, OM
= square on OB + square on OC,

or square on BC + twice rectangle OA, OM
= square on OA + square on OC.

Hence

square on AC + square on BC
= twice square on OA + twice square on OC
= an invariable magnitude.

3. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Prove also that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

Let $ABCD$ (fig. 33) be any quadrilateral in a circle. Join AC.

Then, by the proposition,

the angle in exterior segment $AB + \angle ACB = 2$ right angles,

and .... $AD + \angle ACD = 2$ right angles.

Hence the sum of the angles in the two exterior segments AB, AD, together with the angle BCD, $= 4$ right angles; and, similarly, the sum of the angles in the two exterior segments CB, CD, together with the angle BAD, $= 4$ right angles.

Hence the sum of the angles in the four exterior segments, together with the angles BAD, BCD, $=8$ right angles. But the angles BAD, $BCD=2$ right angles, by the proposition. Hence the sum of the angles in the segments is equal to six right angles.

4. Inscribe a circle in a given triangle.

Circles are inscribed in the two triangles formed by drawing a perpendicular from an angle of a triangle upon the opposite side, and analogous circles are described in relation to the two other like perpendiculars: prove that the sum of the diameters of the six circles together with the sum of the sides of the original triangle is equal to twice the sum of the three perpendiculars.

Draw AP, (fig. 34), at right angles to the base BC of the triangle ABC. Let the circle, inscribed in the triangle ABP, touch AP, BP, AB, in M, N, E, respectively.

Then
$$\begin{aligned} AP+BP &= AM+MP+BN+NP \\ &= AE+MP+BE+MP \\ &= AB + \text{twice the radius.} \end{aligned}$$

Hence, A_b denoting the diameter of the circle,
$$A_b+AB=AP+BP.$$

Similarly, A_c denoting the diameter of the circle inscribed in the triangle ACP,
$$A_c+AC=AP+CP.$$

Hence
$$A_b+A_c+AB+CA=2AP+BC.$$

Similarly, B_c, B_a, and C_a, C_b, denoting analogous diameters in relation to the two other like perpendiculars of the triangle ABC,
$$B_c+B_a+BC+AB=2BQ+CA,$$
and
$$C_a+C_b+CA+BC=2CR+AB.$$

Hence, adding, and taking equals from equals,
$$A_b+A_c+B_c+B_a+C_a+C_b+BC+CA+AB=2(AP+BQ+CR).$$

5. Similar triangles are to one another in the duplicate ratio of their homologous sides.

Any two straight lines, BB', CC', drawn parallel to the base DD' of a triangle ADD', cut AD in B, C, and AD' in B', C': BC', $B'C$, are joined: prove that the area ABC' or $AB'C$ varies as the rectangle contained by BB', CC'.

Area ACB' : area ABB' :: $AC : AB$, by *Euclid* (VI. 1),
:: $CC' : BB'$, by *Euclid* (VI. 4).

But, by *Euclid* (VI. 1),

$CC' : BB'$ :: rectangle BB', CC' : square on BB'.

Hence

area ACB' : area ABB' :: rectangle BB', CC' : square on BB',

whence

area ACB' : rectangle BB', CC' :: area ABB' : square on BB'.

But, by *Euclid* (VI. 19),

area $ABB' \propto$ square on BB':

hence area $ACB' \propto$ rectangle BB', CC':

so also area $ABC' \propto$ rectangle BB', CC'.

6. If two parallel planes be cut by another plane, their common sections with it are parallel.

A triangular pyramid stands on an equilateral base, and the angles at the vertex are right angles; shew that the sum of the perpendiculars on the faces from any point of the base is constant.

Since the angles at the vertex are right angles, each plane face forming the vertex is at right angles to the other two.

Let BAC (fig. 35) be the equilateral base and D the vertex.

Then, since $BA = BC$, and BD is common, and the angles, BDA, BDC, are right angles, $AD = DC$, and, similarly, $= BD$.

Now let P be any point in the base. Through P draw a plane parallel to BDC cutting the planes BDA, CDA, in ba, ac. Then, by the proposition, ba, ac, are respectively parallel to BD, DC. From P draw Pm, Pn, perpendiculars to ac, ab, respectively. Then Pm, Pn, aD, are the perpendicular distances from P on the three faces, and Pm is parallel to DB.

Therefore the triangle Pmc is similar to the triangle BDC.

Hence $\quad Pm = mc$,

$$Pm + Pn = ac = aA, \text{ since } ac \text{ is parallel to } DC,$$

and therefore $\quad Pm + Pn + aD = AD$, and is constant.

8. Prove that, in the parabola, $SY^2 = SP.SA$.

A circle is described on the latus rectum as diameter, and a common tangent QP is drawn to it and the parabola: shew that SP, SQ, make equal angles with the latus rectum.

Let S (fig. 36) be the focus, SL the semi-latus-rectum. Through Q draw QRQ', parallel to the axis of the parabola, meeting SL in R and SP in Q'. Produce PQ, meeting the axis in T. Then, since ST is equal to SP and SQ is common, and the angle SQT is equal to the angle SQP, hence $TQ = QP$, and therefore $SQ' = Q'P$.

But, by the proposition, since SQ is the perpendicular from S on the tangent at P,

$$SQ^2 = SP.SA,$$

or $$4SA^2 = SP.SA, \text{ since } SQ = 2SA;$$

hence $$SP = 4SA = 2SQ,$$

and therefore $$SQ' = \tfrac{1}{2}SP = SQ,$$

and therefore, evidently, since QRQ' is perpendicular to SL, the angle QSL is equal to the angle PSL.

9. Prove that the focal distances of any point of an ellipse make equal angles with the tangent at the point.

PG is a normal to an ellipse, terminating in the major axis; the circle, of which PG is a diameter, cuts SP, HP, in K, L, respectively: prove that KL is bisected by PG, and is perpendicular to it.

Let O (fig. 37) be the point of intersection of PG, KL. Join KG, LG.

Then, in the triangles KPG, LPG, $\angle KPG = \angle LPG$, and, since PG is a diameter, $\angle PKG = \angle PLG$; also, PG is common to both triangles. Hence, by *Euclid* (I. 26), $PK = PL$, $GK = GL$. Again, in the triangles POK, POL, $PK = PL$, PO is common, and $\angle KPO = \angle LPO$: hence, by *Euclid* (I. 4), $KO = LO$. Moreover, in the triangles POK, POL, $PK = PL$, $\angle OPK = \angle OPL$, and PO is common: hence, by *Euclid* (I. 8), $\angle POK = \angle POL$, that is, PG is perpendicular to KL.

10. The perpendiculars from the foci of an ellipse upon the tangent meet the tangent in the circumference of a circle.

Prove also that if from H a line be drawn parallel to SP, it will meet the perpendicular SY in the circumference of a circle.

Let SP (fig. 38) be produced to meet the perpendicular from H upon the tangent at P in H'. Draw HP' parallel to SP, meeting SY in P'. Then $SH'HP'$ is a parallelogram and $HP' = SH' = 2AC$. Therefore the locus of P' is a circle with centre H and radius equal to the major axis.

11. If tangents be drawn at the vertices of the axes of an hyperbola, the diagonals of the rectangle so formed are asymptotes to the four curves.

Prove that a perpendicular, drawn from the focus of an hyperbola to the asymptote, will intersect it in the directrix.

Let CT (fig. 39) be an asymptote, SY a perpendicular from the focus S upon CT. Draw YE at right angles to CS.

Since CT, an asymptote, is a tangent, therefore Y is a point in the circle the centre of which is C and radius equal to CA: hence

$$CY = CA.$$

But $$CS : CY :: CY : CE:$$

hence $$CS : CA :: CA : CE:$$

consequently E is a point in the directrix.

TUESDAY, *Jan.* 6. $1\frac{1}{2}$ *to* 4.

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4. PROVE a rule for extracting the square root of a compound algebraical quantity.

Shew that, if

$$x^4 + ax^3 + bx^2 + cx + d$$

be a complete square, the coefficients satisfy the equation

$$c^2 - a^2d = 0.$$

Is it necessary that the coefficients satisfy any other equation?

Extracting the square root of

$$x^4 + ax^3 + bx^2 + cx + d$$

in the usual manner, we have the following operation;

$$\begin{array}{ll|l}
 & x^4 + ax^3 + bx^2 + cx + d & x^2 + \frac{a}{2}x + \frac{1}{2}\left(b - \frac{a^2}{4}\right) \\
 & \underline{x^4} & \\
2x^2 + \frac{a}{2}x & \quad ax^3 + bx^2 & \\
 & \quad \underline{ax^3 + \frac{a^2}{4}x^2} & \\
2x^2 + ax + \frac{1}{2}\left(b - \frac{a^2}{4}\right) & \quad \left(b - \frac{a^2}{4}\right)x^2 + cx + d & \\
 & \quad \underline{\left(b - \frac{a^2}{4}\right)x^2 + \frac{a}{2}\left(b - \frac{a^2}{4}\right)x + \frac{1}{4}\left(b - \frac{a^2}{4}\right)^2} & \\
 & \quad \left\{c - \frac{a}{2}\left(b - \frac{a^2}{4}\right)\right\}x + d - \frac{1}{4}\left(b - \frac{a^2}{4}\right)^2. &
\end{array}$$

Now, if the expression be a complete square, this remainder must vanish; and, that it may vanish for general values of x, we must have

$$c - \frac{a}{2}\left(b - \frac{a^2}{4}\right) = 0 \dots\dots\dots\dots\dots\dots (1),$$

$$d - \frac{1}{4}\left(b - \frac{a^2}{4}\right)^2 = 0 \dots\dots\dots\dots\dots\dots (2);$$

whence, eliminating $b - \frac{a^2}{4}$, we obtain

$$c^2 - a^2 d = 0 \dots\dots\dots\dots\dots\dots (3).$$

The coefficients must satisfy the equations (1) and (2), and therefore either of these equations together with the equation (3) which results from them.

6. Find the number of permutations of n things taken r together.

If the number of permutations of n things taken r together be denoted by the symbol

$$^nP_r;$$

shew that the number of such permutations, in which p particular things occur, will be

$$^rP_p \,.\, ^{n-p}P_{r-p}.$$

Let the p particular things be removed; then there will be $n-p$ things remaining, which will admit of

$$\frac{(n-p)(n-p-1)\dots\{n-p-(r-p)+1\}}{1.2\dots(r-p)}$$

combinations with $r-p$ things in each group. Now, if the p things be restored to each group, we shall have

$$\frac{(n-p)(n-p-1)\dots(n-r+1)}{1.2\dots(r-p)}$$

combinations of n things, taken r together, in which p particular things occur; and, since each group admits of $1.2\dots r$ permutations, the corresponding number of permutations will be

$$1.2\dots r \times \frac{(n-p)\dots(n-r+1)}{1.2\dots(r-p)},$$

or $\quad r(r-1)\ldots(r-p+1)\times(n-p)(n-p-1)\ldots(n-r+1),$

or $$^rP_p \,.\, ^{n-p}P_{r-p}.$$

8. Define the sine of an angle, and prove from your definition that for all values of θ numerically less than π, $\sin(\pi-\theta)=\sin\theta$.

Trace the variation in sign of the expression

$$\cos(\pi\sin\theta).\cos(\pi\cos\theta),$$

as θ varies from 0 to $\dfrac{\pi}{2}$.

$$\begin{array}{llll}
& \cos(\pi\sin\theta) \text{ is positive} & \text{from } \theta=0 & \text{to } \theta=\frac{\pi}{6}, \\
& \quad\ldots \text{ negative} \ldots\ldots & \theta=\frac{\pi}{6} & \ldots\ \theta=\frac{\pi}{2}; \\
& \cos(\pi\cos\theta) \ldots \text{ negative} \ldots\ldots & \theta=0 & \ldots\ \theta=\frac{\pi}{3}, \\
& \quad\ldots \text{ positive} \ldots\ldots & \theta=\frac{\pi}{3} & \ldots\ \theta=\frac{\pi}{2}; \\
\text{therefore the product} & \ldots \text{ negative} \ldots\ldots & \theta=0 & \ldots\ \theta=\frac{\pi}{6}, \\
& \quad\ldots \text{ positive} \ldots\ldots & \theta=\frac{\pi}{6} & \ldots\ \theta=\frac{\pi}{3}, \\
& \quad\ldots \text{ negative} \ldots\ldots & \theta=\frac{\pi}{3} & \ldots\ \theta=\frac{\pi}{2}.
\end{array}$$

9. Find an expression for all the angles which have the same sine. Hence, if $\sin 3\theta$ be given, find the number of values of $\tan\theta$ which will be generally obtained; and illustrate the result geometrically.

Let α be the least positive angle, the sine of which is the given sine;

then $$3\theta=n\pi+(-1)^n.\alpha.$$

and $$\tan\theta=\tan\frac{n\pi+(-1)^n.\alpha}{3}.$$

Now n is of one of the forms $3r$, $3r \pm 1$; and from these several forms we obtain

$$\tan\theta = \tan\left\{r\pi + (-1)^r . \frac{\alpha}{3}\right\} = \pm \tan\frac{\alpha}{3}, \text{ as } r \text{ is even or odd,}$$

$$\tan\theta = \tan\left\{r\pi + \frac{\pi - (-1)^r . \alpha}{3}\right\} = \tan\frac{\pi \mp \alpha}{3} \ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\tan\theta = \tan\left\{r\pi - \frac{\pi + (-1)^r . \alpha}{3}\right\} = -\tan\frac{\pi \pm \alpha}{3} \ldots\ldots\ldots\ldots\ldots\ldots;$$

there are therefore, in general, six different values.

11. Determine the expression for the cosine of an angle of a triangle in terms of the sides, and deduce the expression for the sine.

If θ and ϕ be the greatest and least angles of a triangle, the sides of which are in arithmetic progression, prove that

$$4(1 - \cos\theta)(1 - \cos\phi) = \cos\theta + \cos\phi.$$

Let $a - b$, a, $a + b$, be the sides of the triangle;

then $$\cos\theta = \frac{a^2 + (a-b)^2 - (a+b)^2}{2a . (a - b)}$$

$$= \frac{a - 4b}{2(a - b)},$$

and similarly $$\cos\phi = \frac{a + 4b}{2(a + b)}.$$

Hence $$(2\cos\theta - 1)\,a = (2\cos\theta - 4)\,b,$$

$$(2\cos\phi - 1)\,a = (4 - 2\cos\phi)\,b;$$

and therefore $$\frac{2\cos\theta - 1}{2\cos\phi - 1} = \frac{\cos\theta - 2}{2 - \cos\phi};$$

whence $$4\cos\theta . \cos\phi - 5(\cos\theta + \cos\phi) + 4 = 0,$$

and therefore $$4(1 - \cos\theta)(1 - \cos\phi) = \cos\theta + \cos\phi.$$

12. A quadrilateral can be inscribed in a circle; find the tangent of half of one of its angles in terms of its sides. If a circle can be inscribed in the quadrilateral, shew that the fourth

root of the product of its sides is a mean proportional between its semi-perimeter and the radius of the inscribed circle.

If a circle can be inscribed in the quadrilateral, the sums of its opposite sides are equal, and therefore, if a, b, c, d, be the lengths of the sides AB, BC, CD, DA,

$$a + c = b + d.$$

Let r be the radius of the inscribed circle: then it is easily shewn that

$$a = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right),$$

$$c = r\left(\cot\frac{C}{2} + \cot\frac{D}{2}\right);$$

therefore $$a + c = r\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} + \cot\frac{D}{2}\right).$$

But $$\tan^2\frac{A}{2} = \frac{(s-a).(s-d)}{(s-b).(s-c)} = \frac{bc}{ad},$$

and $\tan\frac{B}{2}$, &c. are given by similar forms; therefore

$$(a+c) = r\left\{\sqrt{\left(\frac{ad}{bc}\right)} + \sqrt{\left(\frac{ab}{cd}\right)} + \sqrt{\left(\frac{bc}{ad}\right)} + \sqrt{\left(\frac{cd}{ab}\right)}\right\}$$

$$= r\,.\,\frac{ad + bc + ab + cd}{\sqrt{(abcd)}}$$

$$= r\,.\,\frac{(a+c)^2}{\sqrt{(abcd)}},$$

whence

$$\sqrt{(abcd)} = r\,.\,(a+c)$$

$$= r\,.\,(\text{semi-perimeter}).$$

WEDNESDAY, *Jan.* 7. 9 *to* 12.

1. ASSUMING that the resultant of two forces, acting at a point, is represented in direction by the diagonal of a parallelogram, the sides of which represent the forces in direction and magnitude; shew that the diagonal will also represent the resultant in magnitude.

Shew that within a quadrilateral, no two sides of which are parallel, there is but one point, at which forces, acting towards the corners and proportional to the distances of the point from them, can be in equilibrium.

Let $ABCD$ (fig. 40) be the quadrilateral, and suppose P such a point that the forces represented in direction and magnitude by PA, PB, PC, PD, are in equilibrium. Bisect AB in m, and CD in n; and join Pm, Pn; Pm, Pn, are the semi-diagonals of the parallelograms, the sides of which are PA, PB, and PC, PD, respectively, and therefore will represent in magnitude and direction the resultants of the forces represented by PA, PB, and by PC, PD. Hence Pm, Pn, must be equal to each other and in the same straight line, or P must be the middle point of the line joining the points of bisection of two opposite sides of the quadrilateral. And since, by a known theorem, the lines joining the middle points of the opposite sides of a quadrilateral mutually bisect each other, there is one such point and one only.

2. Shew that if three forces acting in one plane hold a body in equilibrium, they either pass through a point or are parallel to each other.

A heavy equilateral triangle, hung up on a smooth peg by a string the ends of which are attached to two of its angular points, rests with one of its sides vertical; shew that the length of the string is double the altitude of the triangle.

Let ABC (fig. 41) be the triangle, A, B, the points of attachment of the string, and consequently BC the vertical side, E the peg. Draw AD perpendicular to BC, and take AG two-thirds of AD; G is the centre of gravity of the triangle.

Since the triangle is held in equilibrium by the tensions of the string acting along AE, BE, and by its weight acting through G, the directions of these forces must pass through the same point E; and, since the tension of the string is the same throughout, the vertical through G must bisect the angle AEB.

Produce EB to meet AD produced in F, and join GB; then, in the triangles AGE, FGE,

$$\text{right } \angle AGE = \text{right } \angle FGE, \quad \angle AEG = \angle FEG,$$

and the side EG is common; therefore $AG = GF$, and $AE = EF$.

But $AG = 2GD$; therefore $GF = 2GD$; and, since GE and DB are parallel, $EB = BF$, and $AE = 2EB$.

Also, since G and B are the middle points of AF and EF, GB is parallel to AE.

Hence $\qquad \angle BEG = \angle AEG = \angle EGB$;

and therefore $\qquad EB = BG = AG = \frac{2}{3}AD$.

And the length of the string, which is AE together with EB,

$$= 3EB$$

$$= 2AD.$$

3. Find the relation of the Power to the Weight in the single moveable pully, when the strings are not parallel.

An endless string hangs at rest over two pegs in the same horizontal plane, with a heavy pully in each festoon of the string: if the weight of one pully be double that of the other, shew that the angle between the portions of the upper festoon must be greater than 120°.

Let W, $2W$, be the weights of the pullies; 2θ, 2ϕ, the inclinations of the portions of the upper and lower festoon, (fig. 42); T the tension of the string, which is the same throughout.

For the equilibrium of the upper pully, resolving vertically, we have

$$W = 2T\cos\theta;$$

and similarly for the lower pully,

$$2W = 2T\cos\phi.$$

Hence $$2\cos\theta = \cos\phi.$$

Now ϕ can never be zero, and consequently $\cos\phi$ must be always less than 1; therefore $\cos\theta$ must be less than $\frac{1}{2}$, or θ greater than $\frac{1}{3}\pi$, and consequently the angle between the portions of the upper festoon greater than 120°.

5. Define the centre of gravity of a heavy body; and determine the position of the centre of gravity of a pyramid on a triangular base.

Find the centre of gravity of the solid included between two right cones on the same base, the vertex of one cone being within the other; and determine its limiting position if the vertices approach to coincidence.

Since the cones which bound the solid have the same base, their volumes will be proportional to their altitudes; and therefore, if h, h', be those altitudes, and x the height of the centre of gravity of the solid above the common base, we shall have

$$x(h-h') + \frac{h'}{4}\cdot h' = \frac{h}{4}\cdot h;$$

whence $$x = \frac{1}{4}\frac{h^2 - h'^2}{h-h'},$$

$$= \tfrac{1}{4}(h+h');$$

and, when the vertices of the cones approach to coincidence, h' approaches to h as its limit, and consequently x to $\frac{1}{2}h$.

6. State the laws of friction; and explain what is meant by the term 'coefficient of friction.'

A uniform rod is held at a given inclination to a rough horizontal table by a string attached to one of its ends, the other end resting on the table; find the greatest angle at which the string can be inclined to the vertical without causing the end of the rod to slide along the table.

Let AB (fig. 43) be the rod, G its centre of gravity, W its weight, R the vertical reaction of the plane, F the friction, BC the string, inclined at an angle θ to the vertical, β the angle which the rod makes with the plane.

Also, let S be the resultant of R and F, α the angle which its direction makes with the vertical, so that $\tan\alpha = \frac{F}{R}$.

Since there is equilibrium, the direction of S must pass through C, the intersection of the string and the vertical through G; and therefore

$$\frac{GC}{AG} = \frac{\sin(\frac{1}{2}\pi - \alpha - \beta)}{\sin\alpha}, \quad \frac{GC}{BG} = \frac{\sin(\beta + \frac{1}{2}\pi - \theta)}{\sin\theta}.$$

But, since the rod is uniform, $AG = BG$, and therefore

$$\frac{\cos(\alpha+\beta)}{\sin\alpha} = \frac{\cos(\theta-\beta)}{\sin\theta},$$

whence
$$\cot\theta = \cot\alpha - 2\tan\beta.$$

Now θ has its greatest value when $\cot\theta$ is least, and therefore when $\cot\alpha$ is least, or the ratio $\frac{R}{F}$ is least, that is, when the end of the rod is on the point of sliding. Hence, if $\tan\varepsilon$ be the coefficient of friction between the rod and the plane, we have
$$\alpha = \varepsilon, \text{ and } \cot\theta = \cot\varepsilon - 2\tan\beta,$$
for determining the value of θ.

If the string be held on the other side of the vertical, the end of the rod will tend to slide in the contrary direction, and we must replace ε in the formula by $-\varepsilon$. If θ' be the corresponding value of θ, we shall have

$$\cot\theta' = \cot(-\varepsilon) - 2\tan\beta,$$

or
$$\cot(-\theta') = \cot\varepsilon + 2\tan\beta.$$

Hence the greatest angle at which the string can be inclined to the vertical, will be when the string and the rod are on opposite sides of the vertical, in which case the inclination will be

$$\cot^{-1}(\cot\epsilon + 2\tan\beta).$$

7. Define uniform motion and uniformly accelerated motion, and explain how they are measured.

If f be the measure of a uniform acceleration, when t minutes and a feet are taken as the units of time and space, and f' the measure of the same acceleration, when a' feet are taken as the unit of space, find the number of minutes in the unit of time.

If the motion of a point be uniformly accelerated, its acceleration is measured by the increase of the velocity in a unit of time, the velocity being referred to the same unit.

