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SENATE-HOUSE PROBLEMS AND RIDERS

FOR THE YEAR 1875.

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SOLUTIONS

OF THE

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Senate-House Problems and Riders

FOR THE YEAR 1875.

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PREFACE.

This book of the Solutions of the Questions in the Mathematical Tripos of 1875 has been undertaken at the request of several persons who are interested in the teaching of mathematics at Cambridge.

The solution of each question has, in general, been furnished by the maker of the question, or is a solution that has been given by a candidate in the Examination.

By the new regulations for the Examination several new subjects have been introduced, which have not been treated of in preceding collections of Senate-House Solutions.

Complete solutions of the questions on the higher subjects have been given in all cases where it was felt that a reference to the Text-books in ordinary use was not sufficient.

An apology must be given for the unavoidable delay in the appearance of the book.

ERRATUM.

Page 95, in lines 8 and 9, *read* "If the first three planes meet the fourth plane in *BC*, *CA*, *AB*, &c."

SOLUTIONS OF SENATE-HOUSE PROBLEMS AND RIDERS

FOR THE YEAR EIGHTEEN HUNDRED AND SEVENTY-FIVE.

MONDAY, *Jan.* 4, 1875. 9 to 12.

6o

MR. GREENHILL. Arabic numbers.
MR. WRIGHT. Roman numbers.

1. PARALLELOGRAMS and triangles upon the same base and between the same parallels are equal.

A', B', C' are the middle points of the sides of the triangle ABC, and through A, B, C are drawn three parallel straight lines meeting $B'C'$, $C'A'$, $A'B'$ in a, b, c respectively; prove that the triangle abc is half the triangle ABC and that bc passes through A, ca through B, ab through C.

For (fig. 1)

$$\triangle ABa + \triangle Aac = \triangle ABa + \triangle AaC = \tfrac{1}{2}\triangle ABC = \triangle ABB' = \triangle ABc;$$

therefore Bac is a straight line.

Similarly it may be proved that Cab is a straight line.

And $\triangle Bbc = \triangle BbC$, taking away the common part $\triangle Bba$; therefore

$$\triangle abc = \triangle aBC = \tfrac{1}{2}\triangle ABC.$$

Also $\triangle Aab + \triangle Aac = \triangle AaB + \triangle AaC = \tfrac{1}{2}\triangle ABC = \triangle abc;$

therefore bAc is a straight line.

2. The angles in the same segment of a circle are equal to one another.

If the diagonals AC, BD of the quadrilateral $ABCD$, inscribed in a circle the centre of which is at O, intersect at right angles in a fixed point P, prove that the feet of the perpendiculars drawn from O and P to the sides of the quadrilateral lie on a fixed circle, the centre of which is at the middle point of OP.

Let Q (fig. 2) be the middle point of OP; a, b, c, d the feet of the perpendiculars from P on AB, BC, CD, DA, and a', b', c', d' the middle points of AB, BC, CD, DA respectively.

Then $a'b'c'd'$ is a rectangle.

Also $$\angle APa = \angle ABP = \angle PCD = \angle CPc';$$

therefore aPc' is a straight line, and similarly it may be proved that bPd', cPa', and dPb' are straight lines.

Therefore $Pa'Oc'$ is a parallelogram, and Q, which is the centre of the parallelogram, is the centre of the rectangle $a'b'c'd'$.

Therefore a circle can be described with centre Q passing through a, a'; b, b'; c, c'; d, d'.

Also $$2\,(\text{rad.})^2 + 2\,OQ^2 = Oa'^2 + a'P^2 = Oa'^2 + a'A^2 = O^2A,$$

and therefore the radius of the circle is constant.

This circle is analogous to the nine-pointic circle of a triangle.

3. Upon a given straight line describe a segment of a circle which shall contain an angle equal to a given rectilineal angle.

Through a fixed point O any straight line OPQ is drawn cutting a fixed circle in P and Q, and upon OP and OQ as chords are described circles touching the fixed circle at P and Q. Prove that the two circles so described will intersect on another fixed circle.

If C (fig. 3) be the centre of the fixed circle and OM, ON be drawn parallel to CQ and CP, forming the parallelogram $OMCN$, then M and N are the centres of the segments.

Therefore if R be the point of intersection of the segments, OR is perpendicular to MN, and since MN bisects OC, therefore MN is parallel to RC.

Therefore the angle ORC is a right angle, and therefore the segments intersect on a circle described on OC as diameter.

iv. Describe an isosceles triangle having each of the angles at the base double of the third angle.

Prove that the circle drawn through the middle points of the sides of this triangle will intercept portions of the equal sides such that a regular pentagon can be inscribed in the circle having these portions as two of its sides.

If D, E, F (fig. 4) be the middle points of the sides, the triangle FBD will be similar to the triangle ABC, and the circle through D, E, F will be the small circle of Euclid's construction relatively to the triangle FBD, and will cut AB, AC in points MN, such that

$$BD = DN = NF = FE,$$

and therefore $DMEFN$ is a regular pentagon.

v. Equal triangles which have one angle of the one equal to one angle of the other have their sides about the equal angles reciprocally proportional; and triangles which have one angle of the one equal to one angle of the other and their sides about the equal angles reciprocally proportional are equal to one another.

If ABC, ADE be two such triangles placed so that BA, AE are in a straight line, as also CA and AD; and if BC, DE produced meet in F, prove that FA will bisect CE and BD.

Since (fig. 5) the $\triangle BAC = \triangle DAE$;

therefore $\triangle BCE = \triangle DCE$,

and therefore CE is parallel to BD; therefore

$$BC : BF :: DE : DF;$$

therefore $\triangle BCA : \triangle BFA :: \triangle DEA : \triangle DFA$, but $\triangle BCA = \triangle DEA$;

therefore $\triangle BFA = \triangle DFA$,

therefore perpendiculars from B and D on FA are equal, and therefore FA bisects BD, and its parallel CE.

vi. If a straight line stand at right angles to each of two straight lines at the point of their intersection, it shall also be at right angles to the plane which passes through them, that is to the plane in which they are.