Hence, in the case proposed, taking t' for the unknown unit,

the velocity per t minutes added in t minutes $= fa$ (in feet);

$\therefore$ .... per minute .... $= \dfrac{fa}{t}$,

and .... one minute $= \dfrac{fa}{t^2}$,

.... per t' minutes .... $= \dfrac{fa}{t^2} \cdot t'$,

.... t' minutes $= \dfrac{fa}{t^2} \cdot t'^2$;

and therefore, since $f'a'$ also represents (in feet) the velocity per t' minutes acquired in t' minutes,

$$f'a' = fa \cdot \frac{t'^2}{t^2},$$

whence

$$t' = t\left(\frac{f'a'}{fa}\right)^{\frac{1}{2}}.$$

8. State the second law of motion; and apply it to prove that a force, of uniform intensity and direction, acting on a given particle originally at rest, produces a uniform acceleration of its motion.

State the convention with respect to units which is necessary, in order that the equation $P = Mf$ may represent the relation between the numerical measures of force, mass and acceleration; and supposing the unit of force to be 5lbs. and the unit of acceleration, referred to a foot and a second as units, to be 3, find the unit of mass.

It appears, as the result of experimental facts, that $P \propto Mf$, and therefore that $P = CMf$, the constant C depending on the units assumed. The equation $P = Mf$ implies that the unit of mass is the mass of a body in which the unit of force produces the unit of acceleration, that is, two of the units being given the assumption that $C = 1$ defines the third.

Let m measure the mass of a body whose weight is 5lbs.

Then, since a force 5lbs. produces in m an acceleration g, where g, referred to a foot and a second as units, is 32·2 approximately, and since, when P is given, $f \propto \frac{1}{m}$, it would produce an acceleration '1' in a mass gm, and therefore an acceleration 3 in a mass $\frac{gm}{3}$.

Hence the unit of mass required is to m as g is to 3, and is therefore the mass of a body the weight of which is $\frac{32{\cdot}2}{3} \times 5$lbs. or 53·5lbs. nearly.

Taking 1000 oz. as the weight of a cubic foot of water, the volume of water representing the unit of mass will be $\frac{53{\cdot}5 \times 16}{1000}$ or $\frac{107}{125}$th of a cubic foot.

9. An elastic ball A, moving with a given velocity on a smooth horizontal plane, impinges directly on a ball B of the same radius, at rest; determine the velocity of each after the impact, indicating at what points of your reasoning any law of motion or other result of experiment is assumed.

Shew that, if B afterwards impinge perpendicularly on a smooth wall, the original distance of which from the nearest point of B is given, the time, which elapses between the first and second impact of the balls, will be independent of their radius.

If h be the given distance, and x the distance from the wall of the nearest point of B when it meets the ball A, and if d be the diameter of B, the space over which A has moved in the interval between the two impacts

$$=d+h-x-d=h-x.$$

Hence, if v, v' be the velocities of A and B at first,

$$\frac{h}{v'}+\frac{x}{ev'}=\text{the time contemplated}=\frac{h-x}{v};$$

an equation which shews that x, and therefore the time, is independent of the diameter.

10. Shew that a particle, projected in any direction not vertical, and acted upon by gravity only, will describe a parabola.

An inclined plane passes through the point of projection; find the condition that the particle may impinge perpendicularly on the plane; and, in that case, shew that its range on the plane is equal to

$$\frac{2v^2}{g}\cdot\frac{\sin\alpha}{1+3\sin^2\alpha},$$

where v is the velocity of projection, and α the inclination of the plane to the horizon.

If the particle fall perpendicularly on the plane, the vertical plane of its path must be perpendicular to the plane, and therefore perpendicular to the horizontal line in the plane through the point of projection.

Let θ be the angle which the direction of projection makes with the plane, then the time of flight

$$=\frac{2v\,.\sin\theta}{g\,.\cos\alpha},$$

and the velocity parallel to the plane being in this time destroyed, we have

$$0=v\,.\cos\theta-g\,.\sin\alpha\,.\frac{2v\,.\sin\theta}{g\,.\cos\alpha},$$

or $$2\,.\tan\theta = \cot\alpha;$$

and the range $$= \frac{v^2\,.\cos^2\theta}{2g\,\sin\alpha}$$

$$= \frac{v^2}{2g\,\sin\alpha}\cdot\frac{1}{1+\frac{1}{4}\cot^2\alpha}$$

$$= \frac{2v^2}{g}\cdot\frac{\sin\alpha}{1+3\,\sin^2\alpha}.$$

11. Two given weights are connected by an inextensible string, which passes over a smooth pully; determine the motion of each weight and the tension of the string.

The system being initially at rest, find the weight which, let fall at the beginning of the motion from a point vertically above the ascending weight, so as to impinge upon it, will instantaneously reduce the system to rest. Will the system afterwards remain at rest?

Let P and Q be the two weights, P being greater than Q; and let the weight R at the time t impinge vertically on Q; then the velocity of Q at the instant before impact being $\frac{P-Q}{P+Q}.gt$, and of R, gt, the impulsive action on R

$$= \frac{R}{g}.gt = Rt,$$

if the system, and therefore R, be supposed to be reduced instantaneously to rest.

Hence, since Q is reduced to rest, the impulsive tension of the string

$$= R.t - \frac{Q}{g}\cdot\frac{P-Q}{P+Q}.gt$$

$$= \left(R - Q\cdot\frac{P-Q}{P+Q}\right)t;$$

and, since this impulsive tension reduces P to rest, it also

$$= \frac{P}{g}\cdot\frac{P-Q}{P+Q}\cdot gt.$$

Therefore, equating these values, we obtain

$$R = P - Q,$$

and the system will obviously remain at rest. It may be noticed that the principle of the conservation of the motion of the centre of gravity would give this result at once.

WEDNESDAY, *Jan.* 7. $1\frac{1}{2}$ *to* 4.

1. GIVE the meanings of the several symbols which are employed in the formula $p = g\rho z$.

If one second be the unit of time, what must be the unit of length, in order that the above formula may give the pressure in pounds, supposing the unit of volume of the standard substance to weigh 16lbs.?

The unit of weight understood in the formula $p = g\rho z$, is the g^{th} part of the weight of a unit of volume of the standard substance,

$$= \left(\frac{1}{g}\right)^{\text{th}} \text{ of 16lbs., by the question,}$$

$$= \text{1lb., by the question;}$$

therefore, g must be equal to 16.

But, when one foot and one second are taken as the units of length and time, $g = 32$ nearly; therefore, if the unit of time be still one second, the unit of length must be nearly 2 feet, in order that g may be equal to 16, *i.e.*, in order that the unit employed in the formula $p = g\rho z$ may be one pound weight.

4. A body of given volume is immersed totally in a given fluid; find the magnitude and direction of the resultant fluid pressure.

A body is floating in a fluid; a hollow vessel is inverted over it and depressed: what effect will be produced in the position of the body, (1) with reference to the surface of the fluid within the vessel, (2) with reference to the surface of the fluid outside?

By depressing the vessel the density of the air within the vessel is increased.

Now before depression, the weight of air displaced, together with the weight of water displaced, is equal to the weight of the floating body.

Suppose the body in the same position relatively to the surface of the fluid within the vessel after depression as before.

Then the resultant fluid pressure on the body is equal to the weight of water displaced together with the weight of air displaced; but the weight of the same volume of air is greater when the vessel is depressed than before; therefore the resultant fluid pressure upon the body, supposed in the same position relatively to the surface of the fluid after depression as before, would be greater than the weight of the body, and therefore the body, if free to move, will rise relatively to the interior surface.

Whether it will rise or sink, with reference to the exterior surface, will depend upon the relation between the density of the body, of the fluid, and of the compressed air, and therefore, in order to determine the effect in any particular case, it would be necessary to know the volumes of the vessel and of the body.

5. Describe the Diving Bell, and find the volume of the air in the bell at any depth below the surface.

If P be the weight of the bell, P' of a mass of water the bulk of which is equal to that of the material of the bell, and W of a mass of water the bulk of which is equal to that of the interior of the bell, prove that, supposing the bell to be too light to sink without force, it will be in a position of unstable equilibrium, if pushed down until the pressure of the enclosed air is to that of the atmosphere as W to $P-P'$.

Conceive the bell to be depressed to a certain depth below the surface of the fluid. Let $u=$ the volume of the interior of the bell which is free from water, $v=$ the volume of the whole interior of the bell, $x=$ the depth of the surface of the water in the bell below the surface of the fluid, $h=$ the altitude of a water barometer, and $\rho=$ the density of water.

Then, by Boyle's law,

$$\frac{u}{v}=\frac{h}{h+x} \qquad \text{.........................} (1).$$

Also, if the bell be in equilibrium,

$$\begin{aligned} P &= g\rho u + P' \\ &= \frac{g\rho hv}{h+x} + P', \quad \text{by (1)}, \\ &= \frac{g\rho h}{g\rho (h+x)} \cdot g\rho v + P' \\ &= \frac{p}{p'} \cdot W + P', \end{aligned}$$

p being the pressure of the atmosphere and p' of the air enclosed in the bell.

Hence $$\frac{p'}{p}=\frac{W}{P-P'}.$$

If the bell be depressed lower, the displaced water being less than in the position of equilibrium, the bell will sink: if the bell be elevated, the displaced water being greater, the bell will rise: thus the position of equilibrium is unstable.

7. Find the geometrical focus (1) of a pencil of rays incident directly upon a plane refracting surface, and (2) of a pencil incident directly upon a refracting plate.

A ray, passing through a point Q, is incident upon a refracting plate; q is the intersection of the emergent ray, produced backwards, with the normal to the plate through Q: if the angle of incidence be equal to $\tan^{-1}\mu$, and t be the thickness of the plate, prove that

$$Qq=\frac{\mu^2-1}{\mu^2}.t.$$

Let $QRST$ (fig. 44) be the path of the ray.

Let ϕ = the angle of incidence on the plate, ϕ' = the angle of first refraction, t = the thickness of the plate, μ = the index of refraction.

Draw RE, parallel to Qq, to intersect Sq in E. Then, since ST, the emergent ray, is parallel to QR, the incident ray, we see that

$$Qq = RE = RS.\frac{\sin(\phi-\phi')}{\sin\phi}$$

$$= t.\frac{\sin(\phi-\phi')}{\cos\phi'\ \sin\phi}$$

$$= t(1-\tan\phi'\ \cot\phi).$$

But $\tan\phi=\mu$, and $\sin\phi=\mu\ \sin\phi'$; hence

$$\cos\phi=\sin\phi',\quad \phi'=\tfrac{1}{2}\pi-\phi,$$

and therefore $$\tan\phi'=\cot\phi=\frac{1}{\mu}.$$

Hence $$Qq=\frac{\mu^2-1}{\mu^2}.t.$$

8. A ray of light passes through a prism in a plane perpendicular to its edge: shew that, if ϕ and ψ be the angles of incidence and emergence and i the refracting angle of the prism, the deviation is equal to

$$\phi\pm\psi-i,\ \text{or}\ \psi-\phi-i,$$

according as the incident ray makes an acute angle with the face of the prism towards the thicker end or the edge. Under what convention will these expressions for the deviation be all represented by $\phi+\psi-i$, and with this convention for what value of ϕ will ψ change sign?

If the ray be incident between the normal at the point of incidence and the thicker end, it may emerge on either side of the normal at the point of emergence. Now, taking the notation given in the question,

the deviation at the 1st surface $=\phi-\phi'$,

.... 2nd .... $=\psi-\psi'$;

therefore the total deviation (being always from the edge)

$$=\phi-\phi'\pm(\psi-\psi')$$

$$=\phi\pm\psi-(\phi'\pm\psi'),$$

the positive or negative sign being taken according as the emergent ray lies on the thicker-end or thinner-end side of the normal,

or the total deviation in this case $= \phi \pm \psi - i$.

If the ray be incident between the normal and the thin end,

the deviation at incidence will be $\phi - \phi'$ towards the edge,

.... emergence .... $\psi - \psi'$ from ....,

and the total deviation from the edge $= \psi - \psi' - (\phi - \phi') = \psi - \phi - i$.

If it be agreed to consider angles measured from the normal towards the thicker end as positive, and those measured towards the thinner end negative, all these expressions will be included in the formula $\phi + \psi - i$.

The value of ψ will change sign at that value of ϕ which gives the incidence on the second surface direct; *i.e.* which gives $\phi' = i$.

But $$\sin\phi = \mu \sin\phi' = \mu \sin i;$$

therefore, the value of ϕ required $= \sin^{-1}\mu \sin i$.

9. Explain the formation of an image by reflection, and find the magnitude and position of the image of a given object placed before a plane mirror.

The faces of two walls of a room, meeting at right angles, are covered with plane mirrors; shew that a person will be able to see but one complete image of himself in either wall.

Let CA, CB, (fig. 45) be the intersections of the walls with a horizontal plane through P, any point of the observer. Draw Pn_1 perpendicular to CB and produce it to P_1, taking $n_1P_1 = n_1P$: P_1 is the first image of P formed by reflection at CB.

Join PC and produce it. Through P_1 draw $P_1n_2P_2$ perpendicular to AC and meeting AC produced in n_2 and PC in P_2.

Then, since $Pn_1 = n_1P_1$,

$$PC = CP_2 \text{ and } P_1n_2 = P_2n_2;$$

therefore P_2 is the image of P_1 reflected at CA, and similarly it may be proved to be the image of the image of P at CA formed at CB.

Therefore all rays from a given point will after two reflections, whether first reflected at CA, or CB, diverge from a common focus.

There will then be two *first* images, one formed by reflection at each wall and visible to the observer looking directly into either mirror; and one *common* second image visible to the observer looking towards the edge. Also, since the second images of all points of the observer in a vertical plane through the intersection of the walls and the eye of the observer lie in this plane, part of the second image of the observer will lie to the right of this plane and part to the left, and these two parts will be visible, the one in the right hand mirror and the other in the left. Hence part only of the second image will be visible in either wall. Moreover, since no rays reflected from CA, as from P_2, can fall on CB, and none reflected from CB, as from P_2, can fall on CA, P_2 is the *last* image that can be formed of P.

Hence no more than one complete image of himself *can* be visible to the observer, whether he looks directly into either mirror or towards their common edge.

10. A diverging pencil of rays is incident directly upon a concave spherical refractor: find the geometrical focus of the refracted pencil.

A short object is placed perpendicularly on the axis of the refractor, and at a distance from it equal to $\frac{f}{\mu}$, f being the focal length: prove that the linear magnitude of the virtual image is half that of the object.

Let PQ (fig. 46) be the object, C the centre of the refractor: let CA be the radius through Q: join CP. Let q be the image of Q and p of P. Let $AQ=u$, $Aq=v$. Then

$$\frac{\mu}{v}=\frac{\mu-1}{r}+\frac{1}{u};$$

but $$u=\frac{f}{\mu}, \text{ and } \frac{\mu}{f}=\frac{\mu-1}{r},$$

whence also $$u=\frac{r}{\mu-1}:$$

hence $$\frac{\mu}{v} = \frac{2(\mu-1)}{r};$$

and therefore
$$\frac{pq}{PQ} = \frac{r-v}{r-u}$$
$$= \frac{2(\mu-1)r-\mu r}{2(\mu-1)r-2r}$$
$$= \frac{\mu-2}{2\mu-4} = \frac{1}{2}.$$

11. Describe the human eye as an optical instrument. When a pencil of rays is refracted through the eye, at what point of its passage does it experience its principal modification of form; and what is the most probable hypothesis in regard to the change of configuration of the eye by which it adjusts itself to distinct vision at different distances?

An eye is placed close to a sphere of glass, a portion of the surface of which, most remote from the eye, is silvered: prove that, assuming eight inches to be the least distance of distinct vision, the eye cannot see a distinct image of itself unless the diameter of the sphere be at least ten inches in length.

Let A (fig. 47) be the position of the eye, B of the silvered portion of the sphere, Q of the image of the eye after reflection of rays at B, and Q' of the image of Q after the refraction of rays into the air. Let $AB = 2r$, $BQ = u$, $AQ' = v$.

Then $$\frac{1}{2r} + \frac{1}{u} = \frac{2}{r},$$
$$u = \tfrac{2}{3}r, \quad AQ = \tfrac{4}{3}r:$$

also $$\frac{2}{3v} = \frac{\frac{2}{3}-1}{r} + \frac{1}{\frac{4}{3}r},$$
$$\frac{1}{v} = -\frac{1}{2r} + \frac{9}{8r} = \frac{5}{8r},$$
$$v = \frac{8r}{5}.$$

Hence vision will be indistinct unless v be at least equal to 8 inches: hence, the least value of $2r$, for distinct vision, is 10 inches.

THURSDAY, *Jan.* 8. 9 *to* 12.

1. ENUNCIATE and prove Newton's fourth Lemma.

Apply this Lemma to shew that the volume of a right cone is one third of that of the cylinder on the same base and of the same altitude.

Let the triangle ABC (fig. 48), and the rectangle CD generate by revolution round AC the cone and cylinder.

Let P, Q, be two points near each other; then the volume generated by the rectangle PN is equal to

$$\pi . PM^2 . MN;$$

and that generated by Pn is equal to

$$\pi (QN^2 - PM^2) . AM$$
$$= \pi . (QN + PM) . mn . AM.$$

But $\quad MN : mn = AM : PM;$

therefore the ratio of these volumes

$$= PM : (QN + PM)$$
$$= 1 : 2, \text{ ultimately, when } P \text{ and } Q \text{ coincide.}$$

Hence, by Lemmas III. and IV., the volumes generated by the triangles ADB, ACB, are in the ratio $2 : 1$, and therefore the volumes of the cone and cylinder are in the ratio $1 : 3$.

2. Enunciate Lemma XI., and prove it when the subtenses are parallel.

An arc of continuous curvature PQQ', is bisected in Q; PT is the tangent at P; shew that ultimately, as Q' approaches P, the angle $Q'PT$ is bisected by QP.

Draw QT, $Q'T'$, parallel subtenses; produce TQ to meet PQ' in R, and join PQ. Then, by the Lemma,

$$Q'T' : QT :: PT'^2 : PT^2, \text{ ultimately,}$$
$$:: (\text{arc}\,PQ')^2 : (\text{arc}\,PQ)^2$$
$$:: 4 : 1.$$

But $$Q'T' : RT :: PT' : PT :: PQ' : PQ, \text{ ultimately,}$$
$$:: 2 : 1;$$

therefore $$RT = 2QT, \quad RQ = TQ:$$

also $$PR : PT :: PQ' : PT' :: 1 : 1, \text{ ultimately;}$$

therefore $$PR : PT :: RQ : TQ, \text{ ultimately;}$$

or, ultimately, TPR is bisected by PQ.

4. State and prove Proposition I.

Will the velocity of the body or the rate at which areas are swept out about the centre of force be affected by any sudden change in the law of force?

A body moves in a parabola about a centre of force in the vertex; shew that the time of moving from any point to the vertex varies as the cube of the distance of the point from the axis of the parabola.

Let A (fig. 49) be the vertex of the parabola, P any point in it, and T the point in which the tangent parallel to AP meets the axis; then the time from P to A

$$\propto \text{curvilinear area } APQ$$
$$\propto \text{parallelogram } PT$$
$$\propto AT \times PD$$
$$\propto AT^{\frac{3}{2}}, \text{ since } PD^2 = 4AS \,.\, QV,$$
$$\propto AN^{\frac{3}{2}}$$
$$\propto PN^3.$$

6. If any number of bodies revolve in ellipses about a common centre, and the centripetal force varies inversely as the square of the distance; the squares of the periodic times are proportional to the cubes of the major axes.

A particle moves in an ellipse about the centre of force in the focus S: when the particle is at B, the extremity of the minor axis, the centre of force is changed to S' in SB, so that $S'B$ is one-fifth of SB, and the absolute force is diminished to one-eighth of its original value; shew that the periodic time is unaltered, and that the new minor axis is two-fifths of the old.

Let the accented symbols denote the elements of the new orbit. Then, since the velocity of the particle is unchanged,

$$v_1^2 = \mu\left(\frac{2}{SB} - \frac{1}{AC}\right)$$

$$> \frac{2\mu}{SB}$$

$$> \frac{2\dfrac{\mu}{8}}{\dfrac{SB}{5}}$$

$$> \frac{2\mu'}{S'B};$$

therefore the particle will continue to describe an ellipse.

Also $$\mu\left(\frac{2}{SB} - \frac{1}{AC}\right) = \mu'\left(\frac{2}{S'B} - \frac{1}{A'C'}\right),$$

or $$\frac{\mu}{AC} = \frac{\mu}{8}\cdot\left(\frac{10}{AC} - \frac{1}{A'C'}\right);$$

therefore $$A'C' = \frac{AC}{2};$$

or the new major axis is one-half of the old.

Again, by the proposition,

$$\frac{P'^2}{P^2} = \frac{\dfrac{A'C'^3}{\mu'}}{\dfrac{AC^3}{\mu}} = \frac{\mu}{\mu'}\,\frac{A'C'^3}{AC^3} = 8 \times \frac{1}{8} = 1;$$

and therefore the periodic time is unaltered.

Hence $\dfrac{2\pi AC.BC}{h} = \dfrac{2\pi A'C'.B'C'}{h'} = \dfrac{2\pi AC.B'C'}{2h'}$,

or $$B'C' : BC :: 2h' : h.$$

But $h' : h$ :: perpendicular from S' on the tangent : perpendicular from S, that is, as $\frac{1}{8}$ to 1;

therefore $$B'C' : BC :: \tfrac{2}{8} : 1.$$

7. Define the term, "zenith" and explain some method for determining the zenith of a given observatory.

How would an increase in the Earth's velocity of rotation affect the latitude of a given place, supposing the form of the Earth to remain unaltered?

The latitude of a given place is the angular distance of the zenith of the place from the equator.

Now an alteration in the velocity of rotation of the earth would alter the direction of the vertical at any place, and therefore, altering the position of the zenith, would alter the latitude. An *increase* in the velocity of rotation would cause an increase of the latitude of a given place.

8. What conditions must be satisfied in order that the transit instrument may be in accurate adjustment?

Shew how, by aid of this instrument, the difference in right ascension of two stars may be determined; and state the principal astronomical assumptions on which the truth of this determination depends.

The principal assumptions made are that the Earth revolves uniformly about an axis which is fixed in direction, and that the stars are fixed relatively to each other.

9. Explain the phrases "mean solar time" and "equation of time".

Shew that in the month of February the equation of time is additive.

Account for the fact that the time of the Sun's setting as given in the ordinary Almanacs is not the latest on the longest day?

The equation of time due to eccentricity can be shewn to be additive from perigee to apogee, *i.e.* from the beginning of January to the beginning of June.

The equation of time due to obliquity can be proved to be additive from a Solstice to an Equinox, and therefore from Dec. 21 to March 21. Therefore the effect due to both causes is additive in February, and therefore the total equation of time is then additive.

The time of the Sun's setting is given in the ordinary Almanacs in *mean* time, and therefore, as at the longest day the daily increment of the equation of time is greater than the daily decrement in the Sun's declination, it follows that the mean time of the Sun's setting is not greatest when the true time is so, that is, on the longest day.

10. Prove that generally the apparent place of a star will depend upon the ratio of the velocity of the Earth in her orbit to the velocity of light.

Find the least diurnal velocity of rotation of the Earth, which will render sensible to an observer at the equator the aberration due to this cause, the least appreciable angle being $1''$.

The greatest value of the aberration

$$= \frac{\text{velocity of the earth}}{\text{velocity of light}} \times \text{the unit of circular measure};$$

therefore, if $2\pi x$ be the least angle of diurnal rotation required

$$1'' = \frac{2\pi x \cdot 4000}{24 \times 60 \times 60 \times 190000} \cdot \frac{180^\circ}{\pi},$$

taking 4000 miles as the radius of the Earth, and 190000 miles per second as the velocity of light;

therefore $$1 = \frac{2 \times x \times 4 \times 18}{24 \times 19},$$

and $$x = \frac{3 \times 19}{18} = 3\tfrac{1}{6};$$

or the earth must revolve rather more than three times as fast as it does at present.

11. Describe the apparent motion of the Moon among the stars, and the real motion of its centre of gravity about the Sun, illustrating the latter description by a figure.

What is inferred from the fact that, with slight variations, the same portion of the Moon's surface is always presented to the Earth? How much should the Moon's rate of rotation about its centre of gravity be increased, in order that its whole surface might be seen in the course of one orbital revolution?

It will be easily seen, that, if the moon's axis be supposed at rest, the unseen portion of its surface would be presented to the eye by a half-rotation, that is, by a rotation through 180 degrees. If then, in the course of an orbital revolution, the Moon were to rotate on its axis through 540°, instead of 360°, the whole of its surface would be seen. The rate of rotation should therefore be increased in the ratio 3 : 2.

12. Explain the method of determining the longitude by means of Lunar Distances.

On January 1st 1855, at the mean time 9 hrs. 42 min. 8 secs. P.M., the distance of α Arietis from the Moon's centre was calculated from observations to be $45° 30' 16''$: at noon and at 3 P.M. Greenwich mean time, the distances are $44° 56' 11''$, and $46° 23' 39''$ respectively: find the longitude of the place of observation.

Subtracting $44° 56' 11''$ from $46° 23' 39''$ it appears that the Moon's distance increases by $1° 27' 28''$ or $5248''$ in 3 hours.

Also $$45° 30' 16'' - 44° 56' 11'' = 34' 5'' = 2045''.$$

Therefore the time from noon, at Greenwich, at which the Moon's distance is $45° \, 30' \, 16'' = \frac{2045}{5248} \times 3$ hours, or 1 hr. 10 min. $8\frac{1}{2}$ sec. very nearly.

The difference of the times at the two places, when the Moon's centre is at the same distance from the star, is therefore 8 hrs. $31' \, 59\frac{1}{2}''$, and multiplying by 15, this gives the difference of longitudes, $127° \, 59' \, 52\frac{1}{2}''$; and, the time being later, the place of observation is East of Greenwich.

MONDAY, *Jan.* 19. 9 *to* 12.

1. DEFINE a couple, and find the condition that two couples acting on a body in the same plane may hold it in equilibrium.

Find the moment of the couple which is sufficient to sustain a right cone, with its vertex on a rough plane of given inclination and its base parallel to the plane; the roughness of the plane being just sufficient to prevent the vertex from sliding.

Let α be the inclination of the plane, h the altitude of the cone, W its weight, R the normal reaction of the plane, and μR the friction.

Let L be the moment of the couple sufficient to sustain the cone on the required position; then the cone will be held in equilibrium by the couple L and the forces W, R, μR. Hence, taking moments about the vertex of the cone, and resolving the equilibrating forces parallel and perpendicular to the plane, we have

$$L + W.\tfrac{3}{4}h \sin\alpha = 0,$$

$$W \sin\alpha - \mu R = 0,$$

$$W \cos\alpha - R = 0;$$

therefore $$\tan\alpha = \mu,$$

and $$L = -\tfrac{3}{4}h \sin\alpha . W.$$

3. A heavy string of uniform density and thickness is suspended from two given points; find the equation of the curve in which the string hangs when it is at rest.

Compare the curvatures at the lowest points of two catenaries formed by an inextensible and by an extensible string, the tension at the lowest point of each catenary being τ, and the modulus of elasticity w.

In the catenary formed by the inextensible string, if m be the mass of a unit of length, $\tau + \delta t$ the tension at each extremity of the lowest element of the string, δs the length of the element, and $\delta\theta$ the angle between the normals at its extremities, we have, resolving vertically,

$$2(\tau + \delta t) \sin\frac{\delta\theta}{2} = mg\,\delta s;$$

and therefore, if c be the curvature at the lowest point,

$$c = \frac{d\theta}{ds} = \frac{mg}{\tau}.$$

Similarly, if μ' be the mass of a unit of length of the stretched string, and c' the curvature, at the lowest point,

$$c' = \frac{\mu' g}{\tau}.$$

But, if μ be the mass of a unit of length of the string before it is stretched, and $\delta\sigma$ the length of the element which is stretched into $\delta\sigma'$ by the tension τ, we have

$$\mu'\delta\sigma' = \mu\delta\sigma,$$

and
$$\frac{\delta\sigma' - \delta\sigma}{\delta\sigma} = \frac{\tau}{w};$$

therefore
$$\mu' = \frac{\mu}{1 + \frac{\tau}{w}},$$

and
$$c' = \frac{\frac{\mu g}{\tau}}{1 + \frac{\tau}{w}}.$$

Hence
$$c : c' :: m\left(1 + \frac{\tau}{w}\right) : \mu;$$

and, if the mass of a unit of length of the inextensible string be the same as that of the extensible before it is stretched,

$$c : c' :: 1 + \frac{\tau}{w} : 1.$$

4. A particle of mass m describes a plane curve under the action of forces of which the components parallel to the tangent and normal are $m.T$ and $m.N$: shew that

$$T=\frac{d^2s}{dt^2}, \quad N=\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2.$$

If ϕ be the angle which the tangent at any point of the path makes with a fixed line, the differential equation of the path will be

$$\frac{d}{ds}\left(N\frac{ds}{d\phi}\right)=2T.$$

From the proposition

$$N\rho = \left(\frac{ds}{dt}\right)^2,$$

or

$$N\frac{ds}{d\phi} = \left(\frac{ds}{dt}\right)^2;$$

therefore

$$\frac{d}{ds}\left(N\frac{ds}{d\phi}\right) = \frac{d}{dt}\left(\frac{ds}{dt}\right)^2 \frac{dt}{ds}$$

$$= 2\frac{d^2s}{dt^2};$$

therefore

$$\frac{d}{ds}\left(N\frac{ds}{d\phi}\right) = 2T;$$

which, being a relation not involving t, is the differential equation to the path.

5. A heavy particle, suspended from a fixed point by an elastic string, makes vertical oscillations in a medium of which the resistance varies as the square of the velocity: determine the velocity of the particle for any position, neglecting the weight of the string and supposing the motion to commence when the string is unstretched, and the particle to have no initial velocity.

Deduce the greatest extension of the string, supposing the motion to take place in a vacuum.

If a be the unstretched length of the string, w the modulus of elasticity of the string, k the coefficient of resistance of the medium, m the mass of the particle, τ the tension of the string, and x the extension at the time t, the equation of motion of the descending particle will be

$$\frac{d^2(a+x)}{dt^2} = g - \frac{\tau}{m} - k\frac{dx^2}{dt^2};$$

but

$$\frac{x}{a} = \frac{\tau}{w},$$

and therefore the equation of motion becomes

$$\frac{d^2x}{dt^2} + k\frac{dx^2}{dt^2} + \frac{w}{m}\frac{x}{a} - g = 0.$$

Multiply by $2\varepsilon^{2kx}\dfrac{dx}{dt}$, and integrate; then

$$\varepsilon^{2kx}\frac{dx^2}{dt^2} + \frac{w}{ma}\varepsilon^{2kx}\left(\frac{x}{k} - \frac{1}{2k^2}\right) - \frac{g}{k}\varepsilon^{2kx} = C.$$

At the commencement of motion, $x = 0$, $\dfrac{dx}{dt} = 0$, therefore

$$C = -\frac{w}{ma}\cdot\frac{1}{2k^2} - \frac{g}{k},$$

and

$$\frac{dx^2}{dt^2} = \frac{g}{k} + \frac{w}{2mak^2} - \frac{wx}{mak} - \varepsilon^{-2kx}\left(\frac{w}{2mak^2} + \frac{g}{k}\right);$$

whence the greatest extension will be found by putting $\dfrac{dx}{dt} = 0$, that is, from the equation

$$\frac{2amkg(1 - e^{-2kx}) - 2kwx + w(1 - e^{-2kx})}{2amk^2} = 0 \ldots\ldots\ldots (1).$$

The value of x for a vacuum will be obtained by finding the limiting form of this equation when $k = 0$.

Now, differentiating the numerator and denominator twice with respect to k, and then putting $k = 0$, (1) becomes

$$\frac{8amgx - 4wx^2}{4am} = 0;$$

hence

$$x = 0, \text{ or } x = 2\frac{mg}{w}.a,$$

which is the greatest extension of the string in a vacuum.

6. Two particles connected by a stretched inextensible string are constrained to move in a fine curvilinear tube in a vertical plane: determine the motion.

If the tube be cycloidal, the axis of the cycloid being vertical and the vertex upwards; shew that the tension of the string is constant throughout the motion.

Let T be the tension of the string supposed stretched throughout, l its length, m, m', the masses of the particles P and P', xy, $x'y'$, the coordinates of P and P' respectively, the axis of the cycloid being taken as the axis of x, and the vertex being the origin.

Then the equations of motion of P and P' are

$$m\frac{d^2s}{dt^2} = mg\frac{dx}{ds} + T \dots\dots\dots\dots\dots\dots (1),$$

$$m'\frac{d^2s'}{dt^2} = m'g\frac{dx'}{ds'} - T \dots\dots\dots\dots\dots\dots (2),$$

and
$$s' - s = l \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3);$$

therefore, multiplying (1) by m', (2) by m, and subtracting, we get, by virtue of (3),

$$(m' + m)\, T = mm'g.\left(\frac{dx'}{ds'} - \frac{dx}{ds}\right).$$

But, since the curve is a cycloid,

$$s^2 = ax;$$

therefore,
$$\frac{dx}{ds} = \frac{2s}{a},\ \frac{dx'}{ds'} = \frac{2s'}{a},$$

and
$$T = \frac{mm'g}{m+m'}.\frac{2.(s'-s)}{a},$$

$$= 2\frac{m}{m+m'}.\frac{l}{a}.m'g,$$

$$= \text{constant}.$$

8. A body floats in a fluid: determine the position of its metacentre with reference to a vertical plane of displacement dividing the body symmetrically through its centre of gravity.

A cylindrical diving bell is suspended with its axis vertical at a depth such that the water rises half way up the bell: find the least distance of the centre of gravity of the bell from the centre of its upper surface, consistent with the condition that the equilibrium may be stable with reference to an angular displacement of the axis.

Let C (fig. 50) be the centre of the upper surface of the bell, CD the axis of the bell, G the centre of gravity of the bell, which must be in CD since CD is vertical; and let H be the centre of gravity of the fluid displaced.

Suppose the bell slightly displaced, as directed by the question; then the common surface of the water and air within the bell will still be horizontal, and the volume of water displaced will be unaltered; and therefore, the direction of the resultant of the fluid pressure will be vertical, and will pass through a point m in the axis of the bell, such that

$$Hm = \frac{Ak^2}{V},$$

where Ak^2 is the moment of inertia of the transverse circular section of the bell about a diameter, and V is half the volume of the bell.

The forces now acting on the bell are all vertical, viz.—

W the weight of the bell, through G,

W' .... fluid, the volume of which = $\frac{1}{2}$ that of the bell, through m,

T .... tension of the rope $= W - W'$, at C;

therefore, for stable equilibrium, $W.CG > W'.mC$,

$$> W'.(mH + HC);$$

now, if a be the radius of the bell, $2b$ its height,

we have $$Hm = \frac{Ak^2}{V} = \frac{\frac{1}{4}\pi a^4}{\pi a^2 b} = \frac{a^2}{4b};$$

therefore the condition for stability is that

$$W.CG > W'\left(\frac{a^2}{4b}+\frac{b}{2}\right),$$

$$> W'\frac{a^2+2b^2}{4b},$$

and the required least distance $= \dfrac{W'}{W} \cdot \dfrac{a^2+2b^2}{4b}$.

9. A small pencil of rays is incident obliquely on a plane refracting surface; find the positions of the primary and secondary foci of the refracted pencil.

If the pencil consist of common light, shew that the primary foci of the pencils of different colours will lie on a curve of the third order.

Let the intersection of the primary plane with the refracting surface be taken as the axis of y, (fig. 51), and the normal to the surface through the point of incidence A as the axis of x; and let $AM=a$, $Am=x$, $MQ=b$, $mq_1=y$.

Employing the usual notation, we have

$$\frac{\mu\cos^2\phi^1}{v_1} - \frac{\cos^2\phi}{u} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

Hence, substituting for μ its value from the relation $\sin\phi = \mu\sin\phi'$,

$$v_1\cos^2\phi = \frac{\sin\phi}{\sin\phi'}, u\cos^2\phi';$$

therefore $$v_1^2 \cdot \frac{u^2\cos^2\phi}{u\sin\phi} = u^2 \cdot \frac{v_1^2\cos^2\phi'}{v_1\sin\phi'},$$

or $$(x^2+y^2)\frac{a^2}{b} = (a^2+b^2)\frac{x^2}{y};$$

and the required locus is

$$a^2y(x^2+y^2) = x^2b(a^2+b^2),$$

a curve of the third order.

MONDAY, *Jan.* 19. $1\frac{1}{2}$ *to* 4.

3. PROVE that the base of Napier's system of logarithms is incommensurable.

Prove also that it cannot be a root of a quadratic equation the coefficients of which are rational.

Take any quadratic
$$ax^2 - cx + b = 0,$$
where a is a positive integer, and b, c, integers, either positive or negative.

Assume that $x = e$: then
$$ae + be^{-1} = c,$$
whence
$$a\left(1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\right) + b\left(1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \ldots\right) = c.$$
Multiplying both sides by $1.2.3\ldots n$, we find
$$\frac{a}{n+1}\left(1 + \frac{1}{n+2} + \ldots\right) \pm \frac{b}{n+1}\left(1 - \frac{1}{n+2} + \ldots\right) = \mu,$$
μ being an integer.

But we can always make $\pm\dfrac{b}{n+1}$ positive, by taking n even in $1.2.3\ldots n$, when b is negative, and odd, when b is positive.

If n be exceedingly great, the left-hand member of the equation, being an exceedingly small positive quantity, will lie between 0 and 1, while the right-hand member is integral; which is impossible. Hence the truth of the proposition is established.

Liouville: *Journal de Mathèmatiques*, tome cinquième, p. 192.

4. Prove that impossible roots enter rational equations by pairs.

If $e^{\alpha\sqrt{(-1)}}$ be a root of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + p_3 x^{n-3} + \ldots + p_n = 0,$$

prove that

$$p_1 \sin\alpha + p_2 \sin 2\alpha + p_3 \sin 3\alpha + \ldots + p_n \sin n\alpha = 0.$$

Dividing the proposed equation by x^n, and then putting $e^{x\sqrt{(-1)}}$ for x, we have

$$1 + p_1 e^{-\alpha\sqrt{(-1)}} + p_2 e^{-2\alpha\sqrt{(-1)}} + p_3 e^{-3\alpha\sqrt{(-1)}} + \ldots + p_n e^{-n\alpha\sqrt{(-1)}} = 0,$$

$$1 + p_1 \{\cos\alpha - \sqrt{(-1)}\sin\alpha\} + p_2 \{\cos 2\alpha - \sqrt{(-1)}\sin 2\alpha\}$$
$$+ p_3 \{\cos 3\alpha - \sqrt{(-1)}\sin 3\alpha\} + \ldots + p_n \{\cos n\alpha - \sqrt{(-1)}\sin n\alpha) = 0,$$

and therefore, equating the impossible terms to zero, we have

$$p_1 \sin\alpha + p_2 \sin 2\alpha + p_3 \sin 3\alpha + \ldots + p_n \sin n\alpha = 0.$$

7. Shew that the equations

$$x = a \sec\phi, \quad y = b \tan\phi,$$

represent an hyperbola, and give a geometrical interpretation of the angle ϕ.

If P, Q, be points in the one, and P', Q', in the other of two confocal hyperbolas, and if the values of ϕ at P', Q', be respectively equal to those at P, Q; prove that PQ' is equal to $P'Q$.

Let the coordinates of P, Q, be, respectively,

$$\begin{Bmatrix} a \sec\phi \\ b \tan\phi \end{Bmatrix}, \quad \begin{Bmatrix} a \sec\psi \\ b \tan\psi \end{Bmatrix}:$$

and those of P', Q', respectively,

$$\begin{Bmatrix} a' \sec\phi \\ b' \tan\phi \end{Bmatrix}, \quad \begin{Bmatrix} a' \sec\psi \\ b' \tan\psi \end{Bmatrix}.$$

Then

$$PQ'^2 = (a \sec\phi - a' \sec\psi)^2 + (b \tan\phi - b' \tan\psi)^2,$$

$$P'Q^2 = (a \sec\psi - a' \sec\phi)^2 + (b \tan\psi - b' \tan\phi)^2,$$

and therefore

$$\begin{aligned} PQ'^2 - P'Q^2 &= \quad a^2 (\sec^2\phi - \sec^2\psi) + \quad a'^2 (\sec^2\psi - \sec^2\phi) \\ &+ \quad b^2 (\tan^2\phi - \tan^2\psi) + \quad b'^2 (\tan^2\psi - \tan^2\phi) \\ &= (a^2+b^2)(\tan^2\phi - \tan^2\psi) + (a'^2+b'^2)(\tan^2\psi - \tan^2\phi), \end{aligned}$$

whence, since $a^2 + b^2 = a'^2 + b'^2$, we have

$$PQ'^2 - P'Q^2 = 0, \quad PQ' = P'Q.$$

8. State Napier's rules for the solution of right-angled spherical triangles, and prove them for the case in which the complement of the hypotenuse is the middle part.

If three arcs of great circles AP, BQ, CR, intersect at right angles the sides BC, CA, AB, in P, Q, R, respectively, prove that they all pass through the same point O, and that

$$\frac{\tan AP}{\tan OP}, \quad \frac{\tan BQ}{\tan OQ}, \quad \frac{\tan CR}{\tan OR},$$

are respectively equal to

$$1 + \frac{\cos A}{\cos B . \cos C}, \quad 1 + \frac{\cos B}{\cos C . \cos A}, \quad 1 + \frac{\cos C}{\cos A . \cos B}.$$

Let the arcs BQ, CR, (fig. 52) intersect AP in O', O, respectively.

Then $$\tan OP = \frac{\sin CP}{\cot BCR}$$

$$= \frac{\sin CP . \cot CBR}{\cos BC}$$

$$= \frac{\tan AP . \cot C . \cot B}{\cos BC},$$

$$\frac{\tan AP}{\tan OP} = \tan B \tan C \cos BC:$$

but $$\cos A = -\cos B \cos C + \sin B \sin C \cos BC,$$

$$1 + \frac{\cos A}{\cos B \cos C} = \tan B \tan C \cos BC:$$

hence $$\frac{\tan AP}{\tan OP} = 1 + \frac{\cos A}{\cos B . \cos C}.$$

Interchanging B, C, and writing O' for O,

$$\frac{\tan AP}{\tan O'P} = 1 + \frac{\cos A}{\cos C . \cos B}.$$

Hence O' coincides with O.

9. Find the polar equation of the tangent at a point of the conic section

$$\frac{c}{r} = 1 + e \cos\theta.$$

Find the polar equation of the straight line through the foot of the directrix perpendicular to the tangent, and shew that the locus of its intersection with the radius vector at the point of contact is a circle.

Let $\frac{c}{r} = A \cos\theta + B \sin\theta$ be the equation of the straight line required; then, since it is perpendicular to the tangent, the equation of which is

$$\frac{c}{r} = e \cos\theta + \cos(\theta - \alpha) = (e + \cos\alpha) \cos\theta + \sin\alpha \sin\theta,$$

we must have

$$A(e + \cos\alpha) + B \sin\alpha = 0.$$

Also, since it passes through the foot of the directrix, the coordinates of which are $\left(\frac{c}{e}, 0\right)$,

$$e = A;$$

therefore

$$B = -\frac{e(e + \cos\alpha)}{\sin\alpha},$$

and the equation is

$$\frac{c}{r} = \frac{e}{\sin\alpha} \{\sin(\alpha - \theta) - e \sin\theta\}.$$

At the point of intersection of this line with the radius vector of the point of contact, $\theta = \alpha$; and therefore

$$r = -\frac{c}{e^2}.$$

The point of intersection is therefore on the other side of the focus and at a constant distance from it.

11. Shew that through any point of the surface

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{2x}{a},$$

two straight lines can be drawn, entirely coincident with the surface.

Prove that the points on the surface, the straight lines through which, coincident with the surface, are at right angles to each other, lie in a plane parallel to the plane yz, and at a distance from it equal to

$$\frac{c^2 - b^2}{2a}.$$

If the straight line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

lie wholly in the given surface, we must have

$$\left.\begin{aligned} \frac{m^2}{b^2} - \frac{n^2}{c^2} \qquad\quad &= 0 \\ \frac{m\beta}{b^2} - \frac{n\gamma}{c^2} - \frac{l}{a} &= 0 \\ \frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2} - \frac{2\alpha}{a} &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots \text{(A)};$$

whence $$l : m : n :: a\left(\pm\frac{\beta}{b} - \frac{\gamma}{c}\right) : \pm b : c,$$

giving two sets of values for the ratios of the direction cosines.

The two lines thus determined are therefore at right angles if

$$-a^2\left(\frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2}\right) - b^2 + c^2 = 0,$$

or, from the third of the equations (A), if

$$\alpha = \frac{c^2 - b^2}{2a}.$$

12. Investigate the positions of the centric circular sections of an ellipsoid.

If θ, ϕ, be the inclinations of the normal of a centric plane section of the ellipsoid

$$x^2\left(\frac{1}{b^2}-\frac{1}{f^2}\right)+\frac{y^2}{b^2}+z^2\left(\frac{1}{b^2}+\frac{1}{f^2}\right)=1$$

to the normals of the planes of the circular sections, find the equation of the trace of the plane of the section on the plane of zx.

Generally, the equation of the plane centric sections of an ellipsoid

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}+\frac{z^2}{c_1^2}=1,$$

is

$$z=\pm\frac{c_1}{a_1}\left(\frac{a_1^2-b_1^2}{b_1^2-c_1^2}\right)^{\frac{1}{2}}x \text{ .... (1).}$$

In the present case,

$$\frac{1}{a_1^2}=\frac{1}{b^2}-\frac{1}{f^2},\quad \frac{1}{b_1^2}=\frac{1}{b^2},\quad \frac{1}{c_1^2}=\frac{1}{b^2}+\frac{1}{f^2},$$

and therefore

$$\frac{a_1^2-b_1^2}{a_1^2b_1^2}=\frac{1}{f^2}=\frac{b_1^2-c_1^2}{b_1^2c_1^2}.$$

Hence the equation (1) becomes

$$x\pm z=0 \text{ ....(2).}$$

Let the equation of the required plane be

$$lx+my+nz=0 \text{ .... (3).}$$

Then, by the question,

$$l+n=\sqrt{2}.\cos\theta,\quad l-n=\sqrt{2}.\cos\phi,$$

and therefore the equation of the required trace is

$$lx+nz=0=(\cos\theta+\cos\phi)\,x+(\cos\theta-\cos\phi)\,z,$$

or

$$(z+x)\cos\theta=(z-x)\cos\phi.$$

THURSDAY, *Jan.* 20. $1\frac{1}{2}$ *to* 4.

2. EXPLAIN the nature of the difficulty which prevents the formation of a completely achromatic combination of lenses.

A pencil of light is refracted, centrically, and with small obliquity, through two thin lenses in contact; find the condition of achromatism. If such a combination be used as a microscope, determine which of the lenses has the greater dispersive power.