If P be a point equidistant from the angles A, B, C of a right-angled triangle of which A is the right angle and D the middle point of BC, prove that PD is at right angles to the plane of ABC. Prove also that the angle between the planes PAC, PBC and the angle between the planes PAB, PBC are together equal to the angle between the planes PAC, PAB.

For (fig. 6) $PA^2 = PB^2 = PD^2 + DB^2 = PD^2 + DA^2$;

therefore PDA is a right angle, and since PDB is also a right angle, therefore PD is at right angles to the plane ABC.

And since $PA = PB$ and $DA = DB$, therefore the angle between the planes PAB, PBC is equal to the angle between the planes PAD, PAB.

Similarly the angle between the planes PAC, PBC is equal to the angle between the planes PAD, PAC.

Therefore the angle between the planes PAB, PBC, and the angle between the planes PAC, PBC are together equal to the angle between the planes PAC, PAB.

Compare the rider to question 2, Thursday afternoon, Jan. 7.

vii. The ordinate to the diameter through any point on a parabola is a mean proportional between the corresponding abscissa and four times the focal distance of that point.

If through a fixed point A a straight line be drawn meeting two fixed lines OD, OE in B and C respectively, and on it a point P be taken such that $AC.AP = AB^2$; prove that the locus of P is a parabola which passes through

A and O and has its axis parallel to OD and the tangent at A parallel to OE.

Let the straight line through A (fig. 7) parallel to OD meet OE in F, and draw PM parallel to OE to meet AF in M.

By similar triangles

$$PM^2 : OF^2 :: AP^2 : AB^2 :: AP : AC :: AM : AF;$$

therefore the locus of P is the parabola stated.

viii. If the tangent and ordinate at any point P of an ellipse meet the axis major in T and N respectively, then $CT.CN = CA^2$.

If any circle be drawn through N and T, prove that it is cut at right angles by the auxiliary circle of the ellipse.

Let the circle cut the circumscribing circle in P, then

$$CT.CN = CA^2 = CP^2;$$

therefore CP is a tangent to the circle, and therefore it cuts the auxiliary circle at right angles.

ix. Tangents to an ellipse or hyperbola at right angles to each other intersect on a fixed circle.

If any rectangle circumscribe an ellipse, prove that the perimeter of the parallelogram formed by joining the points of contact is equal to twice the diameter of the circle which is the locus of the point of intersection of tangents at right angles.

If $PQRS$ (fig. 8) be a circumscribing rectangle and C the centre of the ellipse.

Join CP and let it cut LM one side of the parallelogram in N.

C is the centre of the rectangle and of the parallelogram and P, Q lie on the director circle; therefore $\triangle\, CPQ$ is isosceles, hence so also is $\triangle\, LNP$; therefore $LN = NP$, and CN is parallel to another side of parallelogram and equal to half of it; therefore $CP = \frac{1}{4}$ perimeter of parallelogram or perimeter of parallelogram $= 4\,CP =$ twice diameter of director circle.

10. In a central conic the tangent makes equal angles with the focal distances, and the sum or difference of the focal distances is constant.

Given a focus, the length of the transverse axis and that the second focus lies on a fixed straight line, prove that the conic will touch two fixed parabolas having the given focus for focus.

If S (figs. 9 and 10) be the given point, CH the given straight line, draw DK, $D'K'$ parallel to CH at distances equal to the given transverse axis, and draw SC perpendicular to CH to meet it in C, and DK, $D'K'$ in D and D'.

Then if H be the second focus of a conic and PP' be drawn through H parallel to SC to meet the conic in P, P' and DK, $D'K'$ in K, K', it is evident that

$$SP = PK \text{ and } SP' = P'K'.$$

Therefore the conic touches two fixed parabolas having the common focus S and the directrices DK, $D'K'$ at P and P'.

If SC (fig. 9) is less than the given transverse axis, the parabolas are turned in opposite directions and intersect at right angles on CH in B and B', such that $SB = SB' =$ given transverse axis.

Then if H be taken between B and B', the conic is an ellipse; but if H be taken beyond B or B', the conic is a hyperbola, and the same branch of it touches the parabolas.

If SC (fig. 10) is greater than the given transverse axis, the parabolas are turned in the same direction and do not intersect, and the conic is always a hyperbola of which different branches touch the parabolas.

11. The tangents to a conic section drawn from a point subtend equal angles at either focus.

If PP' be a chord of a conic parallel to the transverse axis and the two circles be drawn through a focus S touching the conic at P and P' respectively, prove that F the second point of intersection of the circles will be at the intersection of PP' and ST, where T is the point of intersection of the tangents at P and P'.

Prove also that the locus of F for different positions of PP' will be a parabola with its vertex at S.

A circle can be described about $TPSP'$ (fig. 11) and therefore $\angle TPP' = \angle FSP$; therefore the circle described on SP touching PT at P passes through F.

Similarly the circle described on SP' touching $P'T$ at P' passes through F.

Also if FM, FN be drawn perpendicular to the transverse and conjugate axes to meet them in M, N respectively, $CT.CN = CB^2$; therefore

$$FM^2 : CB^2 :: CN : CT :: SM : SC;$$

therefore the locus of F is a parabola with its vertex at S and passing through the ends of the conjugate axis.

12. If a right circular cone be cut by a plane, the distance of any point on the curve of section from a certain point bears a constant ratio to the distance from a certain straight line.

If any sphere be inscribed in the cone, the length of the tangent line drawn from any point on the curve of section to the sphere will bear the same constant ratio to the distance of the point from the line of intersection of the plane section and the plane of the circle of contact of the sphere and cone.

The rider is proved incidentally in the book work.

(If O be the centre of the sphere and OD be drawn perpendicular to the plane of the section, it can easily be shewn that the length of any tangent line from a point in the plane to the sphere is equal to the distance of this point from a fixed point S in OD such that $SD^2 = OD^2 - (\text{rad. of sphere})^2$.

The locus of S is called the focal conic of the conic section.)

Monday, *Jan.* 4, 1875. $1\frac{1}{2}$ to 4.

Mr. Freeman. Arabic numbers.
Mr. Cookshot. Roman numbers.