The position of the geometrical focus of a pencil refracted through the two lenses is given by the equation

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2};$$

and the condition of achromatism is

$$\frac{\delta\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1} + \frac{\delta\mu_2}{\mu_2 - 1} \frac{1}{f_2} = 0,$$

or
$$\frac{\varpi_1}{f_1} + \frac{\varpi_2}{f_2} = 0;$$

hence f_1 and f_2 are of contrary signs.

If the combination be used as a microscope,

$$v > u \text{ and therefore } \frac{1}{v} < \frac{1}{u};$$

therefore
$$\frac{1}{f_1} + \frac{1}{f_2} \text{ or } \frac{1}{f_1}\left(1 - \frac{\varpi_1}{\varpi_2}\right) \quad \text{is negative.}$$

Let f_1 refer to the convex lens; then $1-\frac{\varpi_1}{\varpi_2}$ is positive and therefore $\varpi_2 > \varpi_1$, *i.e.* the concave lens has the greater dispersive power.

5. A particle, acted upon by given forces, moves on a given smooth surface; shew how to determine its motion, and the pressure on the surface.

If the surface be a smooth cone, placed with its axis vertical and vertex downwards, and if gravity be the only force acting, shew that the differential equation of the projection on the horizontal plane of the path of the particle, is

$$\frac{d^2u}{d\theta^2}+u\sin^2\alpha=\frac{g\sin\alpha\cos\alpha}{h^2u^2},$$

where u is the reciprocal of the distance of the particle from the axis, θ the angle between this distance and a fixed vertical plane, h constant, and α the semi-vertical angle of the cone.

Let P (fig. 53) be the position of the particle at the time t, $PN=r$, its distance from the axis, $ON=z$, and let θ be the angle between the planes ONP, ONA.

Then, if PG be the normal at P, $\angle GPN=\alpha$, and the equations of motion are

$$\left.\begin{aligned}\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2&=-\frac{R}{m}\cos\alpha,\\ r^2\frac{d\theta}{dt}&=h,\\ \frac{d^2z}{dt^2}&=\frac{R}{m}\sin\alpha-g.\end{aligned}\right\}$$

Also $$z=r\cot\alpha.$$

Eliminating R from the first and third of the above equations, and taking account of the second and fourth, we get

$$\frac{d^2r}{dt^2}-\sin^2\alpha\,.\,\frac{h^2}{r^3}=-g\sin\alpha\cos\alpha.$$

But, if $\qquad r = \frac{1}{u}, \quad \frac{dr}{dt} = \frac{dr}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta},$

and therefore $\qquad \frac{d^2r}{dt^2} = -\frac{h^2}{r^2} \cdot \frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}.$

Hence, by substituting for $\frac{d^2r}{dt^2}$ and reducing, the given equation is at once obtained.

7. Investigate the equations of fluid motion, referred to rectangular axes.

An elastic fluid, not acted upon by any impressed forces, flows uniformly through a cylindrical tube; compare the pressures of the fluid for two different velocities, and hence explain the following experiment.

To one end of a tube is fitted a plane disc which is capable of sliding on wires projecting from the end of the tube in directions parallel to the axis: if the disc be placed at a small distance from the end, and a person blow steadily into the other end, the disc will remain nearly stationary.

Assuming the motion in the cylindrical tube to be rectilinear, the equation of steady motion is

$$v\frac{dv}{dx} = -\frac{1}{\rho} \cdot \frac{dp}{dx} = -\frac{\kappa}{p} \cdot \frac{dp}{dx},$$

where x is the distance from a fixed point in the axis of a section of the tube, κ a constant, and v the velocity of the particles of fluid passing through the section.

Hence $\qquad \kappa \, . \log p = C - \frac{v^2}{2};$

and, if $p = p'$ when $v = v'$,

$$\frac{p}{p'} = \varepsilon^{\frac{v'^2 - v^2}{2}}:$$

it appears from this equation that if v be increased, p is diminished. The pressure of atmospheric air in motion is therefore less than that of the same air at rest, if the change caused by variations of temperature, consequent on motion, be neglected.

This explains, in the experiment detailed, the apparent anomaly that the disc is not driven off by blowing through the tube. In general the disc will oscillate slightly about a position near the end of the tube.

The experiment may be performed easily by fastening a straw with sealing-wax to a piece of cardboard having a small hole in it. If a piece of paper be placed over the hole and the experimenter blow through the straw, the paper will bend so as to allow the egress of the air, but will not be detached from the card.

10. Assuming the following equation for determining the Moon's longitude,

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2},$$

where $$\frac{P}{h^2u^2} = \frac{\mu}{h^2} - \frac{m'u'^3}{2h^2u^3}\left[1 + 3\cos\{(2-2m)\,\theta - 2\beta\}\right],$$

find the term of the second order, in the expression for θ, of which the period is one year.

Considering only the effect of this term, and assuming $e' = \frac{1}{60}$ and $\sin m\pi = \frac{1}{4}$, find approximately in minutes the difference between the greatest and least periodic times of the Moon.

The value of θ obtained from the given equation is

$$\theta = pt - 3me' \sin(mpt + \beta - \xi);$$

therefore $$\frac{d\theta}{dt} = p\,\{1 - 3m^2e' \cos(mpt + \beta - \xi)\}.$$

Hence the greatest angular velocity of the Moon $= p(1 + 3m^2e')$,

least .... $= p(1 - 3m^2e')$.

Therefore, considering these angular velocities constant for a month, we shall have

the longest month $= \dfrac{2\pi}{p\,(1 - 3m^2e')} = \dfrac{2\pi}{p}\,(1 + 3m^2e')$, nearly,

the shortest .... $= \dfrac{2\pi}{p\,(1 + 3m^2e')} = \dfrac{2\pi}{p}\,(1 - 3m^2e')$, ...;

and the required difference $= \dfrac{12\pi}{p} m^2 e'$;

but $\dfrac{2\pi}{p}$ = a mean month = 28 days, say;

therefore the difference $= 6 \times 28 \times 24 \times 60 \times \frac{1}{13} \times \frac{1}{13} \times \frac{1}{60}$ minutes

$$= \frac{6 \times 28 \times 24}{169} \text{ minutes}$$

$$= 23 \text{ minutes nearly.}$$

WEDNESDAY, *Jan.* 21. $1\frac{1}{2}$ *to* 4.

3. SHEW that the values of x, which render $\phi(x)$, a continuous function of x, a maximum or a minimum, are given by the condition that $\phi'(x)$ for such values, vanishes or is infinite; and shew how to distinguish between a maximum and a minimum.

Determine in each case the sign of $\phi''(x)$ for values of x very nearly equal to those which make $\phi'(x)$ infinite.

If $\phi(x)$ be a maximum when $x=a$, and if $\phi'(a)=\infty$, it is clear, since $\phi'(x)$ is positive for values of x less than a and negative for values greater than a, that $\phi'(x)$ *increases* to $+\infty$, as x increases to a, and, when x is greater than a, increases from $-\infty$: therefore $\phi''(x)$ is *positive* for values of x nearly equal to a.

And similarly, if $\phi(a)$ be a minimum when $x=a$, $\phi'(a)$ being infinite, $\phi''(a)$ is *negative* for such values.

The figures (54, 55) will illustrate these two cases; for, if $y=\phi(x)$ be the equation of a curve, it is clear that, in the first figure, $\tan PTx$, *i.e.* $\phi'(x)$, increases algebraically, as x increases, on each side of the point A, and that the reverse is the case in the other figure.

4. Give some definition of an asymptote of a curve, and employ it to shew how to determine the asymptotes of polar curves.

If the equation of the curve be

$$u=\frac{1}{r}=f(\theta),$$

shew that there may be as many asymptotes as there are unequal roots of the equation $f(\theta)=0$: and that, if α be one of these roots, the equation of the corresponding asymptote will be

$$u=f'(\alpha)\sin(\theta-\alpha).$$

Let α be a single root of the equation $f(\theta)=0$: then α will not satisfy the limiting equation $f'(\theta)=0$, and therefore $\frac{1}{f'(\alpha)}$ cannot be infinite.

But $-\frac{1}{f'(\theta)}$ is the length of the subtangent at the point u, θ; therefore, when $\theta=\alpha$, since $u=0$ and $\frac{1}{f'(\alpha)}$ is finite, the radius vector is infinite, and the subtangent is finite, and the value α for θ corresponds to an asymptote. Hence to every *unequal* root of $f(\theta)=0$ belongs an asymptote of the curve $u=f(\theta)$.

But, if α were one of two or more equal roots of $f(\theta)=0$, $f'(\alpha)$ would be zero, and therefore $\frac{1}{f'(\alpha)}$ would be infinite, *i.e.* the subtangent would be infinite, or there would be no asymptote.

Therefore there can be no more asymptotes to the curve, $u=f(\theta)$, than there are unequal roots of $f(\theta)=0$. Since different roots may give the same direction and the same subtangent, we cannot say that there must be, but only that there may be, as many asymptotes as there are unequal roots.

5. Find the magnitude and position of the circle which has the closest possible contact with the curve $y=f(x)$ at a given point; and shew that it generally cuts the curve at the point.

Prove that the chord of curvature, parallel to the axis of x, of the curve

$$\sec\frac{y}{a}=\varepsilon^{\frac{x}{a}}$$

is constant, and that

$$\sec\left(\frac{3y}{a}\right)^{\frac{1}{3}}=\varepsilon^{\frac{x-a}{a}}$$

approximately represents the evolute of this curve for the part near the origin.

The length of the chord of curvature parallel to the axis of x is

$$2\rho\,\frac{dy}{ds} = 2\,\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\cdot\frac{\frac{dy}{dx}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}}$$

$$= 2\,\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}.$$

Now if $$\sec\frac{y}{a} = \varepsilon^{\frac{x}{a}},$$

$$\tan\frac{y}{a}.\sec\frac{y}{a}.\frac{dy}{dx} = \varepsilon^{\frac{x}{a}},$$

or $$\frac{dy}{dx} = \cot\frac{y}{a};$$

therefore $$\frac{d^2y}{dx^2} = -\frac{1}{a}.\operatorname{cosec}^2\frac{y}{a}\,\frac{dy}{dx},$$

and the length of the chord $$= 2\,\frac{\left(1+\cot^2\frac{y}{a}\right)\frac{dy}{dx}}{\frac{1}{a}.\operatorname{cosec}^2\frac{y}{a}.\frac{dy}{dx}}$$

$$= 2a, \text{ a constant.}$$

Let ξ, η, be the coordinates of the centre of curvature and c this chord. Then

$$\xi = x+\frac{c}{2}, \quad \eta = y-\frac{c}{2}.\frac{dx}{dy},$$

or $$\xi = x+a, \quad \eta = y - a\tan\frac{y}{a}$$

$$= -\frac{1}{3}\,\frac{y^3}{a^2},$$

neglecting powers of $\frac{y}{a}$ above the third.

Therefore, near the origin,

$$y^3 = -3a^2\eta, \quad x = \xi - a;$$

and substituting in $\sec\frac{\eta}{a} = \varepsilon^{\frac{x}{a}}$, we get

$$\sec\left(\frac{3\eta}{a}\right)^{\frac{1}{3}} = \varepsilon^{\frac{x-a}{a}},$$

which therefore represents the evolute for the part near the origin.

6. Investigate the analytical conditions for the existence of multiple points in a curve of which the equation is $u = 0$, u being a rational function of x and y; and shew how the degree of multiplicity may be determined.

Prove that, if

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0,$$

at a double point, the coordinates of which are x, y, the two branches of the curve are at right angles to each other; and that, if the point be the origin, the equation of the tangents to the branches will be

$$\frac{d^2u}{dx^2}(\eta^2 - \xi^2) + 2\frac{d^2u}{dx\,dy}\eta\xi = 0,$$

ξ, η, being current coordinates of the tangent.

At a double point the values of $\frac{dy}{dx}$ are given by the equation

$$\left(\frac{dy}{dx}\right)^2 + 2\frac{\left(\frac{d^2u}{dx\,dy}\right)}{\left(\frac{d^2u}{dy^2}\right)}\frac{dy}{dx} + \frac{\frac{d^2u}{dx^2}}{\frac{d^2u}{dy^2}} = 0;$$

and therefore, if θ_1, θ_2, be the directions of the two tangents

$$\tan\theta_1 . \tan\theta_2 = \frac{\frac{d^2u}{dx^2}}{\frac{d^2u}{dy^2}}$$

$$= -1, \text{ if } \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0,$$

which is the condition that the two branches may be at right angles.

The equation of the two tangents through the origin is

$$(\eta - \tan\theta_1 . \xi)(\eta - \tan\theta_2 . \xi) = 0,$$

or $$\eta^2 - (\tan\theta_1 + \tan\theta_2)\,\xi\eta + \tan\theta_1 \tan\theta_2 \xi^2 = 0 \text{ ...... (1)};$$

but $$\tan\theta_1 \tan\theta_2 = -1, \quad \tan\theta_1 + \tan\theta_2 = -2\frac{\dfrac{d^2u}{dxdy}}{\dfrac{d^2u}{dy^2}}:$$

whence (1) becomes

$$\frac{d^2u}{dy^2}(\eta^2 - \xi^2) + 2\left(\frac{d^2u}{dxdy}\right)\eta\xi = 0.$$

8. Find the equation of the locus of tangent lines at a point (x, y, z) of a surface, the equation of which is $u = f(x, y, z) = 0$.

Common tangent planes are drawn to the ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1;$$

shew that the perpendiculars upon them from the origin lie in the surface of the cone

$$(a^2 - a'^2)\,x^2 + (b^2 - b'^2)\,y^2 + (c^2 - c'^2)\,z^2 = 0.$$

The equation of the tangent plane at the point (xyz) of the first surface, is

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1;$$

which may be transformed into

$$l\xi + m\eta + n\zeta = l^2a^2 + m^2b^2 + n^2c^2,$$

where l, m, n, are the direction-cosines of the normal.

If this be also a tangent plane of the second surface,

$$l^2a^2 + m^2b^2 + n^2c^2 = l^2a'^2 + m^2b'^2 + n^2c'^2.$$

But, if (x, y, z) be the coordinates of a point in the perpendicular on the plane from the origin,

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

and therefore $(a^2 - a'^2)\,x^2 + (b^2 - b'^2)\,y^2 + (c^2 - c'^2)\,z^2 = 0.$

9. Shew how to integrate the equation

$$\frac{dy}{dx} + Py = Q,$$

P and Q being functions of x.

The normal at a point P of a curve meets the axis of x in G, and the locus of the middle point of PG is the parabola $y^2 = lx$; find the equation of the curve, supposing it to pass through the origin.

If O be the origin, and (xy) the coordinates of P; then OG (fig. 56) $= x + y\dfrac{dy}{dx}$, and the coordinates of the middle point of PG are

$$\frac{1}{2}\left\{2x + y\frac{dy}{dx}\right\}, \text{ and } \tfrac{1}{2}y.$$

Hence
$$\frac{y^2}{4} = \frac{l}{2}\left(2x + y\frac{dy}{dx}\right),$$

or
$$\frac{d(y^2)}{dx} - \frac{y^2}{l} = -4x:$$

therefore
$$y^2 = 4l(x + l) + C\varepsilon^{\frac{x}{l}};$$

and, since the curve passes through the origin,

$$0 = 4l^2 + C.$$

Hence the equation is

$$y^2 = 4l\left(x + l - l\varepsilon^{\frac{x}{l}}\right).$$

10. Prove that the differential equation of the surfaces generated by a straight line which passes through the axis of z and through a given curve, and which makes a constant angle α with the axis of z, is

$$x\frac{dz}{dx} + y\frac{dz}{dy} = (x^2 + y^2)^{\frac{1}{2}} \cot\alpha.$$

Let
$$\frac{x}{l} = \frac{y}{m} = \frac{z - \gamma}{n}$$

be the equations of one of the generating lines, l, m, n, being its direction-cosines: then

$$n = \cos\alpha, \quad \text{and } l^2 + m^2 = \sin^2\alpha.$$

From the above equations

$$y=\frac{m}{l}x, \quad z=\gamma+\frac{n}{l}x;$$

and if $$\frac{m}{l}=\mu, \qquad l^2=\frac{\sin^2\alpha}{1+\mu^2},$$

we have $$y=\mu x, \qquad z=\gamma+x\cot\alpha\,(1+\mu^2)^{\frac{1}{2}}.$$

Hence, eliminating x, y, and z between these equations and the equations of the curve, we obtain

$$\gamma=f(\mu);$$

but $$\mu=\frac{y}{x}, \quad \gamma=z-\frac{n}{l}x=z-\cot\alpha\,(x^2+y^2)^{\frac{1}{2}};$$

therefore $$z-\cot\alpha\,(x^2+y^2)^{\frac{1}{2}}=f\left(\frac{y}{x}\right).$$

Differentiating this equation with respect to x and y separately, the function may be eliminated and the result will be the given equation.

THURSDAY, *Jan.* 22. $1\frac{1}{2}$ *to* 4.

1. ONE plane curve rolls on another, the planes of the two curves coinciding, and their convexities being opposed to each other: if r, r', are the radii of curvature of the fixed and rolling curves respectively, at their point of contact, ρ the distance of any point P in the moving plane from the point of contact, α the angle between ρ and the common normal to the two curves, prove that the corresponding radius of curvature of P's path is equal to

$$\frac{\frac{1}{r}+\frac{1}{r'}}{\frac{1}{r}+\frac{1}{r'}-\frac{\cos\alpha}{\rho}}\cdot\rho.$$

Shew also that the directions of motion of all the points in the moving plane, fixed relatively to the rolling curve, which at any instant are going through points of inflection in their respective paths, pass through a single point.

Let A (fig. 57) be the point of contact of the fixed and rolling curves at any instant: let AB be an elementary arc of the rolling curve, each point of which, during the next element of time, comes into contact with a corresponding point of the elementary arc AB' of the fixed curve. Let $A'P'$ be the new position of the line AP, fixed in the rolling area, at the end of the element of time: let O be the intersection of AP, $A'P'$.

Produce PA, $P'B'$, to meet in C. Then, since A, B', are respectively motionless, at the beginning and at the end of the element of time, CP, CP', are normals at the ends P, P', of the elementary arc PP'. Draw AM at right angles to CP'. Let the arc $AB = c = AB'$.

Now the angle between the normals at A, B, of the rolling curve, is equal to $\frac{c}{r}$, and that between the normals at A, B', of the fixed curve is equal to $\frac{c}{r'}$. Hence, while rolling takes place from A to B', every line, fixed relatively to the rolling curve, revolves through an angle $\frac{c}{r} + \frac{c}{r'}$: hence

$$\angle POP' = \frac{c}{r} + \frac{c}{r'},$$

and therefore, OP being normal to PP',

$$PP' = OP\left(\frac{c}{r} + \frac{c}{r'}\right) = \rho\left(\frac{c}{r} + \frac{c}{r'}\right), \text{ ultimately.}$$

Again, by similar triangles,

$$\frac{CP}{CA} = \frac{PP'}{AM},$$

and therefore
$$\frac{CP}{AP} = \frac{PP'}{PP' - AM}.$$

But, α being ultimately the inclination of CB' to the normal at B',

$$AM = c\cos\alpha:$$

hence
$$\frac{CP}{\rho} = \frac{\rho\left(\frac{c}{r} + \frac{c}{r'}\right)}{\rho\left(\frac{c}{r} + \frac{c}{r'}\right) - c\cos\alpha},$$

and therefore CP, the radius of curvature of P's path, is equal to

$$\frac{\frac{1}{r} + \frac{1}{r'}}{\frac{1}{r} + \frac{1}{r'} - \frac{\cos\alpha}{\rho}}\cdot\rho.$$

An expression for the radius of curvature of P's path, substantially the same as this, is given in Jullien's *Problèmes de Mécanique Rationelle*, tom. I, p. 184.

At a point of inflection of P's path, $CP = \infty$: hence

$$\rho = \frac{rr'}{r + r'}\cos\alpha,$$

or the locus of a point of inflection is a circle of which the diameter AB is normal to the fixed or rolling curve at A and is equal to $\frac{rr'}{r + r'}$, the motion of P being in the direction BP.

6. A free rigid body, the mass of which is m, is at rest: its moments of inertia about the principal axes through its centre of gravity are A, B, C: supposing the body to be struck by an impulsive force R through its centre of gravity, and by an impulsive couple G, prove that it will revolve for an instant about an axis, the velocity of which is in the direction of its length and is equal to

$$\frac{\dfrac{L.X}{A.m} + \dfrac{M.Y}{B.m} + \dfrac{N.Z}{C.m}}{\left(\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}\right)^{\frac{1}{2}}},$$

X, Y, Z, being the components of R, and L, M, N, of G, along the principal axes.

If θ be the inclination of R's direction to the spontaneous axis, prove that

$$\cos\theta = \frac{\dfrac{L.X}{A.R} + \dfrac{M.Y}{B.R} + \dfrac{N.Z}{C.R}}{\left(\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}\right)^{\frac{1}{2}}}.$$

See Walton's *Mechanical Problems*, Second Edition, p. 519.

FRIDAY, *Jan.* 23. 9 *to* 12.

1. IF $\alpha=0$, $\beta=0$, $\gamma=0$, be the equations of the sides of a triangle, shew that the equation of a conic touching the sides of the triangle is

$$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0.$$

If
$$\alpha=0,\quad \beta=0,\quad \gamma=0,$$

$$\frac{\alpha}{a_1}+\frac{\beta}{b_1}+\frac{\gamma}{c_1}=0,\quad \frac{\alpha}{a_2}+\frac{\beta}{b_2}+\frac{\gamma}{c_2}=0,\quad \frac{\alpha}{a_3}+\frac{\beta}{b_3}+\frac{\gamma}{c_3}=0,$$

be the equations of the sides of a hexagon which circumscribes a conic, shew that

$$a_1(b_2c_3-c_2b_3)+a_2(b_3c_1-c_3b_1)+a_3(b_1c_2-c_1b_2)=0.$$

If the line

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=0$$

touch the conic

$$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0,$$

the equation which results from combining them, must be the equation of two coincident lines. Hence the left-hand side of the equation

$$\{(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}\}^2+n\left(\frac{c}{a}\,\alpha+\frac{c}{b}\,\beta\right)=0$$

must be a complete square in α and β; therefore

$$\left(l+n\,\frac{c}{a}\right)\left(m+n\,\frac{c}{b}\right)=lm,$$

and
$$la+mb+nc=0.$$

Now the three sides of the hexagon $\alpha=0$, $\beta=0$, $\gamma=0$, touch the conic

$$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0;$$

and that the other three sides

$$\frac{\alpha}{a_1}+\frac{\beta}{b_1}+\frac{\gamma}{c_1}=0, \quad \frac{\alpha}{a_2}+\frac{\beta}{b_2}+\frac{\gamma}{c_2}=0, \quad \frac{\alpha}{a_3}+\frac{\beta}{b_3}+\frac{\gamma}{c_3}=0,$$

also may touch it, we must have

$$la_1+mb_1+nc_1=0,$$
$$la_2+mb_2+nc_2=0,$$
$$la_3+mb_3+nc_3=0;$$

whence, eliminating l, m, n, by cross multiplication, we obtain

$$a_1(b_2c_3-c_2b_3)+a_2(b_3c_1-c_3b_1)+a_3(b_1c_2-c_1b_2)=0,$$

as the necessary condition.