1. For a house occupied by B, A pays a rent of £40 per annum by equal payments at the end of each quarter. B pays A by equal payments in advance at the beginning of each month. How much a month ought B to pay in order that at the end of the year, with simple interest reckoned at $3\frac{1}{3}$ per cent. per annum, A may have recovered the value of his own four payments with one-tenth additional?

The answer is £3. 12*s*. $11\frac{1}{7}\frac{4}{3}\frac{5}{3}$.

2. Shew how to find the lowest common multiple of three algebraical expressions.

If l_1, l_2, l_3 are the lowest common multiples of B and C, of C and A, of A and B, respectively; if g_1, g_2, g_3 are the highest common divisors of the same pairs; and if L, G are the lowest common multiple and highest common divisors of A, B, and C; prove that $\frac{L}{G} = \left(\frac{l_1 l_2 l_3}{g_1 g_2 g_3}\right)^{\frac{1}{2}}$.

Let α, β, γ be the powers of a simple factor P in A, B, C respectively. Suppose them to be in descending order of magnitude but without excluding the cases in which some of them are equal or zero.

Then the index of P in $\frac{L}{G}$ is $\alpha - \gamma$, and in

$$\left(\frac{l_1 l_2 l_3}{g_1 g_2 g_3}\right)^{\frac{1}{2}} \text{ is } \tfrac{1}{2}(\beta + \alpha + \alpha - \gamma - \gamma - \beta) = \alpha - \gamma.$$

The same argument applies to any factor and therefore $\frac{L}{G} = \left(\frac{l_1 l_2 l_3}{g_1 g_2 g_3}\right)^{\frac{1}{3}}$.

3. (α) Find the simplest expression for

$$\frac{(a+p)(a+q)}{(a-b)(a-c)(a+x)} + \frac{(b+p)(b+q)}{(b-c)(b-a)(b+x)} + \frac{(c+p)(c+q)}{(c-a)(c-b)(c+x)}.$$

(β) If the letters all denote positive quantities, prove that

$$\frac{(a+b)xy}{ay+bx} \text{ is never greater than } \frac{ax+by}{a+b}.$$

(α) $\dfrac{(p-x)(q-x)}{(a+x)(b+x)(c+x)}$.

(β) $\dfrac{(a+b)xy}{ay+bx} - \dfrac{ax+by}{a+b} = \dfrac{-ab(x-y)^2}{(a+b)(ay+bx)}$,

which is never positive.

4. Find in terms of the coefficients the sum and product of the roots of the equation

$$ax^2 + 2bx + c = 0.$$

Find the condition that the roots of $ax^2 + 2bx + c = 0$ may be formed from those of $a'x^2 + 2b'x + c' = 0$ by adding the same quantity to each root.

The difference of the roots in each equation being the same, therefore

$$\frac{b^2 - ac}{a^2} = \frac{b'^2 - a'c'}{a'^2},$$

the required condition.

5. Solve the equations

(α) $(c+a-2b)x^2 + (a+b-2c)x + (b+c-2a) = 0.$

(β) $ax + yz = ay + zx = az + xy = b^2.$

(α) $x=1$ is obviously one root, and therefore the other root is $\dfrac{b+c-2a}{c+a-2b}$.

(β) $$ax+yz=ay+zx=az+xy=b^2;$$

therefore $$(y-z)(x-a)=0;$$

therefore $x=y=z=$ a root of the quadratic

$$x^2+ax=b^2,$$

or $$x=a,\quad y+z=\frac{b^2}{a},\quad yz=\frac{b^2-a^2}{a},$$

with the roots obtained by a cyclical change of x, y, z. Of these six sets of roots three only are distinct, and the roots are a, a, $\dfrac{b^2-a^2}{a}$.

6. Investigate the formula for the number of combinations of n things taken r at a time, without assuming the formula for permutations.

A selection of c things is to be made, part from a group of a things and the remainder from a group of b things. Prove that the number of ways in which such a selection may be made will never be greater than when the number of things taken from the group of a things is the integer next less than $\dfrac{(a+1)(c+1)}{a+b+2}$.

Let x, y be the numbers to be chosen respectively from the groups of a and b things.

Then $$x+y=c,$$

and $$\frac{\lfloor a}{\lfloor x \lfloor a-x} \cdot \frac{\lfloor b}{\lfloor y \lfloor b-y}$$

is to be a maximum for integral values of x and y,

i.e. $$\frac{\lfloor a}{\lfloor x \lfloor a-x} \cdot \frac{\lfloor b}{\lfloor c-x \lfloor b-c+x}$$

is to be a maximum for an integral value of x.

Therefore the factor by which this expression is multiplied when we write $x+1$ for x must be just less than 1.

That is, that value of x must be taken which will first make

$$\frac{a-x}{x+1} \cdot \frac{c-x}{b-c+x+1} < 1,$$

or $$ac-(a+c)x < b-c+1+(b-c+2)x,$$

or $$ac+c-(b+1) < (a+b+2)x.$$

Therefore x is the integer next greater than

$$\frac{c(a+1)-(b+1)}{a+c+2},$$

or next less than $$\frac{(a+1)(c+1)}{a+b+2}.$$

If there were n groups containing $a_1, a_2, a_3 \ldots a_n$ things (the a's being in descending order of magnitude), and if p things were to be chosen, the number of ways in which the selection could be made would never be greater than when the number taken from the group of a_r things was the integer next less than

$$\frac{(a_r+1)(p+n-1)}{a_1+a_2+\ldots+a_n+n}.$$

vii. Shew that corresponding small increments of a number and its logarithm are proportional.

Find n from the following data:

$$\log_{10} 42563 = 4\,6290322,$$
$$\log_{10} 42564 = 4{\cdot}6290424,$$
$$\log_{10} n \quad = 2{\cdot}6290376.$$

Find to how many decimal places n can be determined by this method, given that $\log_{10} e = {\cdot}43429$.

The result is $n=425{\cdot}6353$ correct to four places of decimals, for generally

$$\log_{10}(n+\delta) - \log_{10} n = \frac{\mu\delta}{n} + \ldots$$

where $\mu = \log_{10} e$ is the modulus.

If δ is so small that $\frac{\mu\delta}{n}$ does not come into the tables, δ cannot be determined by the method of proportional parts;

therefore $$\frac{\mu\delta}{n} \nless \frac{1}{10^7},$$

$$\delta \nless \frac{1}{10^7} \cdot \frac{n}{\mu} \nless \frac{1}{10^7} \cdot \frac{425{\cdot}63}{{\cdot}43429}$$

$$\nless \frac{1}{10^4} \cdot \frac{42563}{43429} \nless {\cdot}0001,$$

or the result is not to be relied upon beyond the fourth place of decimals.

viii. Find a general expression in terms of α for all the angles whose cosecants are equal to $\operatorname{cosec}\alpha$.

Find all the solutions of $\sin 3\theta - \cos\theta = 0$.

Which of them will satisfy the equation

$$1 + \sin^2\theta = 3 \sin\theta \cos\theta?$$

$$\sin 3\theta = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right);$$

therefore $$3\theta = n\pi + (-1)^n\left(\frac{\pi}{2} - \theta\right);$$

therefore $$\theta = \frac{m\pi}{2} + \frac{\pi}{8} \text{ or } m\pi + \frac{\pi}{4}.$$

If $$1 + \sin^2\theta = 3 \sin\theta \cos\theta,$$

then $$2 \tan^2\theta - 3 \tan\theta + 1 = 0;$$

therefore $$\tan\theta = 1 \text{ or } \tfrac{1}{2},$$

and the values $\theta = m\pi + \frac{\pi}{4}$ make $\tan\theta = 1$.

ix. Prove that $\cos(A - B) = \cos A \cos B + \sin A \sin B$ for all values of A and B between 0 and 90°.

Shew how the proof may be extended so as to include all values of A and B.

If $A+B+C=90^\circ$, prove that

$$\operatorname{cosec}A \operatorname{cosec}B \operatorname{cosec}C - \cot B \tan C - \cot C \tan B - \cot C \tan A$$
$$- \cot A \tan C - \cot A \tan B - \cot B \tan A = 2.$$

The order of the letters denoting the direction in which a straight line is measured, $AB+BA=0$ and for any three points A, B, C in a straight line $AB+BC=AC$.

Then for any magnitude of the angle POM,

$$\sin POM = \frac{MP}{OP}, \quad \cos POM = \frac{OM}{OP},$$

and
$$\cos(A-B) = \frac{OM}{OP} = \frac{OQ+QM}{OP}$$

$$= \frac{OQ}{ON}\cdot\frac{ON}{OP} + \frac{RP}{NP}\cdot\frac{NP}{OP} = \cos A \cos B + \sin A \sin B,$$

and this will be true however the figure be drawn as in figs. 12 and 13.

The expression

$$\operatorname{cosec}A \operatorname{cosec}B \operatorname{cosec}C - \tan A(\cot B + \cot C) - \ldots$$
$$= \operatorname{cosec}A \operatorname{cosec}B \operatorname{cosec}C - \sin A \operatorname{cosec}B \operatorname{cosec}C - \ldots$$
$$= \operatorname{cosec}A \operatorname{cosec}B \operatorname{cosec}C(1 - \sin^2 A - \sin^2 B - \sin^2 C)$$
$$= \operatorname{cosec}A \operatorname{cosec}B \operatorname{cosec}C(\cos^2 A - \sin^2 B - \sin^2 C)$$
$$= \operatorname{cosec}A \operatorname{cosec}B \operatorname{cosec}C\{\cos(A-B)\cos(A+B) - \sin^2 C\}$$
$$= \operatorname{cosec}A \operatorname{cosec}B\{\cos(A-B) - \cos(A+B)\}$$
$$= 2$$

x. Prove that the sides of a triangle are proportional to the sines of the opposite angles.

Shew that if the squares of the sides of a triangle are in arithmetical progression, the tangents of the angles are in harmonical progression.

If $a^2 - b^2 = b^2 - c^2$,

$$\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C,$$
$$\sin(A-B)\sin(A+B) = \sin(B-C)\sin(B+C),$$
$$\sin(A-B)\sin C = \sin(B-C)\sin A;$$

therefore $\cot B - \cot A = \cot C - \cot B$;

therefore $\dfrac{1}{\tan A} + \dfrac{1}{\tan C} = \dfrac{2}{\tan B}$.

xi. Find the radius of the inscribed circle of a triangle in terms of the angles and one side.

If R, r, r_1, r_2, r_3 are the radii of the circumscribed, inscribed, and escribed circles of a triangle, prove that

$$r_1 + r_2 + r_3 - r = 4R.$$

For $$r_1 + r_2 + r_3 - r$$

$$= a\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} + b\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}} + c\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

$$- a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{a}{\sin\frac{A}{2}\cos\frac{A}{2}}\left(\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\right.$$

$$\left. + \cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= \frac{2a\sin\frac{A+B+C}{2}}{\sin A} = \frac{2a}{\sin A} = 4R.$$

xii. Prove that $\sin\theta$ lies between θ and $\theta - \dfrac{\theta^3}{4}$ if θ is the circular measure of an angle between 0 and 90°.

If a triangle be solved from the observed parts $C = 75°$, $b = 2$, $a = \sqrt{(6)}$, shew that an error of 10″ in the value of C would cause an error of about 3″·66 in the calculated value of B.

The ultimate ratio of the error in B to the error in C (fig. 14)

$$= lt\left(\frac{AN}{AB} : \frac{AA'}{AC}\right) = \frac{b}{c}\, lt\,\frac{AN}{AA'} = \frac{b}{c}.\cos A$$

$$= \frac{b^2 + c^2 - a^2}{2c^2} = \frac{1}{2} - \frac{1}{c^2}.$$

And $$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= 6 + 4 - \sqrt{(6)}\,\{\sqrt{(5)} - 1\};$$

therefore $$\text{ratio} = \cdot 5 - \cdot 134 = \cdot 366;$$

therefore error in B is $3''\cdot 66$.

TUESDAY, *Jan.* 5, 1875. 9 to 12.

MR. COCKSHOTT. Arabic numbers.
MR. FREEMAN. Roman numbers.