2. Transform the triple integral $\iiint f(\alpha,\beta,\gamma)\,d\alpha\,d\beta\,d\gamma$ into one in which x, y, z, are the independent variables, having given

$$\alpha=F_1(x,y,z), \quad \beta=F_2(x,y,z), \quad \gamma=F_3(x,y,z).$$

If $$\alpha x=yz, \quad \beta y=zx, \quad \gamma z=xy,$$

shew that

$$\iiint f(\alpha,\beta,\gamma)\,d\alpha\,d\beta\,d\gamma=4\iiint f\left(\frac{yz}{x},\frac{zx}{y},\frac{xy}{z}\right)dx\,dy\,dz.$$

The formula for the transformation of the given triple integral is, (Todhunter's *Integral Calculus*, p. 245 et seq.),

$$\iiint f(\alpha,\beta,\gamma)\,d\alpha\,d\beta\,d\gamma$$

$$\begin{aligned}=\iiint F(x,y,z)\left\{\frac{d\alpha}{dx}\left(\frac{d\beta}{dy}\cdot\frac{d\gamma}{dz}-\frac{d\gamma}{dy}\cdot\frac{d\beta}{dz}\right)\right.&\\ +\frac{d\beta}{dx}\left(\frac{d\gamma}{dy}\cdot\frac{d\alpha}{dz}-\frac{d\alpha}{dy}\cdot\frac{d\gamma}{dz}\right)&\\ \left.+\frac{d\gamma}{dx}\left(\frac{d\alpha}{dy}\cdot\frac{d\beta}{dz}-\frac{d\beta}{dy}\cdot\frac{d\alpha}{dz}\right)\right\}dx\,dy\,dz.&\end{aligned}$$

Now, in the given example,

$$\alpha = \frac{yz}{x}, \qquad \beta = \frac{zx}{y}, \qquad \gamma = \frac{xy}{z};$$

therefore
$$\frac{d\alpha}{dx} = -\frac{yz}{x^2}, \quad \frac{d\alpha}{dy} = \frac{z}{x}, \quad \frac{d\alpha}{dz} = \frac{y}{x},$$

$$\frac{d\beta}{dx} = \frac{z}{y}, \quad \frac{d\beta}{dy} = -\frac{zx}{y^2}, \quad \frac{d\beta}{dz} = \frac{x}{y},$$

$$\frac{d\gamma}{dx} = \frac{y}{z}, \quad \frac{d\gamma}{dy} = \frac{x}{z}, \quad \frac{d\gamma}{dz} = -\frac{xy}{z^2};$$

whence, substituting, we obtain

$$\iiint f(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma = 4 \iiint f\left(\frac{yz}{x}, \frac{zx}{y}, \frac{xy}{z}\right) dx\, dy\, dz.$$

3. Shew how to integrate the equation of differences,

$$u_{x+n} + p_1 u_{x+n-1} + \ldots + p_n u_x = f(x),$$

where $p_1, p_2, \ldots p_n$, are independent of x.

Shew that a solution of the equation

$$u_{x+n} \cdot u_{x+n-1} \ldots u_{x+1} \cdot u_x = a\,(u_{x+n} + u_{x+n-1} + \ldots + u_{x+1} + u_x),$$

is included in that of

$$u_{x+n+1} - u_x = 0,$$

and is consequently

$$u_x = C_1 \alpha^x + C_2 \alpha^{2x} + \ldots + C_{n+1} \alpha^{(n+1)x},$$

where α is one of the imaginary $(n+1)^{\text{th}}$ roots of unity, the $n+1$ constants being subject to an equation of condition.

In the equation

$$u_{x+n} \cdot u_{x+n-1} \ldots u_{x+1} \cdot u_x = a\,(u_{x+n} + u_{x+n-1} + \ldots + u_{x+1} + u_x) \ldots (1),$$

change x into $x+1$: then

$$u_{x+n+1} \cdot u_{x+n} \ldots u_{x+2} \cdot u_{x+1} = a\,(u_{x+n+1} + u_{x+n} + \ldots + u_{x+2} + u_{x+1}) \ldots (2);$$

subtracting (1) from (2), we have

$$u_{x+n} \cdot u_{x+n-1} \ldots u_{x+1}\,(u_{x+n+1} - u_x) = a\,(u_{x+n+1} - u_x),$$

or
$$(u_{x+n+1} - u_x)\,(u_{x+n} \cdot u_{x+n-1} \ldots u_{x+1} - a) = 0.$$

Hence a solution of the equation (1) is included in that of

$$u_{x+n+1} - u_x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the auxiliary equation of which is

$$z^{n+1} - 1 = 0;$$

and, if α be one of the imaginary roots of this auxiliary equation, since the remaining roots may be put into the forms

$$\alpha^2,\ \alpha^3,\ \alpha^4,\ \ldots\ \alpha^{n+1},$$

the solution of (3) will be

$$u_x = C_1\alpha^x + C_2\alpha^{2x} + \ldots + C_{n+1}\alpha^{(n+1)x} \ldots\ldots\ldots\ldots (4).$$

But, since the integral of the equation (1) can contain only n constants, the $n+1$ constants of (4) will be subject to the equation of condition amongst the constants, which arises from substituting the values of u_0, u_1, ... u_n, derived from (4), in

$$u_n . u_{n-1} \ldots u_1 . u_0 = a\,(u_n + u_{n-1} + \ldots + u_1 + u_0).$$

4. Shew how to find the differential equation of a class of surfaces, which cuts at right angles all the surfaces represented by the equation

$$f(x, y, z, a) = 0,$$

where a is an arbitrary parameter.

If the class of surfaces have an envelope, shew how we may find it without solving the differential equation.

The equation of condition in this case is

$$1 + pp' + qq' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where p', q', are the partial differential coefficients of q found from the equation

$$f(x, y, z, a) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and p, q, are similar partial differential coefficients in the class of surfaces of which the differential equation is sought.

Now the equation (1) will generally contain the parameter a, by eliminating which between the equations (1) and (2), we obtain the differential equation sought.

Suppose, for example, we obtain from the equation (1)

$$a = F(x, y, z, p, q);$$

then the required differential equation will be

$$f\{x, y, z, F(x, y, z, p, q)\} = 0.$$

If the class of surfaces represented by the general integral of the differential equation have an envelope, its equation will be the singular solution of the differential equation, and therefore will be found by eliminating p and q between the equations

$$f = 0, \quad \frac{df}{dp} = 0, \quad \frac{df}{dq} = 0.$$

5. Disturbances are excited in the air contained in a cylindrical tube of given length by a plate vibrating isochronously at one end, the other end being closed: assuming expressions for the velocity and condensation at any point, find the time of vibration, in order that a musical note may be produced; and determine the points in the tube at which openings may be made without affecting the pitch.

Supposing a vibrating plate also at the closed end, how must the time of vibration of the first plate be modified, and how must the times of vibration of the two plates be related, that musical notes may be produced?

In the first part of the question the time of vibration is determined from the condition that at the closed end there is no velocity, or that the closed end corresponds with a *node*, whilst at the vibrating end there is a loop.

If there be a vibrating plate also at the closed end, there must be a *loop* there instead of a node; and therefore the time of vibration of the first plate must be modified to satisfy this condition; *i.e.*, if l be the length of the tube and v be the velocity of sound, the time of vibration must be $\frac{1}{n} \cdot \frac{l}{v}$, where n is some integer. So, the time of vibration of the second plate must be given by the same formula, and if the two plates vibrate in times obtained by giving any integral value to n in $\frac{1}{n} \cdot \frac{l}{v}$, musical notes will be produced.

FRIDAY, *Jan.* 23. $1\frac{1}{2}$ *to* 4.

1. INVESTIGATE formulæ for the determination of the umbilici of surfaces.

Prove that the radius of normal curvature of the surface $xyz = a^3$ at an umbilicus is equal to the distance of the umbilicus from the origin of coordinates.

The formulæ for the determination of the umbilici of surfaces, given in Gregory's *Solid Geometry*, p. 264, second edition, are

$$\begin{aligned}\frac{P}{\rho} &= u + \frac{U}{VW}(Uu' - Vv' - Ww')\\ &= v + \frac{V}{WU}(Vv' - Ww' - Uu')\\ &= w + \frac{W}{UV}(Ww' - Uu' - Vv'),\end{aligned}$$

P being defined by the relation

$$P^2 = U^2 + V^2 + W^2.$$

Taking the equation $xyz = a^3$, we have

$$\begin{array}{lll} U = yz, & V = zx, & W = xy; \\ u = 0, & v = 0, & w = 0; \\ u' = x, & v' = y, & w' = z. \end{array}$$

Hence
$$\begin{aligned}\frac{P}{\rho} &= -\frac{yz}{x} = -\frac{zx}{y} = -\frac{xy}{z}\\ &= -\frac{a^3}{x^2} = -\frac{a^3}{y^2} = -\frac{a^3}{z^2},\end{aligned}$$

whence also $x^2 = y^2 = z^2$; and therefore, by the equation to the surface,

$$x^2 = y^2 = z^2 = a^2.$$

Hence $$\frac{P}{\rho} = -a.$$

But $$P^2 = y^2z^2 + z^2x^2 + x^2y^2$$
$$= 3a^4:$$

hence $$\rho^2 = \frac{P^2}{a^2} = 3a^2.$$

But since, at an umbilicus, $x^2 = y^2 = z^2 = a^2$, $3a^2$ is equal to the square of the distance of an umbilicus from the origin of coordinates. Hence the proposition is established.

2. Integrate the equations.

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} - \frac{d^2u}{dz^2} + 2\frac{d^2u}{dx\,dy} = xyz \dots\dots\dots\dots (1),$$

$$u_{x+1}\,.\sin x\theta - u_x\,.\sin(x+1)\,\theta = \cos(x-1)\,\theta - \cos(3x+1)\,\theta \dots (2);$$

and find a general value of $\phi(x)$ from the equation

$$\phi(m^2x) - (a+b)\,\phi(mx) + ab\phi(x) = cx \dots\dots\dots\dots (3).$$

(1) The equation may be written

$$\left\{\left(\frac{d}{dx} + \frac{d}{dy}\right)^2 - \left(\frac{d}{dz}\right)^2\right\} u = xyz;$$

therefore $$u = -\left(\frac{d}{dz}\right)^{-2} \frac{1}{1 - \left(\frac{d}{dz}\right)^{-2}\left(\frac{d}{dx} + \frac{d}{dy}\right)^2} \,.\, xyz$$

$$= -\left(\frac{d}{dz}\right)^{-2}\left\{1 + \left(\frac{d}{dz}\right)^{-2}\left(\frac{d}{dx} + \frac{d}{dy}\right)^2 + \dots\right\} xyz$$

$$= -\left(\frac{d}{dz}\right)^{-2}\left\{xyz + 2\,.\left(\frac{d}{dz}\right)^{-2}.\,z\right\}$$

$$= -\frac{xyz^3}{6} - \frac{z^5}{60} + \text{complementary function.}$$

The complementary function is best obtained by a different arrangement of the operating symbols: thus

$$\left(\frac{d}{dx} + \frac{d}{dy} + \frac{d}{dz}\right)^{-1} 0 = \varepsilon^{-x\left(\frac{d}{dy} + \frac{d}{dz}\right)}\,\phi(y, z)$$
$$= \phi(y - x, z - x);$$

therefore

$$\left\{\left(\frac{d}{dx}+\frac{d}{dy}\right)^2-\left(\frac{d}{dz}\right)^2\right\}^{-1}0=\left(\frac{d}{dx}+\frac{d}{dy}-\frac{d}{dz}\right)^{-1}\phi\,(y-x,z-x)$$

$$=\varepsilon^{-x\left(\frac{d}{dy}-\frac{d}{dz}\right)}\int\varepsilon^{x\left(\frac{d}{dy}-\frac{d}{dz}\right)}\phi\,(y-x,z-x)\,dx$$

$$=\varepsilon^{-x\left(\frac{d}{dy}-\frac{d}{dz}\right)}\int\phi(y,z-2x)\,dx$$

$$=\varepsilon^{-x\left(\frac{d}{dy}-\frac{d}{dz}\right)}\{\psi\,(y,z-2x)+\chi\,(y,z)\}$$

$$=\psi\,(y-x,z-x)+\chi\,(y-x,z+x),$$

which is therefore the complementary function.

By changing the forms of the operating functions, apparently different results may perhaps be obtained, but it will be found that all such results are identically the same when proper account is taken of the complementary function.

(2) Since

$$\cos(x-1)\,\theta-\cos 3\,(x+1)\,\theta=2\sin 2x\theta\,.\,\sin(x+1)\,\theta$$
$$=4\sin x\theta\sin(x+1)\,\theta\cos x\theta,$$

the equation may be put in the form

$$\frac{u_{x+1}}{\sin(x+1)\,\theta}-\frac{u_x}{\sin x\theta}=4.\cos x\theta;$$

therefore $$\frac{u_x}{\sin x\theta}=4\,.\,\Sigma'.\cos x\theta$$

$$=2\,.\frac{\sin\left(x\theta-\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}+C.$$

(3) Let $$x=u_z \text{ and } mx=u_{z+1};$$

therefore $$u_{z+1}-mu_z=0,$$

and $$u_z=am^z=x,$$

$$u_{z+2}=am^{z+2}=m^2x,$$

the equation then becomes

$$\phi(u_{z+2}) - (a+b)\,\phi(u_{z+1}) + ab\phi(u_z) = cam^z,$$

or if

$$\phi(u_z) = v_z,$$

$$v_{z+2} - (a+b)\,v_{z+1} + abv_z = cam^z,$$

or, according to the usual notation,

$$(D-a)(D-b)\,v_z = cam^z.$$

Hence

$$v_z = \frac{cam^z}{(m-a)(m-b)} + A.a^z + B.b^z.$$

But

$$z \log m = \log \frac{x}{a},$$

and

$$\log(a^z) = z.\log a = \frac{\log a}{\log m} \log\left(\frac{x}{a}\right);$$

therefore

$$a^z = \left(\frac{x}{a}\right)^{\frac{\log a}{\log m}},$$

and

$$\phi(x) = v_z = \frac{cx}{(m-a)(m-b)} + C.x^{\log_m a} + C'.x^{\log_m b}.$$

It must be observed that the constants here involved are 'Finite Difference' constants; thus, a, A, B, do not change when z is changed into $z+1$.

3. State and prove the principle of Vis Viva, and describe the different kinds of forces which do not appear in the equation of Vis Viva.

A circular wire ring, carrying a small bead, lies on a smooth horizontal table; an elastic thread, the natural length of which is less than the diameter of the ring, has one end attached to the bead and the other to a point in the wire; the bead is placed initially so that the thread coincides very nearly with a diameter of the ring; find the Vis Viva of the system when the string has contracted to its natural length.

Let A be the point of the ring to which the thread is fastened, and P the position of the bead at any time during the motion.

The forces acting horizontally on the ring, are the tension (T) at A and the action at P in direction of its centre; and the forces acting on the bead are the tension at P, and the reaction at P.

The virtual velocities of the action and reaction at P are clearly equal and opposite, but the displacement of the bead resolved in the direction AP consists of two parts, one the same as the displacement of A resolved in that direction, and the other due to the contraction of the thread, and, relative to A, in the direction of the tension at P. The resultant "virtual moment" of the tensions in the system is therefore T (the contraction of the thread), and the Vis Viva therefore, if $AP=r$,

$$\begin{aligned}&= 2\int_c^a T(-dr)\\ &= \frac{2w}{a}\cdot\int_a^c (r-a)\,dr\\ &= \frac{w}{a}\cdot(c-a)^2,\end{aligned}$$

where c is the diameter of the ring, a the natural length of the thread, and w the modulus of elasticity.

4. If V be a given function of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... find the conditions that $\int Vdx$, between given limits, may be a maximum or minimum.

When a particle is attracted towards a fixed centre of force and moves in the brachistochrone, prove that the area described round the centre of force varies as the "action."

Let r, θ, be the polar coordinates of the path of the particle, and v its velocity at any time t. Then the time of describing any arc is equal to

$$\int\frac{(dr^2+r^2d\theta^2)^{\frac{1}{2}}}{v}=\int\left(1+r^2\frac{d\theta^2}{dr^2}\right)^{\frac{1}{2}}\cdot\frac{dr}{v}.$$

Now v is a function of r: thus, under the integral sign, the only unconnected variables are r and $\frac{d\theta}{dr}$: hence, the path being brachistochronous, we know by the Calculus of Variations, that the differential coefficient of

$$\frac{1}{v}\left(1 + r^2\frac{d\theta^2}{dr^2}\right)^{\frac{1}{2}}$$

with regard to $\frac{d\theta}{dr}$ must be constant, that is, that

$$\frac{r^2\frac{d\theta}{dr}}{v\left(1 + r^2\frac{d\theta^2}{dr^2}\right)^{\frac{1}{2}}} = \text{constant},$$

or

$$\frac{r^2 d\theta}{vds} = \text{constant}:$$

hence

$$\int r^2 d\theta \propto \int vds$$

$$\propto \int v^2 dt,$$

that is, the area varies as the action.*

6. Define the potential function V, and shew that, at any point (x, y, z), external to the attracting mass, it satisfies the equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0.$$

Hence prove that, if S be any closed surface to which all the attracting mass is external, dS an element of S, and dn an element of the normal drawn outwards at dS,

$$\iint \frac{dV}{dn} . dS = 0,$$

the integral being taken throughout the whole surface S.

* For this rider as well as for the rider to the first question of the afternoon's paper of January 22, the Junior Moderator is indebted to Mr. R. L. Ellis, of Trinity College.

If $dxdydz$ be an element of the space within the closed surface, it follows, since the equation $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0$ holds for all points within the surface, that

$$\iiint\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dxdydz=0.$$

The transformation of this equation, which is here required, is given by Professor Stokes in the *Cambridge and Dublin Mathematical Journal*, Vol. IV. p. 201.

7. Assuming the formulæ

$$l\alpha+m\beta+n\gamma=0,$$

$$\frac{l}{\alpha(v^2-a^2)}=\frac{m}{\beta(v^2-b^2)}=\frac{n}{\gamma(v^2-c^2)},$$

investigate the equation of the wave-surface in a biaxal crystal.

Prove that the direction of the vibration at any point of this surface coincides with the projection of the distance of the point from the centre of the surface upon the tangent plane at the point.

In the investigation of the equation of the wave-surface (see Griffin's *Treatise on Double Refraction*) we establish the formulæ

$$\frac{v^2-a^2}{r^2-a^2}.x=vl,\quad \frac{v^2-b^2}{r^2-b^2}.y=vm,\quad \frac{v^2-c^2}{r^2-c^2}.z=vn,$$

whence

$$vl-x=\frac{v^2-r^2}{r^2-a^2}.x,\quad vm-y=\frac{v^2-r^2}{r^2-b^2}.y,\quad vn-z=\frac{v^2-r^2}{r^2-c^2}.z.$$

Hence the formulæ

$$\frac{l}{\alpha(v^2-a^2)}=\frac{m}{\beta(v^2-b^2)}=\frac{n}{\gamma(v^2-c^2)}$$

give us the relations

$$\frac{x}{\alpha(r^2-a^2)}=\frac{y}{\beta(r^2-b^2)}=\frac{z}{\gamma(r^2-c^2)},$$

and therefore

$$\frac{vl - x}{\alpha} = \frac{vm - y}{\beta} = \frac{vn - z}{\gamma};$$

which formulæ, vl, vm, vn, being the coordinates of the end of the perpendicular v from the origin on the tangent plane, establish the truth of the proposition.

Stenarmont: *Liouville, Journal de Mathématiques*, tome VIII. p. 372. année, 1843.

EXAMINATION PAPERS FOR THE MATHEMATICAL TRIPOS 1857.

Moderators:

WILLIAM MAGAN CAMPION, M.A., Queens' College.
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TUESDAY, *Jan.* 6. 9—12.

1. PARALLELOGRAMS upon the same base, and between the same parallels, are equal to one another.

ABC is an isosceles triangle, of which A is the vertex: AB, AC, are bisected in D and E respectively; BE, CD, intersect in F: shew that the triangle ADE is equal to three times the triangle DEF.

2. In any triangle, the square on the side subtending either of the acute angles is less than the sum of the squares on the sides including this angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular drawn to this side, produced if necessary, from the opposite angular point.

The base of a triangle is given and is bisected by the centre of a given circle, the circumference of which is the locus of the vertex: prove that the sum of the squares on the two sides of the triangle is invariable.

3. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Prove also that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

4. Inscribe a circle in a given triangle.

Circles are inscribed in the two triangles formed by drawing a perpendicular from an angle of a triangle upon the opposite side, and analogous circles are described in relation to the two other like perpendiculars: prove that the sum of the diameters of the six circles together with the sum of the sides of the original triangle is equal to twice the sum of the three perpendiculars.

5. Similar triangles are to one another in the duplicate ratio of their homologous sides.

Any two straight lines, BB', CC', drawn parallel to the base DD' of a triangle ADD', cut AD in B, C, and AD' in B', C': BC', $B'C$, are joined: prove that the area ABC' or $AB'C$ varies as the rectangle contained by BB', CC'.

6. If two parallel planes be cut by another plane, their common sections with it are parallel.

A triangular pyramid stands on an equilateral base, and the angles at the vertex are right angles; shew that the sum of the perpendiculars on the faces from any point of the base is constant.

7. SP is the focal distance, PT the tangent, and PG the normal of any point P of a parabola: state the characteristic property of the tangent, and shew that $SP = ST = SG$, and that the subnormal of P is equal to the semi-latus rectum.

If the triangle SPG is equilateral, prove that SP is equal to the latus rectum.

If the ordinate of a point P bisects the subnormal of a point P', prove that the ordinate of P is equal to the normal of P'.

8. Prove that, in the parabola, $SY^2 = SP.SA$.

A circle is described on the latus rectum as diameter, and a common tangent QP is drawn to it and the parabola: shew that SP, SQ, make equal angles with the latus rectum.

9. Prove that the focal distances of any point of an ellipse make equal angles with the tangent at the point.

PG is the normal to an ellipse, terminating in the major axis; the circle, of which PG is a diameter, cuts SP, HP, in K, L, respectively: prove that KL is bisected by PG, and is perpendicular to it.

10. The perpendiculars from the foci of an ellipse upon the tangent meet the tangent in the circumference of a circle.

Prove also that if from H a line be drawn parallel to SP, it will meet the perpendicular SY in the circumference of a circle.

11. If tangents be drawn at the vertices of the axes of an hyperbola, the diagonals of the rectangle so formed are asymptotes to the four curves.

Prove that a perpendicular, drawn from the focus of an hyperbola to the asymptote, will intersect it in the directrix.

12. Shew that all sections of a right cone, made by planes parallel to a tangent plane of the cone, are parabolas, and that the foci lie on a cone having with the first a common vertex and axis.

TUESDAY, *Jan.* 6. $1\frac{1}{2}$...4.

1. THE imperial gallon contains 277·27 cubic inches, and a cubic foot of water at its maximum density weighs 62·42 lbs.; find the weight of a pint of water correctly to two places of decimals.

2. Supposing the cost of digging a trench to vary as the depth to which it is sunk and the quantity of earth taken out, and that the cost of digging a trench 3 feet broad by 8 feet deep is 9 pence per yard, what should be the cost of digging a trench 120 yards long, 5 feet broad, and 10 feet deep?

3. Define a fraction; and from your definition prove a rule for adding together two fractions with different denominators.

Add together the fractions,

$$\frac{a^2-bc}{(a+b)(a+c)},\quad \frac{b^2-ca}{(b+c)(b+a)},\quad \frac{c^2-ab}{(c+a)(c+b)}.$$

4. Prove a rule for extracting the square root of a compound algebraical quantity.

Shew that, if

$$x^4+ax^3+bx^2+cx+d$$

be a complete square, the coefficients satisfy the equation

$$c^2-a^2d=0.$$

Is it necessary that the coefficients satisfy any other equation?