1. FIND an expression for the magnitude of the resultant of any number of given forces acting at a point in given directions in a plane.

Prove that the resultant of forces 7, 1, 1 and 3 acting from one angle of a regular pentagon towards the other angles taken in order is $\sqrt{(71)}$.

$$R^2 = \left(2\cos\frac{\pi}{10} + 10\cos\frac{3\pi}{10}\right)^2 + \left(4\sin\frac{3\pi}{10}\right)^2$$

$$= 4\cos^2\frac{\pi}{10} + 40\cos\frac{\pi}{10}\cos\frac{3\pi}{10} + 84\cos^2\frac{3\pi}{10} + 16$$

$$= 60 + 22\cos\frac{\pi}{5} + 20\cos\frac{2\pi}{5} + 42\cos\frac{3\pi}{5}$$

$$= 60 + 22\left(\cos\frac{\pi}{5} - \sin\frac{\pi}{10}\right)$$

$$= 60 + 22\left\{\frac{\sqrt{(5)}+1}{4} - \frac{\sqrt{(5)}-1}{4}\right\} = 71.$$

2. State the conditions for the equilibrium of any number of forces acting upon a body in one plane and prove that they are necessary and sufficient.

If six forces acting on a body be completely represented, three by the sides of a triangle taken in order, and three by the sides of the triangle formed by joining the middle points of the sides of the original triangle, prove that they will be in equilibrium if the parallel forces act in the same direction and the scale on which the first three forces are represented be four times as large as that on which the last three are represented.

Each set of three forces will be equivalent to a couple, and the two couples will balance.

3. If a system of parallel forces act at given points in a plane, find the distance of the centre of the system from a given straight line in that plane.

A triangular lamina is supported at its three angular points and a weight equal to that of the triangle is placed upon it; find the position of the weight if the pressures on the points of support are proportional to $4a+b+c$, $a+4b+c$, $a+b+4c$, where a, b, c are the lengths of the sides of the triangle.

The resultant of $a+b+c$ acting at each angular point is $3(a+b+c)$ acting at the centre of inertia of the triangle, and the resultant of $3a$, $3b$, $3c$ acting at the angular points is $3(a+b+c)$ acting at the centre of the inscribed circle.

Therefore the weight must be placed at the centre of the inscribed circle.

4. Describe the common steelyard, and shew that the distances between the graduations are proportional to the differences of the weights to which they belong.

In a weighing machine constructed on the principle of the common steelyard the pounds are read off by graduations reaching from 0 to 14, and the stones by weights hung at the end of the arm; if the weight corresponding to one stone be 7 oz., the moveable weight $\frac{1}{2}$ lb. and the length of the arm one foot, prove that the distances between the graduations are $\frac{3}{4}$ in.

If ACB be the beam, C the fulcrum, and O the zero of graduations, then $\frac{1}{2}$ at O and $\frac{7}{16}$ at B will have the same moment about C as $\frac{1}{2}$ at B; therefore

$$\tfrac{1}{2}CO+\tfrac{7}{16}CB=\tfrac{1}{2}CB;$$

therefore $$OB = \tfrac{7}{8} CB;$$

therefore $$\frac{OB}{14} = \frac{CB}{16} = \tfrac{3}{4} \text{ inch}.$$

5. Describe the differential axle, and find the ratio of the power to the weight.

If the ends of the chain, instead of being fastened to the axles, are joined together so as to form another loop in which another pulley and weight are suspended, find the least force which must be applied along the chain in order to raise the greater weight, the different parts of the chain being all vertical.

When W the greater weight (fig. 15) is raised, the force that must be applied either downwards on the inner descending chain or upwards on the inner ascending chain is $\frac{a-b}{b} \cdot \frac{W-W'}{2}$, where a, b are the radii of the outer and inner pulley respectively.

The force that must be applied either downwards on the outer descending chain or upwards on the outer ascending chain is $\frac{a-b}{a} \cdot \frac{W-W'}{2}$.

Therefore $\frac{a-b}{b} \cdot \frac{W-W'}{2}$ is the least force.

6. State the laws of friction, and describe some method of verifying them experimentally.

A glass rod is balanced partly in and partly out of a cylindrical tumbler with the lower end resting against the vertical side of the tumbler. If α and β are the greatest and least angles which the rod can make with the vertical, prove that the angle of friction is

$$\tfrac{1}{2} \tan^{-1} \frac{\sin^3\alpha - \sin^3\beta}{\sin^2\alpha \cos\alpha + \sin^2\beta \cos\beta}.$$

The laws of friction are

(i) That limiting friction is proportional to the pressure.

(ii) That it is independent of the extent of surfaces in contact.

(iii) That it is independent of the velocity when motion takes place.

(i) and (ii) are the laws of statical friction, and (i), (ii), and (iii) are the laws of dynamical friction.

To prove experimentally the laws of friction a plane is taken which is capable of being inclined at different angles to the horizon and a number of blocks of the same substance, but of different weights and shapes, are placed on it.

It will be found that the blocks begin to slide at the same angle of inclination of the plane to the horizon, (i) whatever be their weights, (ii) whatever be their shapes, which proves the laws (i) and (ii) of statical friction.

This angle of inclination of the plane is called the angle of friction between the substance of the blocks and the substance of the plane.

If the plane be fixed at an angle to the horizon greater than the angle of friction, and the blocks be let slide freely, it will be found that they all slide together with the same acceleration, (i) whatever be their weights, (ii) whatever be their shapes, and (iii) that the acceleration is constant, which proves laws (i), (ii), and (iii) of dynamical friction.

The surfaces in contact must be well lubricated for these laws to hold.

Let $2a$ be the length of the rod and b the diameter of the tumbler (fig. 16).