5. Solve the equations,

$$\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0 \text{..............(1)}$$

$$(x-1)(x-2)(x-3)-(6-1)(6-2)(6-3)=0 \text{..............(2)}$$

$$\left.\begin{aligned}\frac{x+y}{m}+\frac{m}{x-y}-p=0\\ \frac{x-y}{m}+\frac{m}{x+y}+\frac{1}{p}=0\end{aligned}\right\}\text{............(3).}$$

6. Find the number of permutations of n things taken r together.

If the number of permutations of n things taken r together be denoted by the symbol

$${}^nP_r;$$

shew that the number of such permutations, in which p particular things occur, will be

$${}^rP_p.{}^{n-p}P_{r-p}.$$

7. Define a logarithm, and find $\log_5 3125$.

Prove that $\log_a N=\log_a b.\log_b N$; and, having given $\log_{10}2=\cdot301030$ and $\log_{10}7=\cdot845098$, find $\log_{10}98$, and $\log_{1000}\sqrt[5]{\left(\frac{4}{343}\right)}$.

8. Define the sine of an angle, and prove from your definition that for all values of θ numerically less than π, $\sin(\pi - \theta) = \sin\theta$.

Trace the variation in sign of the expression $\cos(\pi \sin\theta).\cos(\pi \cos\theta)$, as θ varies from 0 to $\frac{\pi}{2}$.

9. Find an expression for all the angles which have the same sine. Hence, if $\sin 3\theta$ be given, find the number of values of $\tan\theta$ which will be generally obtained; and illustrate the result geometrically.

10. Prove the formula

$$\cos(A - B) = \cos A.\cos B + \sin A.\sin B,$$

A being greater than B, and each angle less than 90°.

Also shew that

$$\cos a + \cos\beta + \cos\gamma + \cos(a + \beta + \gamma) = 4 \cos\tfrac{1}{2}(\beta + \gamma).\cos\tfrac{1}{2}(\gamma + a).\cos\tfrac{1}{2}(a + \beta),$$

and

$$\frac{\sin a + 2 \sin 3a + \sin 5a}{\sin 3a + 2 \sin 5a + \sin 7a} = \frac{\sin 3a}{\sin 5a}.$$

11. Determine the expression for the cosine of an angle of a triangle in terms of the sides, and deduce the expression for the sine.

If θ and ϕ be the greatest and least angles of a triangle, the sides of which are in arithmetic progression, prove that

$$4(1 - \cos\theta)(1 - \cos\phi) = \cos\theta + \cos\phi.$$

12. A quadrilateral can be inscribed in a circle; find the tangent of half of one of its angles in terms of its sides. If a circle can be inscribed in the quadrilateral, shew that the fourth root of the product of its sides is a mean proportional between its semi-perimeter and the radius of the inscribed circle.

WEDNESDAY, *Jan.* 7. 9...12.

1. ASSUMING that the resultant of two forces, acting at a point, is represented in direction by the diagonal of a parallelogram, the sides of which represent the forces in direction and magnitude; shew that the diagonal will also represent the resultant in magnitude.

Shew that within a quadrilateral, no two sides of which are parallel, there is but one point, at which forces, acting towards the corners and proportional to the distances of the point from them, can be in equilibrium.

2. Shew that if three forces acting in one plane hold a body in equilibrium, they either pass through a point or are parallel to each other.

A heavy equilateral triangle, hung up on a smooth peg by a string the ends of which are attached to two of its angular points, rests with one of its sides vertical; shew that the length of the string is double the altitude of the triangle.

3. Find the relation of the Power to the Weight in the single moveable pully, when the strings are not parallel.

An endless string hangs at rest over two pegs in the same horizontal plane, with a heavy pully in each festoon of the string; if the weight of one pully be double that of the other, shew that the angle between the portions of the upper festoon must be greater than 120°.

4. Find the ratio of the Power to the Weight in the Wheel and Axle, in order that there may be equilibrium.

Explain the meaning of the terms 'mechanical advantage' and 'efficiency', as applied to machines; and shew that, in the Wheel and Axle, what is gained in power is lost in velocity.

5. Define the centre of gravity of a heavy body; and determine the position of the centre of gravity of a pyramid on a triangular base.

Find the centre of gravity of the solid included between two right cones on the same base, the vertex of one cone being within the other; and determine its limiting position if the vertices approach to coincidence.

6. State the laws of friction; and explain what is meant by the term 'coefficient of friction'.

A uniform rod is held at a given inclination to a rough horizontal table by a string attached to one of its ends, the other end resting on the table; find the greatest angle at which the string can be inclined to the vertical without causing the end of the rod to slide along the table.

7. Define uniform motion and uniformly accelerated motion, and explain how they are measured.

If f be the measure of a uniform acceleration, when t minutes and a feet are taken as the units of time and space, and f' the measure of the same acceleration, when a' feet are taken as the unit of space, find the number of minutes in the unit of time.

8. State the second law of motion; and apply it to prove that a force, of uniform intensity and direction, acting on a given particle originally at rest, produces a uniform acceleration of its motion.

State the convention with respect to units which is necessary, in order that the equation $P = Mf$ may represent the relation between the numerical measures of force, mass and acceleration; and supposing the unit of force to be 5 lbs. and the unit of acceleration, referred to a foot and a second as units, to be 3, find the unit of mass.

9. An elastic ball A, moving with a given velocity on a smooth horizontal plane, impinges directly on a ball B of the same radius, at rest;

determine the velocity of each after the impact, indicating at what points of your reasoning any law of motion or other result of experiment is assumed.

Shew that, if B afterwards impinge perpendicularly on a smooth wall, the original distance of which from the nearest point of B is given, the time, which elapses between the first and second impact of the balls, will be independent of their radius.

10. Shew that a particle, projected in any direction not vertical, and acted upon by gravity only, will describe a parabola.

An inclined plane passes through the point of projection; find the condition that the particle may impinge perpendicularly on the plane; and, in that case, shew that its range on the plane is equal to

$$\frac{2v^2}{g} \cdot \frac{\sin a}{1 + 3\sin^2 a},$$

where v is the velocity of projection, and a the inclination of the plane to the horizon.

11. Two given weights are connected by an inextensible string, which passes over a smooth pully; determine the motion of each weight and the tension of the string.

The system being initially at rest, find the weight which, let fall at the beginning of the motion from a point vertically above the ascending weight, so as to impinge upon it, will instantaneously reduce the system to rest. Will the system afterwards remain at rest?

12. A seconds pendulum is carried to the top of a mountain 3000 feet high; assuming that the force of gravity varies inversely as the square of the distance from the Earth's centre, and that the Earth's radius is 4000 miles, find the number of oscillations lost in a day.

Also determine how much the pendulum must be shortened in order that it may oscillate seconds on the mountain.

WEDNESDAY, *Jan.* 7. $1\frac{1}{2}$...4.

1. GIVE the meanings of the several symbols which are employed in the formula $p = g\rho z$.

If one second be the unit of time, what must be the unit of length, in order that the above formula may give the pressure in pounds, supposing the unit of volume of the standard substance to weigh 16 lbs.?

2. Prove that the pressure of a fluid on any surface is equal to the weight of a column of the fluid, the base of which is equal to the area of the surface, and altitude equal to the depth of the centre of gravity of the surface below the surface of the fluid.

The inclinations of the axis of a submerged solid cylinder to the vertical in two different positions are complementary to each other; P is the difference between the pressures on the two ends in the one, and P' in the other position: prove that the weight of the displaced fluid is equal to

$$(P^2 + P'^2)^{\frac{1}{2}}.$$

3. Describe an experiment to shew that the pressure of a given mass of air at a given temperature varies as its density. How is this ratio to be modified when the temperature, as well as the density, varies?

A volume of air of any magnitude, free from the action of force, and of variable temperature, is at rest: if the temperatures at a series of points within it be in arithmetical progression, prove that the densities at these points are in harmonical progression.

4. A body of given volume is immersed totally in a given fluid; find the magnitude and direction of the resultant fluid pressure.

A body is floating in a fluid; a hollow vessel is inverted over it and depressed: what effect will be produced in the position of the body, (1) with reference to the surface of the fluid within the vessel, (2) with reference to the surface of the fluid outside?

5. Describe the Diving Bell, and find the volume of the air in the bell at any depth below the surface.

If P be the weight of the bell, P' of a mass of water the bulk of which is equal to that of the material of the bell, and W of a mass of water the bulk of which is equal to that of the interior of the bell, prove that, supposing the bell to be too light to sink without force, it will be in a position of unstable equilibrium, if pushed down until the pressure of the enclosed air is to that of the atmosphere as W to $P - P'$.

6. Explain the principle of the common Barometer. Given the pressure of the air at a given time on a square inch, shew how to find the height in inches of the barometric column.

Why is the rising or falling of a barometer generally an indication of coming fair or foul weather? Why is a sudden fall a sign of a coming gale?

7. Find the geometrical focus (1) of a pencil of rays incident directly upon a plane refracting surface, and (2) of a pencil of incident directly upon a refracting plate.

A ray, passing through a point Q, is incident upon a refracting plate; q is the intersection of the emergent ray, produced backwards, with the normal to the plate through Q: if the angle of incidence be equal to $\tan^{-1}\mu$, and t be the thickness of the plate, prove that

$$Qq = \frac{\mu^2 - 1}{\mu^2} \cdot t.$$

8. A ray of light passes through a prism in a plane perpendicular to its edge: shew that, if ϕ and ψ be the angles of incidence and emergence and i the refracting angle of the prism, the deviation is equal to

$$\phi \pm \psi - i, \text{ or } \psi - \phi - i,$$

according as the incident ray makes an acute angle with the face of the prism towards the thicker end or the edge. Under what convention will these expressions for the deviation be all represented by $\phi + \psi - i$, and with this convention for what value of ϕ will ψ change sign?

9. Explain the formation of an image by reflection, and find the magnitude and position of the image of a given object placed before a plane mirror.

The faces of two walls of a room, meeting at right angles, are covered with plane mirrors: shew that a person will be able to see but one complete image of himself in either wall.

10. A diverging pencil of rays is incident directly upon a concave spherical refractor: find the geometrical focus of the refracted pencil.

A short object is placed perpendicularly on the axis of the refractor, and at a distance from it equal to $\frac{f}{\mu}$, f being the focal length: prove that the linear magnitude of the virtual image is half that of the object.

11. Describe the human eye as an optical instrument. When a pencil of rays is refracted through the eye, at what point of its passage does it experience its principal modification of form; and what is the most probable hypothesis in regard to the change of configuration of the eye by which it adjusts itself to distinct vision at different distances?

An eye is placed close to a sphere of glass, a portion of the surface of which, most remote from the eye, is silvered: prove that, assuming eight inches to be the least distance of distinct vision, the eye cannot see a distinct image of itself unless the diameter of the sphere be at least ten inches in length.

12. Describe Galileo's telescope, and trace a pencil of rays through it.

State what would be the effect on the image—

(1) of increasing the size of the object-glass } eye-glass unaltered;
(2) „ focal length }

(3) „ size of the eye-glass } object-glass unaltered.
(4) „ focal length }

THURSDAY, *Jan.* 8. 9...12.

1. ENUNCIATE and prove Newton's fourth Lemma.

Apply this Lemma to shew that the volume of a right cone is one third of that of the cylinder on the same base and of the same altitude.

2. Enunciate Lemma XI., and prove it when the subtenses are parallel.

An arc of continuous curvature PQQ', is bisected in Q; PT is the tangent at P; shew that ultimately, as Q' approaches P, the angle $Q'PT$ is bisected by QP.

3. Shew that, if a subtense be drawn from the extremity of an arc of finite curvature, in any direction, the chord of curvature parallel to that direction is the limit of the third proportional to the subtense and the arc.

Hence find the chord of curvature through the focus at any point of an ellipse; and prove that half this chord is a harmonic mean between the focal distances of the point.

4. State and prove Proposition I.

Will the velocity of the body or the rate at which areas are swept out about the centre of force be affected by any sudden change in the law of force?

A body moves in a parabola about a centre of force in the vertex; shew that the time of moving from any point to the vertex varies as the cube of the distance of the point from the axis of the parabola.

5. A body is revolving in an ellipse, find the law of centripetal force tending to the centre of the ellipse.

Shew that the time in which any given area will be swept out by the radius vector is independent of the eccentricity of the ellipse, if the area of the ellipse be given.

6 If any number of bodies revolve in ellipses about a common centre, and the centripetal force varies inversely as the square of the distance; the squares of the periodic times are proportional to the cubes of the major axes.

A particle moves in an ellipse about the centre of force in the focus S: when the particle is at B, the extremity of the minor axis, the centre of force is changed to S' in SB, so that $S'B$ is one-fifth of SB, and the absolute force is diminished to one-eighth of its original value; shew that the periodic time is unaltered, and that the new minor axis is two-fifths of the old.

7. Define the term, "zenith" and explain some method for determining the zenith of a given observatory.

How would an increase in the Earth's velocity of rotation affect the latitude of a given place, supposing the form of the Earth to remain unaltered?

8. What conditions must be satisfied in order that the transit instrument may be in accurate adjustment?

Shew how by aid of this instrument the difference in right ascension of two stars may be determined; and state the principal astronomical assumptions on which the truth of this determination depends.

9. Explain the phrases "mean solar time" and "equation of time."

Shew that in the month of February the equation of time is additive.

Account for the fact that the time of the Sun's setting as given in the ordinary Almanacs is not the latest on the longest day.

10. Prove that generally the apparent place of a star will depend upon the ratio of the velocity of the Earth in her orbit to the velocity of light.

Find the least diurnal velocity of rotation of the Earth, which will render sensible to an observer at the equator the aberration due to this cause, the least appreciable angle being $1''$.

11. Describe the apparent motion of the Moon among the stars, and the real motion of its centre of gravity about the Sun, illustrating the latter description by a figure.

What is inferred from the fact that, with slight variations, the same portion of the Moon's surface is always presented to the Earth? How much should the Moon's rate of rotation about its centre of gravity be increased, in order that its whole surface might be seen in the course of one orbital revolution?

12. Explain the method of determining the longitude by means of Lunar Distances.

On January 1st 1855, at the mean time 9 hrs. 42 min. 8 secs. P.M., the distance of α Arietis from the Moon's centre was calculated from observations to be $45^\circ\ 30'\ 16''$: at noon and at 3 P.M. Greenwich mean time, the distances are $44^\circ\ 56'\ 11''$, and $46^\circ\ 23'\ 39''$ respectively: find the longitude of the place of observation.

THURSDAY, *Jan.* 8. 1...4.

1. THREE circles, A, B, C, intersect in a common point, the other intersections of (B, C), (C, A), (A, B), being, α, β, γ, respectively. If b, c, be points in B, C, respectively, such that b, a, c, lie in a straight line, prove that a, the intersection of $b\gamma$, $c\beta$, produced, lies in the circle A.

2. Shew that the sum of all the harmonic means, which can be inserted between all the pairs of numbers the sum of which is n, is

$$\tfrac{1}{3}(n^2-1).$$

3. Eliminate θ between the equations

$$\frac{x}{a} = \cos\theta + \cos 2\theta,$$

$$\frac{y}{b} = \sin\theta + \sin 2\theta.$$

4. From a point on a hill-side of constant inclination the angle of elevation of the top of an obelisk on its summit is observed to be α, and, a feet nearer to the top of the hill, to be β; shew that, if h be the height of the obelisk, the inclination of the hill to the horizon will be

$$\cos^{-1}\left\{\frac{a}{h}\cdot\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}\right\}.$$

5. Each of three circles, within the area of a triangle, touches the other two, touching also two sides of the triangle: if a be the distance between the points of contact of one of the sides, and b, c, be like distances on the other two sides, prove that the area of the triangle, of which the centres of the circles are the angular points, is equal to

$$\tfrac{1}{4}(b^2c^2 + c^2a^2 + a^2b^2)^{\frac{1}{2}}.$$

6. The acute angles, which the distances of two points of an ellipse from the same focus make with the respective tangents at the points, are complementary to each other: prove that the square on the semi-axis minor is a mean proportional between the areas of the two triangles, of which the two points are the respective vertices, and the distance between the foci the common base.

Shew that the problem is impossible unless the axis minor is less than the distance between the foci.

7. CP, CD, are two conjugate semi-diameters of an ellipse: Tt is a tangent parallel to PD: a straight line CIJ cuts at a given angle PD, Tt, in I, J, respectively: prove that the loci of I, J, are similar curves.

8. A fine string $ACBP$, tied to the end A of a uniform rod AB of weight W, passes through a fixed ring at C, and also through a ring at the end B of the rod, the free end of the string supporting a weight P: if the system be in equilibrium, prove that

$$AC : BC :: 2P + W : W.$$

9. A picture is hung up against a rough vertical wall by a string fastened to a point in its back, so that the picture inclines forwards; apply the principle of the triangle of forces to find the inclination of the string to the wall, when its tension is the least possible.

10. A lamina, cut into the form of an equilateral triangle, is hung up against a smooth vertical wall by means of a string attached to the

middle point of one side, so as to have a corner in contact with the wall; shew that, when there is equilibrium, the reaction of the wall and the tension of the string are independent of the length of the string, and that, if the string be beyond a certain length, equilibrium in such a position is impossible.

11. A ball is projected from the middle point of one side of a billiard table, so as to strike in succession one of the sides adjacent to it, the side opposite to it, and a ball placed in the centre of the table; shew that, if a and b be the lengths of the sides of the table, and e the elasticity of the ball, the inclination of the direction of projection to the side a of the table from which it is projected must be

$$\tan^{-1}\left\{\frac{b}{a}\cdot\frac{1+2e}{1+e}\right\}.$$

12. A perfectly elastic ball is projected at an inclination β to a plane inclined to the horizon at an angle α, so as to ascend it by bounds; find the inclination to the plane at which the ball rises at the n^{th} rebound, and shew that it will rise vertically if

$$\cot\beta = (2n+1)\tan\alpha.$$

13. A string, charged with $n+m+1$ equal weights fixed at equal intervals along it, and which would rest on a smooth inclined plane, with m of the weights hanging over the top, is placed on the plane with the $(m+1)^{\text{th}}$ weight just over the top; shew that, if a be the distance between each two adjacent weights, the velocity which the string will have acquired, at the instant the last weight slips off the plane, will be

$$\{nag\}^{\frac{1}{2}}.$$

14. A perfectly elastic ball is projected with a given velocity from a point between two parallel walls, and returns to the point of projection, after being once reflected at each wall; prove that its angle of projection is either of two complementary angles.

15. A particle is attracted to one centre of force and repelled from another, both forces varying as the distance: prove that, if the absolute intensities of the forces are equal, the path of the particle is a parabola.

16. When a body arrives at a point P of an elliptic orbit, which it is describing about one focus S, the centre of force is suddenly transferred to the other focus H: supposing the orbit to remain the same as before, prove that, μ denoting the absolute force in the former, and μ' in the latter case,

$$\mu : \mu' :: SP^2 : HP^2.$$

17. A solid triangular prism, the faces of which include angles α, β, γ, is placed in any position entirely within an inelastic gravitating fluid: if P, Q, R, be the pressures on the three faces, which are respectively opposite to the angles α, β, γ, prove that

$$P \operatorname{cosec}\alpha + Q \operatorname{cosec}\beta + R \operatorname{cosec}\gamma$$

is invariable so long as the depth of the centre of gravity of the prism is unchanged.

18. A heavy sphere is placed in a vertical cylinder, filled with atmospheric air, which it exactly fits. Find the density of the air in the cylinder when the sphere is in a position of permanent rest.

19. A solid formed of two co-axial right cones, of the same vertical angle, connected at their vertices, is placed with one end in contact with the horizontal base of a vessel: water is then poured into the vessel; shew that if the altitude of the upper cone be treble that of the lower, and the common density of the spindle four-sevenths that of the water, it will be upon the point of rising when the water reaches to the level of its upper end.

20. A fish is floating in a cubical glass tank filled with water, with its head in one corner and its tail towards the one diagonally opposite; describe the appearance which will be presented to an eye looking towards the corner in the direction of the length of the fish, and in the same horizontal plane with it.

21. Two rays emanate from a point in the circumference of a reflecting circle, in the plane of the circle: supposing that their n^{th} points of incidence are coincident, prove that the angle between their original directions is any one of a series of $n - 1$ angles in arithmetical progression.

22. A luminous globe falls from a point above the Earth's surface in a dark night: shew that it will look like a bright falling column, elongating as it descends.

If c_1, c_2, c_3, be the lengths of the apparent column at the ends of times t_1, t_2, t_3, from the commencement of the fall, prove that, gravity being considered constant, and the resistance of the air being neglected,

$$t_1(c_2 - c_3) + t_2(c_3 - c_1) + t_3(c_1 - c_2) = 0.$$

MONDAY, *Jan.* 19. 9...12.

1. DEFINE a couple, and find the condition that two couples acting on a body in the same plane may hold it in equilibrium.

Find the moment of the couple which is sufficient to sustain a right cone, with its vertex on a rough plane of given inclination and its base parallel to the plane; the roughness of the plane being just sufficient to prevent the vertex from sliding.

2. Assuming that any system of forces acting upon a rigid body may be reduced to a resultant force and a resultant couple, find the relation which must subsist among the forces that these resultants may be equivalent to a single force.

When this condition is satisfied, find the locus of a point in the body by fixing which the body will be held at rest.

3. A heavy string of uniform density and thickness is suspended from two given points; find the equation of the curve in which the string hangs when it is at rest.

Compare the curvatures at the lowest points of two catenaries formed by an inextensible and by an extensible string, the tension at the lowest point of each catenary being τ, and the modulus of elasticity w.

4. A particle of mass m describes a plane curve under the action of forces of which the components parallel to the tangent and normal are $m.T$ and $m.N$: shew that

$$T = \frac{d^2s}{dt^2}, \quad N = \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2.$$

If ϕ be the angle which the tangent at any point of the path makes with a fixed line, the differential equation of the path will be

$$\frac{d}{ds}\left(N\frac{ds}{d\phi}\right) = 2T.$$

5. A heavy particle, suspended from a fixed point by an elastic string, makes vertical oscillations in a medium of which the resistance varies as the square of the velocity: determine the velocity of the particle for any position, neglecting the weight of the string and supposing the motion to commence when the string is unstretched, and the particle to have no initial velocity.

Deduce the greatest extension of the string, supposing the motion to take place in a vacuum.

6. Two particles connected by a stretched inextensible string are constrained to move in a fine curvilinear tube in a vertical plane: determine the motion.

If the tube be cycloidal, the axis of the cycloid being vertical and the vertex upwards; shew that the tension of the string is constant throughout the motion.

7. Every point of a fluid at rest is acted upon by impressed forces the resultant of which always tends to a fixed centre: prove that at a point, the distance of which from the fixed centre is r,

$$dp = -\rho F dr,$$

where p is the pressure at the point, ρ the density, and F the resultant of the impressed forces, referred to a unit of mass.

If the resultant force varies as r^{3n}, and the fluid be homogeneous and of given mass, shew that the pressure on a diametral plane varies as ρ^{-n}.

8. A body floats in a fluid: determine the position of its metacentre with reference to a vertical plane of displacement dividing the body symmetrically through its centre of gravity.

A cylindrical diving bell is suspended with its axis vertical at a depth such that the water rises half way up the bell: find the least distance of the centre of gravity of the bell from the centre of its upper surface, consistent with the condition that the equilibrium may be stable with reference to an angular displacement of the axis.

9. A small pencil of rays is incident obliquely on a plane refracting surface; find the positions of the primary and secondary foci of the refracted pencil.

If the pencil consists of common light, shew that the primary foci of the pencils of different colours will lie on a curve of the third order.

10. Obtain an equation for determining the equatoreal interval of a given wire and the mean wire of a transit instrument by observations on the transit of the pole star.

Explain how the determination of this value for all the wires enables an observer to find the time of transit across the mean wire, when the time across some of the wires is not noted.

11. Describe Flamsteed's method of determining the position of the first point of Aries, mentioning the astronomical instruments required for the purpose.

MONDAY, *Jan.* 19. $1\frac{1}{2}$...4.

1. SHEW that, if

$$A + Bx + Cx^2 + \ldots = a + bx + cx^2 + \ldots$$

for all values of x, and if the coefficients do not increase without limit,

$$A = a, \quad B = b, \quad C = c, \quad \&c.$$

Find the sum of the series

$$1^4 + 2^4 + 3^4 + \ldots + n^4;$$

and prove that, when n is indefinitely increased,

$$\Sigma_1^n (r^4) : n^2\Sigma_1^n (r^2) :: 3 : 5.$$

2. Expand $(\cos\theta)^{2n}$ in a series of cosines of multiples of θ, n being a positive integer. Hence deduce the expansion of $(\sin\theta)^{2n}$ in a series of cosines of multiples of θ.

3. Prove that the base of Napier's system of logarithms is incommensurable.

Prove also that it cannot be a root of a quadratic equation the coefficients of which are rational.

4. Prove that impossible roots enter rational equations by pairs.

If $e^{a\sqrt{(-1)}}$ be a root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \ldots + p_n = 0,$$

prove that

$$p_1 \sin a + p_2 \sin 2a + p_3 \sin 3a + \ldots + p_n \sin na = 0.$$

5. Describe Horner's method of approximating to the roots of equations; and apply it to find the cube root of 37 to four places of decimals.

6. Find the expression for the distance of a given point from a straight line of which the equation is given.

The distance of a point (x_1, y_1) from each of two straight lines, which pass through the origin of coordinates, is δ; prove that the two lines are represented by the equation

$$\frac{(x_1y - xy_1)^2}{x^2 + y^2} = \delta^2.$$

7. Shew that the equations

$$x = a \sec\phi, \quad y = b \tan\phi,$$

represent an hyperbola, and give a geometrical interpretation of the angle ϕ.

If P, Q, be points in the one, and P', Q', in the other of two confocal hyperbolas, and if the values of ϕ at P', Q', be respectively equal to those at P, Q; prove that PQ' is equal to $P'Q$.

8. State Napier's rules for the solution of right-angled spherical triangles, and prove them for the case in which the complement of the hypotenuse is the middle part.

If three arcs of great circles AP, BQ, CR, intersect at right angles the sides BC, CA, AB, in P, Q, R, respectively, prove that they all pass through the same point O, and that

$$\frac{\tan AP}{\tan OP}, \quad \frac{\tan BQ}{\tan OQ}, \quad \frac{\tan CR}{\tan OR},$$

are respectively equal to

$$1 + \frac{\cos A}{\cos B . \cos C}, \quad 1 + \frac{\cos B}{\cos C . \cos A}, \quad 1 + \frac{\cos C}{\cos A . \cos B}.$$

9. Find the polar equation of the tangent at a point of the conic section

$$\frac{c}{r} = 1 + e \cos\theta.$$

Find the polar equation of the straight line through the foot of the directrix perpendicular to the tangent, and shew that the locus of its intersection with the radius vector at the point of contact is a circle.

10. The perpendicular from the origin on a plane is of given length and makes given angles with the axes; find the equation of the plane.

Find the equation of the plane which passes through the origin and through the line of intersection of the planes

$$Ax + By + Cz = D, \quad A'x + B'y + C'z = D';$$

and determine the condition that it may bisect the angle between them.

11. Shew that through any point of the surface

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{2x}{a},$$

two straight lines can be drawn, entirely coincident with the surface.

Prove that the points on the surface, the straight lines through which, coincident with the surface, are at right angles to each other, lie in a plane parallel to the plane yz, and at a distance from it equal to

$$\frac{c^2 - b^2}{2a}.$$

12. Investigate the positions of the centric circular sections of an ellipsoid.

If θ, ϕ, be the inclinations of the normal of a centric plane section of the ellipsoid

$$x^2\left(\frac{1}{b^2} - \frac{1}{f^2}\right) + \frac{y^2}{b^2} + z^2\left(\frac{1}{b^2} + \frac{1}{f^2}\right) = 1$$

to the normals of the planes of the circular sections, find the equation of the trace of the plane of the section on the plane of zx.

TUESDAY, *Jan.* 20. 9...12.

1. ELIMINATE x, y, z, between the equations

$$\frac{y}{z} + \frac{z}{y} = a, \quad \frac{z}{x} + \frac{x}{z} = b, \quad \frac{x}{y} + \frac{y}{x} = c.$$

2. From a bag containing a counters, some of which are marked with numbers, b counters are to be drawn; and the drawer is to receive a number of shillings equal to the sum of the numbers on the counters which he draws; if the sum of the numbers on all the counters be n, what will be the value of his chance?

3. O is the middle point of a given straight line AA': BOB' is a straight line perpendicular to AA': P, P', are two points in the plane of AA', BB': perpendiculars from P' upon AP, $A'P$, cut $A'A$, in C, C', respectively: if OC, OC', be equal to each other and of given magnitude, prove that the distances of P, P', from BB' are in a constant ratio.

4. The foci of a given ellipse A lie in an ellipse B, the extremities of a diameter of A being the foci of B: prove that the eccentricity of B varies as the diameter of A.

5. C is the centre of an ellipse, G the foot of a normal at any point P, and O the corresponding centre of curvature: find the distance of P from the axis minor, in order that the area of COG may be the greatest possible.

6. The corners of a leaf of a book are turned down so as to meet and to make the length of one crease always n times that of the other; shew that each corner will describe a portion of the curve

$$x^2 \{y^2 + (c-x)^2\}^3 = n^2 (c-x)^2 (x^2+y^2)^3,$$

the outer edge of the leaf, the length of which is c, being taken as the axis of x, and the lower edge as the axis of y.

7. A heavy ring is suspended from a point by any number of equal strings attached to it symmetrically; and another ring of the same weight but of smaller radius is in equilibrium when resting on the strings at their middle points; if R, r, be the radii of the rings and $2l$ the length of each string, shew that

$$4R^2 - 8Rr + 3r^2 - 3l^2 = 0.$$

8. A thread without weight carrying a heavy bead has its extremities fastened to two points in the same vertical line; if the bead and thread be made to revolve uniformly about this line with an angular velocity ω; shew that, when the bead is in equilibrium relatively to the thread, its distance below the horizontal plane midway between the points of attachment of the thread will be

$$\frac{g}{\omega^2} \cdot \frac{l^2}{l^2 - a^2},$$

$2l$ being the length of the thread, and $2a$ the distance between the points of attachment.

9. A flexible chain, the ends of which are united, hangs over two pegs, in a horizontal line, in the form of two festoons; if P, P', be the tensions at the vertices of the festoons, and a, a', the inclinations of the festoons to the horizon at either peg, prove that the weight of half the chain is equal to

$$P \tan a + P' \tan a'.$$

Prove also that the weight of a piece of the chain, equal in length to the distance between the vertices of the festoons, is equal to $P \sim P'$.

10. A triangular lamina has a small ring at each of its angular points, which slides on a smooth wire occupying the position of the circle circumscribing the triangle; determine the motion of the triangle when the wire is held in any position, and find the time of a small oscillation when the wire is so held that the triangle is nearly in its position of stable equilibrium.

11. Shew, by aid of the formulæ

$$2\cot 2x = \cot x - \tan x, \quad 2\operatorname{cosec} 2x = \cot x + \tan x,$$

that if
$$\tan x = a_1 x + a_3 x^3 + a_5 x^5 + \ldots,$$

then
$$\cot x = \frac{1}{x} - \frac{a_1}{2^2-1}x - \frac{a_3}{2^4-1}x^3 - \frac{a_5}{2^6-1}x^5 - \ldots,$$

and
$$\operatorname{cosec} x = \frac{1}{x} + \frac{2-1}{2^2-1}\frac{a_1}{2}x + \frac{2^3-1}{2^4-1}\frac{a_3}{2^3}x^3 + \frac{2^5-1}{2^6-1}\frac{a_5}{2^5}x^5 + \ldots$$

12. Circles are described upon the radii vectores of the loop of a lemniscate as diameters, passing through the pole; find the locus of their ultimate intersections, and shew that its area is double that of the loop.

13. A semicircular tube of very small bore containing an elastic string fastened to one of its extremities is revolving with a uniform angular velocity ω about a vertical axis through that extremity perpendicular to its plane, and the string in its stretched state subtends an angle a at the centre of the circle the radius of which is a; shew that, if the modulus of elasticity be the weight of a length l of the unstretched string, and $lg = 4a^2\omega^2\cos^2\frac{a}{2}$ the unstretched length of the string will be

$$2a\cos\frac{a}{2}\log\tan\frac{\pi+a}{4}.$$

14. Two spheres, the molecules of which attract according to the law of the inverse square, were originally in contact; if W, W', W'', be the labouring forces which have been expended in pushing them asunder in the line of their centres, when the distances between their centres are respectively a, a', a''; prove that

$$W\left(\frac{1}{a'} - \frac{1}{a''}\right) + W'\left(\frac{1}{a''} - \frac{1}{a}\right) + W''\left(\frac{1}{a} - \frac{1}{a'}\right) = 0.$$

15. Normals to an ellipsoid through a curve traced on its surface intersect a principal plane in a circle of given radius; prove that the projection of the curve on the plane encloses an invariable area.

16. A curve is traced upon a terrestrial globe, of such a form that the longitude of any point is equal to its north polar distance; prove that the whole length of the curve between the north and south poles is equal to the meridian distance between the north and south poles of an oblate spheroid, the eccentricity of which is $\frac{1}{\sqrt{2}}$ and axis equal to the diameter of the globe.

17. A closed vessel in the form of a right cone is placed with its base on a horizontal plane: supposing it to be filled with fluid through a small orifice at its vertex, prove that the horizontal tension of the vessel at any point varies as the area of the circular section through the point.

18. A luminous point is placed at one of the foci of a semi-elliptic arc bounded by the axis major; prove that the whole illumination of the arc varies inversely as the latus rectum.

19. A homogeneous globe is placed upon a perfectly rough table, very near to a centre of force in the surface of the table, the law of attraction being that of the inverse square; prove that the square of the time of an oscillation varies as the volume of the sphere.

20. An inelastic ball, of given radius, is dropped from the window of a carriage, travelling uniformly along a level road, upon the wheel, which it hits at the highest point; determine the subsequent motion of the ball relatively to the carriage, the rim of the wheel being perfectly rough.

TUESDAY, *Jan.* 20. $1\frac{1}{2}$...4.

1. AN object is viewed through a lens by an eye placed on the axis: describe the several defects to which the image is subject, assuming the light to be homogeneous.

Illustrate the defect of angular distortion by a figure, and explain generally how this defect is diminished by using Huyghen's eye-piece.

2. Explain the nature of the difficulty which prevents the formation of a completely achromatic combination of lenses.

A pencil of light is refracted, centrically, and with small obliquity, through two thin lenses in contact; find the condition of achromatism. If such a combination be used as a microscope, determine which of the lenses has the greater dispersive power.

3. Determine the interval from sunrise to sunset at a given place.

If the increase of the sun's declination from noon to noon be $\Delta°$, and t, t', be the times from sunrise to noon and from noon to sunset respectively, shew that

$$t' - t = t \cdot \frac{\Delta}{180} \cdot \frac{\sin l \sec \delta}{\sqrt{\{\cos(l-\delta)\cos(l+\delta)\}}} \quad \text{nearly,}$$

where l is the latitude of the place, and δ is the sun's declination at sunrise.

4. Calculate the aberration in right ascension of a star, the right ascension and declination of which are given, and find its greatest value.

5. A particle, acted upon by given forces, moves on a given smooth surface; shew how to determine its motion, and the pressure on the surface.

If the surface be a smooth cone, placed with its axis vertical and vertex downwards, and if gravity be the only force acting, shew that the differential equation of the projection on the horizontal plane of the path of the particle, is

$$\frac{d^2u}{d\theta^2} + u \sin^2 a = \frac{g \sin a \cos a}{h^2 u^2},$$

where u is the reciprocal of the distance of the particle from the axis, θ the angle between this distance and a fixed vertical plane, h constant, and a the semi-vertical angle of the cone.

6. Determine the motion of a body revolving about a fixed horizontal axis under the action of gravity: and shew that there is a point in the straight line through the centre of gravity perpendicular to the axis such that, if the whole mass of the body were there collected, and hung by a weightless string from the axis, the angular motion of the point would under the same initial circumstances be the same as that of the body.

How would the pressure on the axis be affected by such a supposed change in the arrangement of the mass?

7. Investigate the equations of fluid motion, referred to rectangular axes.

An elastic fluid, not acted upon by any impressed forces, flows uniformly through a cylindrical tube; compare the pressures of the fluid for two different velocities, and hence explain the following experiment.

To one end of a tube is fitted a plane disc which is capable of sliding on wires projecting from the end of the tube in directions parallel to the axis: if the disc be placed at a small distance from the end, and a person blow steadily into the other end, the disc will remain nearly stationary.

8. A perfectly flexible and slightly extensible cord is stretched between two fixed points, at which its extremities are fastened: if a small disturbance be excited in it, obtain equations for calculating the motion; and determine the velocities with which transversal and longitudinal vibrations are respectively propagated along the cord.

9. A particle revolves about a centre of force in an orbit nearly circular; determine approximately the angle between two consecutive apsidal distances.

Hence shew that the mean central disturbing force of the Sun will cause the apses of the Moon's orbit to progress, assuming the Earth's orbit to be circular.

10. Assuming the following equation for determining the Moon's longitude,

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2},$$

where
$$\frac{P}{h^2u^2} = \frac{\mu}{h^2} - \frac{m'u'^3}{2h^2u^3}\left[1 + 3\cos\{(2-2m)\theta - 2\beta\}\right],$$

find the term of the second order, in the expression for θ, of which the period is one year.

Considering only the effect of this term, and assuming $e' = \frac{1}{60}$ and $\sin m\pi = \frac{1}{4}$, find approximately in minutes the difference between the greatest and least periodic times of the Moon.

WEDNESDAY, *Jan.* 21. 9...12.

1. SHEW that the circle, which cuts orthogonally three given circles lying in a plane, has its centre at the radical centre of these circles.

2. If $\dfrac{a\sin^2\theta + b\sin^2\phi}{b\cos^2\theta + c\cos^2\phi} = \dfrac{b\sin^2\theta + c\sin^2\phi}{c\cos^2\theta + a\cos^2\phi} = \dfrac{c\sin^2\theta + a\sin^2\phi}{a\cos^2\theta + b\cos^2\phi}$;

then will $$a^3 + b^3 + c^3 = 3abc.$$

3. A parabola slides between two rectangular axes; find the curve traced out by any point in its axis; and hence shew that the focus and vertex will describe curves of which the equations are

$$x^2y^2 = a^2(x^2 + y^2), \quad x^2y^2(x^2 + y^2 + 3a^2) = a^6,$$

$4a$ being the latus rectum of the parabola.

4. Shew that, if in the equation

$$ax^2 + by^2 + 2cxy - f = 0,$$

the parameter f alone vary, the focus of the conic represented will lie in either of two straight lines; if, either a or b vary, the other coefficients remaining constant, the focus will lie in a rectangular hyperbola; and, if c alone vary, the focus will lie in the curve

$$ab(x^4 + y^4) - (a^2 + b^2)x^2y^2 + f(a - b)(x^2 - y^2) = 0.$$

5. A right vertical cylinder with circular ends carries a hand upon its upper face, equal in length to a radius of the end, and moveable about an axis coincident with the axis of the cylinder: the extremity of the hand is attached by a fine elastic thread to a point in the circumference of the lower end of the cylinder; and, when the thread is vertical, it is stretched to its natural length: if the hand be made to revolve through any angle α, and then let go, find its angular velocity in any subsequent position; and shew that, if the angle of displacement, α, be very small, the time of an oscillation will be

$$n\int_0^\alpha \frac{d\theta}{(\alpha^4 - \theta^4)^{\frac{1}{2}}},$$

where n is constant.

6. A narrow smooth semicircular tube is fixed in a vertical plane with its vertex upwards; and a heavy flexible string, passing through it, hangs at rest; shew that, if the string be cut at one of the ends of the tube, the velocity, which the longer portion of the string will have attained when it is just leaving the tube, will be

$$(ag)^{\frac{1}{2}}\left\{2\pi - \frac{a}{l}(\pi^2 - 4)\right\}^{\frac{1}{2}};$$

l being the length of the longer portion, and a the radius of the tube.

7. If a, b, c, a', b', c', be the cosines of the inclinations of the faces of a tetrahedron, a and a', b and b', c and c', belonging respectively to the edges which do not meet, and a', b', c', to three conterminous edges; shew that

$$1 + a^2a'^2 + b^2b'^2 + c^2c'^2 = a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2 + 2a'bc + 2b'ca + 2c'ab + 2a'b'c' \\ + 2bcb'c' + 2cac'a' + 2aba'b'.$$

8. Shew that the determination of the circular sections of the cone

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$$

may be made to depend upon the solution of the cubic equation

$$abc\mu^3 - (a^2 + b^2 + c^2)\,\mu^2 + 4 = 0;$$

and that the circular sections of the cone

$$\left(\frac{b}{c} + \frac{c}{b}\right) yz + \left(\frac{c}{a} + \frac{a}{c}\right) zx + \left(\frac{a}{b} + \frac{b}{a}\right) xy = 0$$

are parallel to the planes

$$ax + by + cz = 0, \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0.$$

9. Shew that, if

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0,$$

be the equations of three planes which form a trihedral angle, the equation of a cone of the second order, which has its vertex at the angular point and touches two of the planes at their intersections with the third, is

$$\gamma^2 - k\alpha\beta = 0;$$

and that the equation of a surface of the second order enveloped by the cone is

$$\delta^2 + \mu(\gamma^2 - k\alpha\beta) = 0,$$

$\delta = 0$ being the equation of the plane of contact, and μ being constant.

Shew that if the enveloping cone of a series of ellipsoids be the asymptotic cone of a series of hyperboloids of two sheets, the curves of intersection of any ellipsoid with the series of hyperboloids will lie in planes parallel to the plane of contact of the cone and ellipsoid.

10. A rigid spherical shell is filled with homogeneous inelastic fluid every particle of which attracts every other with a force varying inversely as the square of the distance; shew that the difference between the pressures at the surface and at any point within the fluid varies as the area of the least section of the sphere through the point.

11. A uniform beam is revolving uniformly in a vertical plane about a horizontal axis through its middle point; and, at the instant it is passing through its horizontal position, a perfectly elastic ball, the mass of which is one-third that of the beam, is projected horizontally from a point verti-

cally above the axis, so as to hit the beam at one extremity, then to rebound to the other, and so on for ever, bounding and rebounding along the same path; shew that if θ be the angle, on each side of its horizontal position through which the beam revolves, θ will be given by the equation

$$\theta \tan\theta = 1.$$

12. A homogeneous sphere, of elasticity e, rotating uniformly about a horizontal diameter, falls upon a perfectly rough inclined plane through such a height h that its angular velocity is not affected by the first impact, and then proceeds to descend the plane directly by bounds; if u_n be the velocity of the sphere along the plane after the n^{th} impact, shew that

$$u_n = (2gh)^{\frac{1}{2}} \sin\alpha \left(1 + \tfrac{10}{7} \cdot \frac{e - e^n}{1 - e}\right),$$

and that the range which the sphere describes upon the plane before it ceases to hop will be

$$4h \sin\alpha \frac{e}{(1-e)^2} \left(1 - \tfrac{4}{7} \cdot \frac{e^2}{1+e}\right),$$

α being the inclination of the plane to the horizon.

WEDNESDAY, *Jan.* 21. $1\frac{1}{2}$...4.

1. STATE clearly the meaning of "the limit of $f(x)$ when $x = 0$:" and shew that, if two quantities approximate simultaneously to limiting values, and always bear to each other a certain ratio, then their limits are in that ratio. When are two quantities said to be nearly equal? Give some test of the ultimate equality of two quantities which ultimately vanish.

Assuming that $L_{x=0}(1 + x)^{\frac{1}{x}} = \epsilon$, find the differential coefficient, with respect to x, of a^{nx}.

2. Given that, under certain conditions,

$$\frac{F(x_1 + h) - F(x_1)}{f(x_1 + h) - f(x_1)} = \frac{F'(x_1 + \theta h)}{f'(x_1 + \theta h)},$$

derive the equation

$$\phi(x_1) = \frac{x_1^{\,n}}{\lfloor n} \phi^n(\theta x_1),$$

stating fully the conditions for its truth.

Hence find the limiting value of the ratio of $x - \sin x$ to x^3, as x is indefinitely diminished.

3. Shew that the values of x, which render $\phi(x)$, a continuous function of x, a maximum or a minimum, are given by the condition that $\phi'(x)$ for such values, vanishes or is infinite; and shew how to distinguish between a maximum and a minimum.

Determine in each case the sign of $\phi''(x)$ for values of x very nearly equal to those which make $\phi'(x)$ infinite.

4. Give some definition of an asymptote of a curve, and employ it to shew how to determine the asymptotes of polar curves.

If the equation of the curve be

$$u = \frac{1}{r} = f(\theta),$$

shew that there may be as many asymptotes as there are unequal roots of the equation $f(\theta) = 0$: and that, if α be one of these roots, the equation of the corresponding asymptote will be

$$u = f'(\alpha) \sin(\theta - \alpha).$$

5. Find the magnitude and position of the circle which has the closest possible contact with the curve $y = f(x)$ at a given point; and shew that it generally cuts the curve at the point.

Prove that the chord of curvature, parallel to the axis of x, of the curve

$$\sec\frac{y}{a} = \epsilon^{\frac{x}{a}}$$

is constant, and that

$$\sec\left(\frac{3y}{a}\right)^{\frac{1}{3}} = \epsilon^{\frac{x-a}{a}}$$

approximately represents the evolute of this curve for the part near the origin.

6. Investigate the analytical conditions for the existence of multiple points in a curve of which the equation is $u = 0$, u being a rational function of x and y; and shew how the degree of multiplicity may be determined.

Prove that, if

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0,$$

at a double point, the coordinates of which are x, y, the two branches of the curve are at right angles to each other; and that, if the point be the origin, the equation of the tangents to the branches will be

$$\frac{d^2u}{dx^2}(\eta^2 - \xi^2) + 2\frac{d^2u}{dxdy}\eta\xi = 0,$$

ξ, η, being current coordinates of the tangent.

7. Find the values of the integrals,

$$\int\frac{dx}{\sqrt{(x^2 + a^2)}}, \quad \int\frac{dx}{(x + 2)(x^2 + x + 2)}, \quad \int\frac{d\theta}{\sin\theta},$$

$$\int\frac{\tan x\, dx}{1 + m^2 \tan^2 x}, \text{ and } \int_0^{2a}\frac{x^m dx}{(2ax - x^2)^{\frac{1}{2}}}.$$

8. Find the equation of the locus of tangent lines at a point (x, y, z) of a surface, the equation of which is $u = f(x, y, z) = 0$.