In the position of equilibrium the directions of the actions at A and C will intersect on the vertical through G the middle point of the rod.

$$\text{Therefore } a\sin\beta = AE = AD\cos\phi = AC\,\frac{\sin(\frac{1}{2}\pi+\phi)\cos\phi}{\sin(\pi-\beta-2\phi)}$$

$$= \frac{b\cos^2\phi}{\sin\beta\,\sin(\beta+2\phi)}.$$

$$\text{Similarly} \qquad a\sin\alpha = \frac{b\cos^2\phi}{\sin\alpha\,\sin(\alpha-2\phi)};$$

therefore $\sin^2\alpha \sin(\alpha - 2\phi) = \sin^2\beta \sin(\beta + 2\phi)$;

therefore $$\tan 2\phi = \frac{\sin^3\alpha - \sin^3\beta}{\sin^2\alpha \cos\alpha + \sin^2\beta \cos\beta}.$$

vii. Explain how velocity and rate of change of velocity are measured.

A velocity of one foot per second is changed uniformly in one minute to a velocity of one mile per hour. Express numerically the rate of change when a yard and a minute are the units of space and time.

A velocity of one foot per second is a velocity of 20 yards per minute.

A velocity of one mile per hour is a velocity of $\frac{88}{3}$ yards per minute.

The rate of change referred to yards and minutes is therefore

$$\tfrac{88}{3} - 20 = 9\tfrac{1}{3}.$$

viii. When the change of velocity is in a constant direction and its rate uniform, prove that the spaces described from rest are proportional to the squares of the times of describing them.

A train moving at the rate of sixty miles an hour is brought to rest in five minutes by uniform retardation. Find the space traversed by the train during reduction of speed.

Answer $2\frac{1}{2}$ miles.

ix. State Newton's second and third laws of motion.

A bullet is fired in the direction towards a second equal bullet which is let fall at the same instant. Prove that the two bullets will meet, and that if they coalesce the latus-rectum of their joint path will be one quarter of the latus-rectum of the original path of the first bullet.

Refer the motion to two fixed directions, one vertical, the other that of the line which initially joins the two bullets. It is evident, since gravity is the only force, that when the two bullets are in the same vertical line they are also at the same point, having both been drawn down through the same space from the line which initially joined them.

Secondly, the latus rectum of the path of a projectile varies as the square of the horizontal velocity in that path. One of the bullets has no horizontal velocity; hence, after they coalesce, their masses being equal, the joint horizontal velocity is half that of the first bullet, and the latus rectum of the joint path one quarter.

x. Shew how to determine the motion of two elastic spheres after direct impact, and prove that the relative velocity of each of them with regard to the centre of mass of the two is, after the impact, reversed in direction and reduced in the ratio $e : 1$; e being the coefficient of restitution.

A series of n elastic spheres whose masses are $1, e, e^2$, &c. are at rest, separated by intervals, with their centres on a straight line. The first is made to impinge directly on the second with velocity u. Prove that the final kinetic energy of the system is $\frac{1}{2}(1 - e + e^n) u^2$.

Let the sphere whose mass is e^r impinge with velocity v on the sphere whose mass is e^{r+1}. Let v', v'' be the respective velocities of the spheres after impact.

The whole momentum being unchanged

$$e^r v = e^r v' + e^{r+1} v'',$$

or $$v = v' + ev''.$$

Also, the velocity of separation being e times the velocity of approach

$$v'' - v' = ev;$$

therefore $$v'' = v,$$

and $$v' = (1 - e) v;$$

that is to say, each sphere acquires the velocity of the sphere which struck it, and the velocity of the latter is reduced in ratio $1-e:1$.

Hence the final kinetic energy of the system is

$$\tfrac{1}{2}\left\{(1-e)^2u^2+e(1-e)^2u^2+e^2(1-e)^2u^2+\ldots+e^{n-2}(1-e)^2u^2+e^{n-1}u^2\right\}$$

$$=\tfrac{1}{2}\left\{(1-e)^2\frac{1-e^{n-1}}{1-e}+e^{n-1}\right\}u^2=\tfrac{1}{2}(1-e+e^n)u^2.$$

xi. Determine the change in the square of the velocity of a particle which has descended through a given height down a smooth curve under the action of gravity.

A circle is drawn to touch at their middle points the chord and arc of oscillation of a particle which is moving on a vertical circle under the action of gravity. Prove that a point on the first circle in the same horizontal line with the particle moves with velocity equal to $2\sqrt{(gr)}\sin^2\frac{\alpha}{2}\cos\frac{\theta}{2}$, where r is the radius of the circle on which the particle moves, and α, θ are the angles which the radius drawn to the particle makes with the vertical at the instant when it is stationary and at the instant considered.

Let r be the radius of the arc ACB of the vertical circle over which the particle oscillates (fig. 17); ρ the radius of the circle which touches the chord AB and the arc ACB at their middle points D, C.

P, Q simultaneous positions of the particle and corresponding moving point, PQN being always horizontal.

u, v velocities of P and Q, $\angle PEC=\theta$, $\angle QFD=\phi$, $\angle AEC=\alpha$.

Then since $\quad CN=CF+FN=\tfrac{1}{2}CD(1+\cos\phi)$,

$$r(1-\cos\theta)=\tfrac{1}{2}r(1-\cos\alpha)(1+\cos\phi),$$

whence $$\sin^2\frac{\theta}{2}=\sin^2\frac{\alpha}{2}\cos^2\frac{\phi}{2}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

also since the vertical velocities of P and Q are equal

$$u \sin\theta = v \sin\phi \qquad\qquad (2),$$

and $$u^2 = 2gDN = 2gr(\cos\theta - \cos\alpha) \qquad\qquad (3).$$

Therefore

$$v^2 \sin^2\phi = 2gr(\cos\theta - \cos\alpha)\sin^2\theta$$

$$= 16gr\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}$$

$$= 16gr\sin^2\frac{\alpha}{2}\left(1 - \cos^2\frac{\phi}{2}\right)\sin^2\frac{\alpha}{2}\cos^2\frac{\phi}{2}\cos^2\frac{\theta}{2}, \text{ by (1)}$$

$$= 4gr\sin^4\frac{\alpha}{2}\sin^2\phi\cos^2\frac{\theta}{2},$$

whence $$v^2 = 4gr\sin^4\frac{\alpha}{2}\cos^2\frac{\theta}{2}.$$

xii. Find the time in which a particle under the action of gravity describes from rest any arc of a cycloid terminated at the lowest point, the axis of the cycloid being vertical and its vertex downwards.

Hence or from the rider to xi. determine the time of a small oscillation of a simple pendulum.

Pendulums which beat seconds correctly in London ($g = 32{\cdot}19$) and Edinburgh ($g = 32{\cdot}20$) respectively are interchanged in station. If started simultaneously from the vertical position towards the left, after how many seconds will they again be both vertical and moving leftwards?

To deduce the time of oscillation of a simple pendulum from the rider to xi., we observe that when α is small, $\cos\frac{\theta}{2}$ expressed by equation (1), differs from unity by a quantity of the same order as $\sin^2\frac{\alpha}{2}$.

Hence v is approximately constant and equal to

$$2\sqrt{(gr)}\,\sin^2\frac{\alpha}{2}.$$

Thus, time of P over ACB

$$= \text{time of } Q \text{ over circumference of smaller circle}$$

$$= \frac{2\pi.\tfrac{1}{2}r\,(1-\cos\alpha)}{2\sqrt{(gr)}\,\sin^2\dfrac{\alpha}{2}} = \pi\sqrt{\left(\frac{r}{g}\right)}.$$

(It is easy to obtain the next approximation to the time of oscillation of a simple pendulum.

For if the circumference DQC be divided into n equal parts, n being very large, time over element at Q

$$= \frac{\dfrac{\pi}{n}\,r\sin^2\dfrac{\alpha}{2}}{2\sqrt{(gr)}\,\sin^2\dfrac{\alpha}{2}\cos\dfrac{\theta}{2}} = \frac{\pi}{2n}\sqrt{\left(\frac{r}{g}\right)}\sec\frac{\theta}{2}$$

$$= \frac{\pi}{2n}\sqrt{\left(\frac{r}{g}\right)}\frac{1}{\sqrt{\left(1-\sin^2\dfrac{\alpha}{2}\cos^2\dfrac{\phi}{2}\right)}}$$

$$= \frac{\pi}{2n}\sqrt{\left(\frac{r}{g}\right)}\left(1+\tfrac{1}{2}\sin^2\frac{\alpha}{2}\cos^2\frac{\phi}{2}\right) \text{ nearly}$$

$$= \frac{\pi}{2n}\sqrt{\left(\frac{r}{g}\right)}\left(1+\tfrac{1}{4}\sin^2\frac{\alpha}{2}+\tfrac{1}{4}\sin^2\frac{\alpha}{2}\cos\phi\right).$$

Hence time of Q over semi-circumference DQC

$$= \frac{\pi}{2}\sqrt{\left(\frac{r}{g}\right)}\left(1+\tfrac{1}{4}\sin^2\frac{\alpha}{2}\right)$$

$$+ \frac{\pi}{2n}\sqrt{\left(\frac{r}{g}\right)}.\tfrac{1}{4}\sin^2\frac{\alpha}{2}\left\{\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cos\frac{3\pi}{n}+\ldots+\cos\frac{(n-1)\pi}{n}\right\}.$$

Now $$\cos r\frac{\pi}{n}+\cos(n-r)\frac{\pi}{n} = 0.$$

Hence when n is odd the series vanishes, and when n is even is reduced to its middle term which is multiplied by an infinitely small quantity.

In either case the time over the whole circumference $DQCD$

$$=\pi\sqrt{\frac{r}{g}}\left(1+\tfrac{1}{4}\sin^2\frac{\alpha}{2}\right),$$

and this is the approximate time of swing of a simple pendulum, length r through an angle 2α from rest to rest).

(*Second Rider*).

The London pendulum, removed to Edinburgh, swings from rest to rest in $\sqrt{\left(\frac{3219}{3220}\right)}$ of a second. The Edinburgh pendulum, removed to London, swings from rest to rest in $\sqrt{\left(\frac{3220}{3219}\right)}$ of a second.

By the condition of the question, the London pendulum, when at Edinburgh, must make two swings more than the other in the same time.

Hence, if t be the time in seconds,

$$\frac{t}{\sqrt{\left(\frac{3219}{3220}\right)}}-\frac{t}{\sqrt{\left(\frac{3220}{3219}\right)}}=2;$$

therefore,

$$\begin{aligned} t &= 2\sqrt{(3219\times 3220)} \text{ seconds}\\ &= (3219+3220) \text{ seconds nearly}\\ &= 1^{\text{h}}\ 47^{\text{m}}\ 19^{\text{s}}. \end{aligned}$$

TUESDAY, *Jan.* 5, 1875. $1\frac{1}{2}$ to 4.

MR. GREENHILL. Arabic numbers.
MR. WRIGHT. Roman numbers.

1. DEFINE a fluid and prove that the pressure of a fluid at rest is the same in all directions about a point.

Define the measure of the elasticity of a fluid and prove that if the elasticity is equal to the pressure, the pressure of the fluid is inversely proportional to the volume.

A fluid is a substance such that the smallest shearing stress if continued will cause a constantly increasing change of form.

When the fluid is at rest there must be no shearing stress, and therefore the stress must be uniform in all directions about a point, and this will be true whatever be the degree of viscosity of the fluid. (Maxwell, *Heat*, p. 276).

The elasticity of a fluid under any given conditions is the ratio of any small increase of pressure to the cubical compression thereby produced.

If v be the volume of a given quantity of gas when the pressure is p, and v' the volume when the pressure is increased to p', the increase being small, the cubical compression may be measured by $\dfrac{v-v'}{v'}$.

Therefore the elasticity is $\dfrac{p'-p}{\dfrac{v-v'}{v'}}$, and if this is equal to p,

then $$p'v' = pv.$$

This may also be proved by a diagram, as in Maxwell's *Heat*, pp. 107 and 111.

2. If a solid be immersed in a liquid the resultant pressure on the surface immersed is equal and directly opposed to the weight of the displaced liquid.

Deduce the conditions of equilibrium of a floating body.

If an elliptic lamina with its centre of gravity at an excentric point float in liquid, prove that there may be two or four positions of equilibrium and point out which are stable and which unstable.

The lines of floatation will touch a similar ellipse, and the centres of inertia of the segments cut off by the lines of floatation will also lie on a similar ellipse, the tangent to which at any point will be parallel to the corresponding line of floatation.