Common tangent planes are drawn to the ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1;$$

shew that the perpendiculars upon them from the origin lie in the surface of the cone

$$(a^2 - a'^2)\,x^2 + (b^2 - b'^2)\,y^2 + (c^2 - c'^2)\,z^2 = 0.$$

9. Shew how to integrate the equation

$$\frac{dy}{dx} + Py = Q,$$

P and Q being functions of x.

The normal at a point P of a curve meets the axis of x in G, and the locus of the middle point of PG is the parabola $y^2 = lx$; find the equation of the curve, supposing it to pass through the origin.

10. Prove that the differential equation of the surfaces generated by a straight line which passes through the axis of z and through a given curve, and which makes a constant angle α with the axis of z, is

$$x\frac{dz}{dx} + y\frac{dz}{dy} = (x^2 + y^2)^{\frac{1}{2}} \cot\alpha.$$

11. Prove that the expansion of the function

$$\frac{t}{\varepsilon^t - 1}$$

can involve no odd powers of t above the first, and define Bernoulli's numbers.

If B_{2n-1} be the n^{th} of these numbers, prove that it is equal to

$$(-1)^n \{\tfrac{1}{2}\Delta 0^{2n} - \tfrac{1}{3}\Delta^2 0^{2n} + \tfrac{1}{4}\Delta^3 0^{2n} - \&c.\}.$$

THURSDAY, *Jan.* 22. 9...12.

1. If p, p', be the reciprocals of the perpendiculars from the centre of an ellipse upon SP, HD, where S, H, are the foci respectively nearest to P, D, the ends of two conjugate semi-diameters, prove that, b being the reciprocal of the semi-axis minor,

$$\frac{(pp' - b^2)^2}{(p-b)^2 + (p'-b)^2}$$

is a constant quantity.

2. If forces P, Q, R, acting at the centre O of a circular lamina along the radii OA, OB, OC, be equivalent to forces P', Q', R', acting along the sides BC, CA, AB, of the inscribed triangle, prove that

$$\frac{P \cdot P'}{BC} + \frac{Q \cdot Q'}{CA} + \frac{R \cdot R'}{AB} = 0.$$

3. A fine thread just encloses, without tension, the circumference of an ellipse: supposing a centre of force, attracting inversely as the square of the distance, to be placed at one of the foci, prove that the sum of the tensions of the thread at the ends of any focal chord is invariable, and that the normal pressure on the ellipse at any point varies inversely as the cube of the conjugate diameter.

4. Prove that the eccentricity of a section of an ellipsoid, made by a plane through its least axis, varies inversely as the distance, from this axis, of the point in which it cuts a centric circular section.

5. OA', OB', are two quadrants on the surface of a sphere, at right angles to each other: a great circle cuts them in A, B, respectively: from A', B', through any point P of the great circle, are drawn arcs $B'PM$, $A'PN$, cutting OA', OB', in M, N, respectively; if $PN = \phi$, $PM = \psi$, $\angle OAB = \lambda$, $\angle OBA = \mu$, prove that

$$\sin^2\lambda \, . \cos^2\phi - 2\cos\lambda\cos\mu\sin\phi\sin\psi + \sin^2\mu\cos^2\psi = 1.$$

6. If a polygon of a given number of sides be inscribed in the orbit of a planet, such that all its sides subtend equal angles at the Sun, prove that the sum of the angular velocities of the planet about the Sun, at the angular points of the polygon, is independent of the position of the polygon.

7. A uniform homogeneous wire PAP', of which A is the middle point, is bent into the form of an arc of a loop of the lemniscate of which A becomes the vertex: prove that the resultant attraction on the wire, arising from a centre of force at the node O, attracting according to the law of the inverse square, varies as

$$\left(\frac{1}{OP^2} - \frac{1}{OA^2}\right)^{\frac{1}{2}}.$$

8. A small light is placed at the focus of a perfect reflector in the form of a paraboloid of revolution: prove that the brightness, due to reflection, at any point within the volume of the paraboloid, varies inversely as the square of the focal distance of the end of the diameter through the point.

9. A hollow homogeneous cylinder, of given material, which is perfectly brittle and incompressible, is partially inserted into a fixed horizontal tube just wide enough to admit it: prove that the greatest length which the free portion of the cylinder can have, without snapping off, varies as the square root of the radius of its external surface.

10. A centre of force, repelling inversely as the square of the distance, lies below the surface of a homogeneous inelastic fluid, which is also acted on by gravity and is at rest: the intensity of the force, at a point in the surface of the fluid vertically above its centre, is equal to that of gravity:

prove that the external surface of the fluid has a horizontal asymptotic plane, and that the centre of force is environed by an internal cavity, the summit of which is at the external surface of the fluid.

Find the volume of the cavity in terms of its length.

11. A carriage is travelling along any level road: prove that the sum of the squares of the shadows cast on the ground by any two spokes of a wheel, which are at right angles to each other, varies during the journey as the square of the secant of the Sun's zenith distance.

Prove also that, if the road run due east and west,

$$\sin a = \frac{\tan 2\theta}{\tan 2z},$$

a being the azimuth and z the zenith distance of the Sun, and θ the corresponding inclination of a spoke to the horizon when its shadow is greatest or least.

12. OA, OB, OC, are meridians on a surface of revolution, passing through three points A, B, C, which are connected together by the shortest arcs BC, CA, AB: BC cuts OB, OC, at angles λ_1, λ_2: CA cuts OC, OA, at angles λ_3, λ_4; and AB cuts OA, OB, at angles λ_5, λ_6; prove that

$$\sin\lambda_1 \,.\, \sin\lambda_3 \,.\, \sin\lambda_5 = \sin\lambda_2 \,.\, \sin\lambda_4 \,.\, \sin\lambda_6 .$$

13. A little animal, the mass of which is m, is resting on the middle point of a thin uniform quiescent bar, the mass of which is m' and the length $2a$, the ends of the bar being attached by small rings to two smooth fixed rods at right angles to each other in a horizontal plane: supposing the animal to start off along the bar with a velocity V, relatively to the bar, prove that, θ being the inclination of the bar to either rod, the angular velocity initially impressed upon the bar will be equal to

$$\frac{3m}{3m+4m'} \,.\, \frac{V \sin 2\theta}{a} .$$

14. A narrow tube, in the form of a common helix, is wound round an upright cylinder, initially at rest, which is pierced by two smooth fixed rods, parallel to each other and horizontal: supposing a molecule to be placed within the tube, at a point of which the distance from the axis of the cylinder is parallel to the rods, find the velocity of the cylinder when the molecule arrives at any proposed point of the tube.

Prove that, m, m', being the masses of the molecule and cylinder, the velocities which the cylinder has acquired, at the successive arrivals of the molecule at points most distant from the plane in which the axis of the cylinder moves, will have their greatest values when, a being the inclination of the helix to the horizon,

$$\tan^2 a = \frac{m'}{m+m'} .$$

THURSDAY, *Jan.* 22. $1\frac{1}{2}$...4.

1. ONE plane curve rolls on another, the planes of the two curves coinciding, and their convexities being opposed to each other: if r, r', are the radii of curvature of the fixed and rolling curves respectively, at their point of contact, ρ the distance of any point P in the moving plane from the point of contact, a the angle between ρ and the common normal to the two curves, prove that the corresponding radius of curvature of P's path is equal to

$$\frac{\dfrac{1}{r}+\dfrac{1}{r'}}{\dfrac{1}{r}+\dfrac{1}{r'}-\dfrac{\cos a}{\rho}}.\rho.$$

Shew also that the directions of motion of all the points in the moving plane, fixed relatively to the rolling curve, which at any instant are going through points of inflection in their respective paths, pass through a single point.

2. Prove the following relation between the sides and angles of a spherical triangle,

$$\frac{\cos\dfrac{A+B}{2}}{\sin\dfrac{C}{2}}=\frac{\cos\dfrac{a+b}{2}}{\cos\dfrac{c}{2}}.$$

3. Integrate the simultaneous differential equations:

$$\left.\begin{aligned} a\frac{dx}{dt}+b\frac{dy}{dt}+a'x+b'y&=0,\\ a'\frac{dx}{dt}+b'\frac{dy}{dt}+ax+by&=0.\end{aligned}\right\}$$

4. If $\Gamma(n)$ denote $\int_0^\infty \varepsilon^{-x}x^{n-1}dx$, where n is a proper fraction, shew that

$$\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi};$$

and hence deduce the value of $\int_0^\infty \varepsilon^{-ax^2}dx$.

5. Determine the change in the position of the axis and in the eccentricity of the Moon's orbit indicated by the terms

$$e\cos(c\theta-a)+\frac{15}{8}me\cos\{(2-2m-c)\theta-2\beta+a\},$$

in the expression for the Moon's parallax.

6. A free rigid body, the mass of which is m, is at rest: its moments of inertia about the principal axes through its centre of gravity are A, B, C: supposing the body to be struck by an impulsive force R through its centre of gravity, and by an impulsive couple G, prove that it will revolve for an instant about an axis, the velocity of which is in the direction of its length and is equal to

$$\frac{\dfrac{L.X}{A.m}+\dfrac{M.Y}{B.m}+\dfrac{N.Z}{C.m}}{\left(\dfrac{L^2}{A^2}+\dfrac{M^2}{B^2}+\dfrac{N^2}{C^2}\right)^{\frac{1}{2}}},$$

X, Y, Z, being the components of R, and L, M, N, of G, along the principal axes.

If θ be the inclination of R's direction to the spontaneous axis, prove that

$$\cos\theta=\frac{\dfrac{L.X}{A.R}+\dfrac{M.Y}{B.R}+\dfrac{N.Z}{C.R}}{\left(\dfrac{L^2}{A^2}+\dfrac{M^2}{B^2}+\dfrac{N^2}{C^2}\right)^{\frac{1}{2}}}.$$

7. Two luminous points which emit light of the same colour and of equal intensity, are placed very near to each other before a plane screen and at exactly equal distances from it: investigate the appearance on the screen.

8. Prove the following equation for the determination of the major axis of the orbit of a disturbed planet,

$$\frac{da_1}{dt}=-\frac{2na^2}{\mu}\cdot\frac{dR}{d\varepsilon}.$$

FRIDAY, *Jan.* 23. 9...12.

1. IF $\alpha=0$, $\beta=0$, $\gamma=0$, be the equations of the sides of a triangle, shew that the equation of a conic touching the sides of the triangle is

$$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0.$$

If $\alpha=0$, $\beta=0$, $\gamma=0$, $\dfrac{\alpha}{a_1}+\dfrac{\beta}{b_1}+\dfrac{\gamma}{c_1}=0$, $\dfrac{\alpha}{a_2}+\dfrac{\beta}{b_2}+\dfrac{\gamma}{c_2}=0$, $\dfrac{\alpha}{a_3}+\dfrac{\beta}{b_3}+\dfrac{\gamma}{c_3}=0$,

be the equations of the sides of a hexagon which circumscribes a conic, shew that

$$a_1(b_2c_3-c_2b_3)+a_2(b_3c_1-c_3b_1)+a_3(b_1c_2-c_1b_2)=0.$$

2. Transform the triple integral $\iiint f(\alpha,\beta,\gamma)\,d\alpha\,d\beta\,d\gamma$ into one in which x, y, z, are the independent variables, having given

$$\alpha=F_1(x,y,z),\quad \beta=F_2(x,y,z),\quad \gamma=F_3(x,y,z).$$

If
$$\alpha x = yz, \quad \beta y = zx, \quad \gamma z = xy,$$
shew that
$$\iiint f(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma = 4 \iiint f\left(\frac{yz}{x}, \frac{zx}{y}, \frac{xy}{z}\right) dx\, dy\, dz.$$

3. Shew how to integrate the equation of differences,
$$u_{x+n} + p_1 u_{x+n-1} + \ldots + p_n u_x = f(x),$$
where p_1, p_2, p_n are independent of x.

Shew that a solution of the equation
$$u_{x+n} . u_{x+n-1} . \ldots . u_{x+1} . u_x = a\,(u_{x+n} + u_{x+n-1} + \ldots + u_{x+1} + u_x),$$
is included in that of
$$u_{x+n+1} - u_x = 0,$$
and is consequently
$$u_x = C_1\, a^x + C_2\, a^{2x} + \ldots + C_{n+1}\, a^{(n+1)x},$$
where a is one of the imaginary $(n+1)^{\text{th}}$ roots of unity, the $n+1$ constants being subject to an equation of condition.

4. Shew how to find the differential equation of a class of surfaces, which cuts at right angles all the surfaces represented by the equation
$$f(x, y, z, a) = 0,$$
where a is an arbitrary parameter.

If the class of surfaces have an envelope, shew how we may find it without solving the differential equation.

5. Disturbances are excited in the air contained in a cylindrical tube of given length by a plate vibrating isochronously at one end, the other end being closed: assuming expressions for the velocity and condensation at any point, find the time of vibration, in order that a musical note may be produced; and determine the points in the tube at which openings may be made without affecting the pitch.

Supposing a vibrating plate also at the closed end, how must the time of vibration of the first plate be modified, and how must the times of vibration of the two plates be related, that musical notes may be produced?

6. Assuming that the Sun causes an angular acceleration of the Earth, proportional to the sine of twice the Sun's north polar distance, about the equatoreal diameter perpendicular to the line joining the centres of the Sun and the Earth, shew that the line of equinoxes will have a precessional movement.

How will the amount of precession, as deduced from observation, aid in determining the ratio of the mass of the Earth to the mass of a pound weight?

7. What is meant by secular variations of planetary elements?

Shew how to find the condition that the secular variations of the longitudes of the lines of nodes of two mutually disturbing planets may be periodic.

The following equations, connecting the inclinations and longitudes of the nodes, may be assumed:

$$\tan\iota\,\sin\Omega = p, \qquad \tan\iota\,\cos\Omega = q,$$

$$\frac{dp}{dt} = na^2a'm'C(q' - q), \quad \frac{dq}{dt} = na^2a'm'C(p - p'),$$

with corresponding equations for the planet to which the accented symbols refer.

If the squares of the masses of the two planets were to each other inversely as their mean distances, then the nodes would oscillate through equal angles.

8. Describe some method of obtaining a circularly polarised beam of light, from light polarised in one plane.

Circularly polarised light is incident, at a slight inclination, upon a plate of uniaxal crystal cut perpendicularly to its axis, and the emergent pencil is analysed; explain generally the phenomena produced. What will be the effect produced by turning round the analysing plate?

FRIDAY, *Jan.* 23. $1\frac{1}{2}...4$.

1. INVESTIGATE formulæ for the determination of the umbilici of surfaces.

Prove that the radius of normal curvature of the surface $xyz = a^3$ at an umbilicus is equal to the distance of the umbilicus from the origin of coordinates.

2. Integrate the equations

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} - \frac{d^2u}{dz^2} + 2\frac{d^2u}{dx\,dy} = xyz, \text{ ....(1)},$$

$$u_{x+1}.\sin x\theta - u_x.\sin(x+1)\theta = \cos(x-1)\theta - \cos(3x+1)\theta \text{...(2)};$$

and find a general value $\phi(x)$ from the equation

$$\phi(m^2x) - (a+b)\phi(mx) + ab\phi(x) = cx \text{..................(3)}.$$

3. State and prove the principle of Vis Viva, and describe the different kinds of forces which do not appear in the equation of Vis Viva.

A circular wire ring, carrying a small bead, lies on a smooth horizontal table; an elastic thread, the natural length of which is less than the diameter of the ring, has one end attached to the bead and the other to a point in the wire; the bead is placed initially so that the thread coincides very nearly with a diameter of the ring: find the Vis Viva of the system when the string has contracted to its natural length.

4. If V be a given function of $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots$ find the conditions that $\int V dx$, between given limits, may be a maximum or minimum.

When a particle is attracted towards a fixed centre of force and moves in the brachistochrone, prove that the area described round the centre of force varies as the "action."

5. Describe the terms in the expansion of the disturbing function, which are of the greatest importance in calculating the variations in the elements of a planetary orbit, and explain fully what is meant by the long inequality of two planets.

What principle is used to ascertain the disturbing effects produced on a planet by several other planets.

6. Define the potential function V, and shew that, at any point (x, y, z), external to the attracting mass, it satisfies the equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0.$$

Hence prove that, if S be any closed surface to which all the attracting mass is external, dS an element of S, and dn an element of the normal drawn outwards at dS,

$$\iint \frac{dV}{dn} \cdot dS = 0,$$

the integral being taken throughout the whole surface S.

7. Assuming the formulæ

$$l\alpha + m\beta + n\gamma = 0,$$

$$\frac{l}{\alpha (v^2 - a^2)} = \frac{m}{\beta (v^2 - b^2)} = \frac{n}{\gamma (v^2 - c^2)},$$

investigate the equation of the wave-surface in a biaxal crystal.

Prove that the direction of the vibration at any point of this surface coincides with the projection of the distance of the point from the centre of the surface upon the tangent plane at the point.

THE END.

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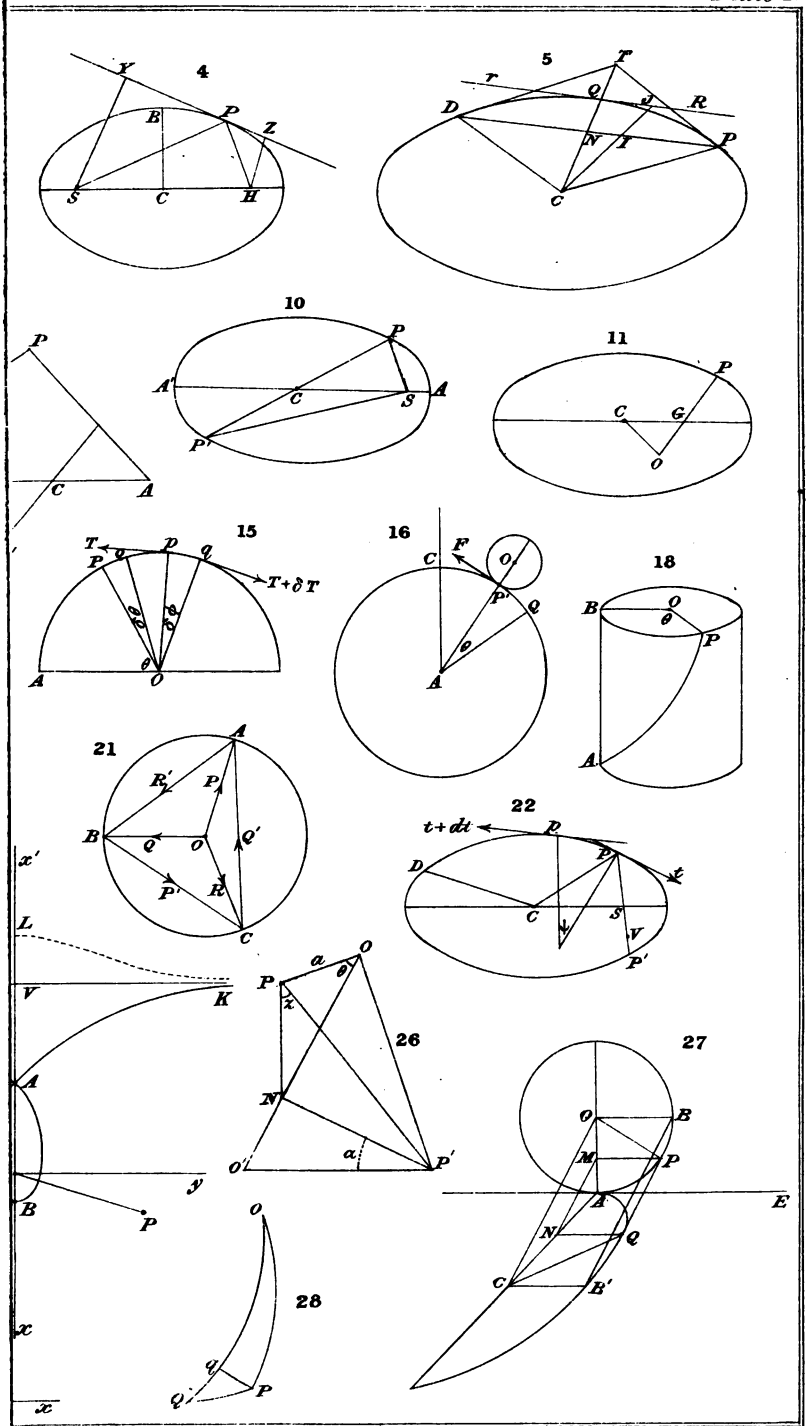

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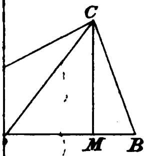

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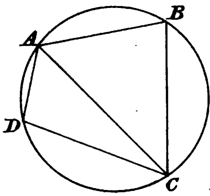

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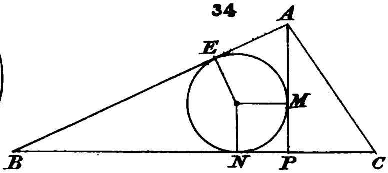

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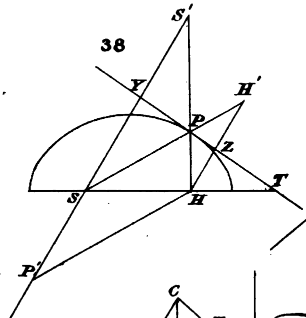

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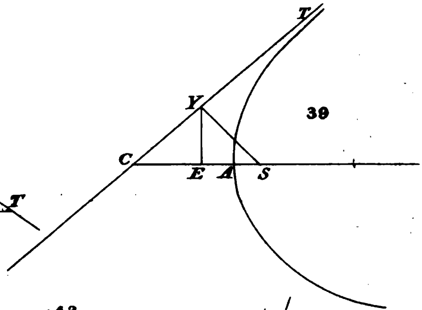

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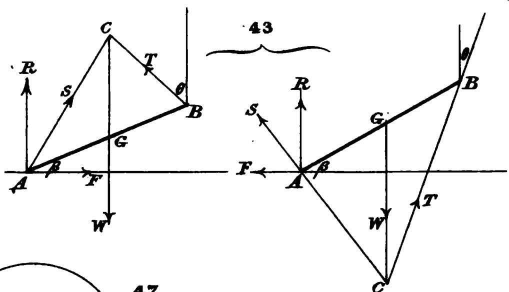

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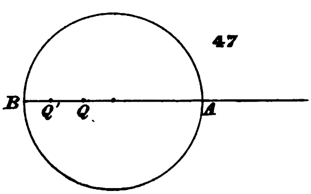

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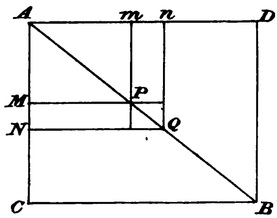

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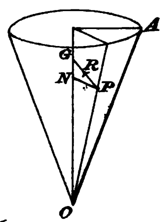

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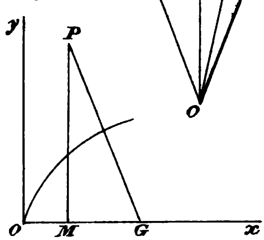

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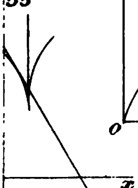

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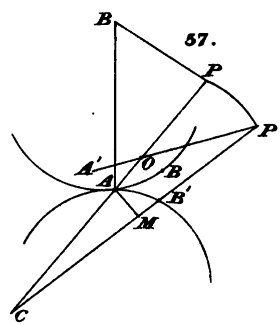

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