If we draw normals from the centre of inertia of the lamina to this last ellipse, these normals will be vertical in the positions of equilibrium.

Two or four normals can be drawn according as the centre of inertia lies outside or inside the evolute of this ellipse.

The points of contact of the normals with the evolute will be the metacentres, and when a normal is vertical the equilibrium will be stable when the point of contact lies above the centre of inertia of the lamina, unstable when it lies below.

3. Find the centre of pressure of a triangular lamina when immersed in liquid (i) with its base in the surface, (ii) with its vertex in the surface and base horizontal.

If a quadrilateral lamina $ABCD$ in which AB is parallel to CD be immersed in liquid with the side AB in the surface, the centre of pressure will be at the point of intersection of AC and BD if $AB^2 = 3CD^2$.

(i) The centre of pressure is at half the depth of the vertex.

(ii) The centre of pressure is at three-fourths the depth of the base.

Let E, F (fig. 18) be the middle points of AB and CD and G, H the centres of pressure of the $\triangle ABD$, and the $\triangle BCD$ respectively.

If the centre of pressure of the quadrilateral lie in BD, taking moments about BD,

$$AB \cdot \tfrac{1}{3} \cdot \tfrac{1}{4}AB = CD \cdot \tfrac{2}{3} \cdot \tfrac{3}{8}CD,$$

$$AB^2 = 3\,CD^2,$$

and the same condition is obtained by taking moments about AC.

4. On the assumptions that "one perfect gas acts as a vacuum with respect to any other perfect gas," and that "the volume of a perfect gas under constant pressure expands uniformly when raised from the freezing to the boiling temperature by the same fraction of itself whatever be the nature of the gas," prove that the pressure of a given quantity of a perfect gas is inversely proportional to the volume and directly proportional to the absolute temperature.

Air is compressed in a vessel at a pressure p and at the same temperature as the atmosphere. An aperture is then opened, and shut the instant the air inside is at the atmospheric pressure P, and it is found that when the air left in the vessel is again at the same temperature as the atmosphere its pressure is p'. Find how much air has issued and the temperature at the instant the aperture was shut.

Explain why p' is greater than P.

(Maxwell's *Heat*, pp. 27, 30, 50, and 179).

The air left in the vessel is $\dfrac{p'}{p}$ of the original amount, and therefore $\dfrac{p-p'}{p}$ of the air has escaped.

If T be the absolute temperature of the atmosphere and t of the air inside the vessel the instant the aperture was shut

$$P : p' :: t : T;$$

therefore
$$t = \frac{PT}{p'}.$$

The air in escaping has done work against the external pressure of the atmosphere; the temperature has consequently fallen, and therefore t being less than T, p' is greater than P.

5. Describe the barometer and explain to what extent the readings are affected by changes of temperature.

If the barometer tube dips into a cylindrical cistern of mercury and is suspended by a string which passes over a pulley and supports a counterpoise, prove that the ratio of the changes of height of the counterpoise to the corresponding changes of height of the barometic column is equal to the ratio of the interior sectional area of the tube to the annular sectional area of the tube.

In the barometer (fig. 19) the height of the column is measured by a scale fixed to a brass rod pointed at the lower end and which can screw up and down.

In taking an observation the point is made just to touch the mercury in the cistern, and the height of the column is read off on the scale.

The only corrections for temperature are then the coefficient of cubical expansion of mercury and the coefficient of linear expansion of the brass rod which carries the scale.

Let x (fig. 20) be the depth of the bottom of the barometer tube and z the depth of the surface of the mercury in the cistern below the level at which the surface would stand if the barometer tube were removed.

Let h be the height of the barometric column, a the internal, A the annular sectional area of the tube, C the internal sectional area of the cistern.

Then $$Ax + (C - A - a)\,z = a\,(h - z),$$

$$z = \frac{ah - Ax}{C - A}.$$

Let W be the weight of the counterpoise, W' of the tube, σ the density of mercury.

The downward force of the air on the top of the tube is $g\sigma h\,(A + a)$ and the upward force of the mercury on the annular bottom of the tube is $g\sigma\,(h + x - z)\,A$.

Therefore the condition of equilibrium of the tube is

$$g\sigma h\,(A + a) - g\sigma\,(h + x - z)\,A = W - W';$$

therefore $$\frac{W-W'}{g\sigma} = ah - A(x-z)$$

$$= \frac{C}{C-A}(ah - Ax).$$

If h', x' be corresponding changes of h and x,

$$ah' - Ax' = 0;$$

therefore $$\frac{x'}{h'} = \frac{a}{A}.$$

6. Define the density and specific gravity of a body and shew how the specific gravity may be determined by the hydrostatic balance.

If P, P' be the weights which balance the body when suspended in air and in water respectively when the absolute temperature is t and the corrected height of the barometer is h, prove that the density of the body at the temperature T is

$$\{1 + k(t-T)\}\left(\frac{Pa_t}{P-P'} - \frac{P'b}{P-P'} \cdot \frac{h}{H} \cdot \frac{T}{t}\right),$$

where a_t is the density of water at the temperature t, k is the coefficient of expansion of the body, and b is the density of air when the absolute temperature is T and the corrected height of the barometer is H.

The density of a body is measured by the number of units of mass in the unit of volume.

The specific gravity of a body is the ratio of its density to that of some standard substance, generally water.

Let σ be the correction for the buoyancy of the weights P and P' in air; v_t, d_t the volume and density of the body at the temperature t.

The density of the air at the time of observation is $b \cdot \frac{hT}{Ht}$;

therefore $$P(1-\sigma) = v_T d_T - v_t b \cdot \frac{hT}{Ht},$$

$$P'(1-\sigma) = v_T d_T - v_t a_t,$$

eliminating σ, $d_T = \frac{v_t}{v_T}\left(\frac{Pa_t}{P-P'} - \frac{P'b}{P-P'}\cdot\frac{hT}{Ht}\right)$

$$= \{1 + k(t-T)\}\left(\frac{Pa_t}{P-P'} - \frac{P'b}{P-P'}\cdot\frac{hT}{Ht}\right).$$

vii. State the laws for the reflexion and refraction of a ray of light at a surface.

Prove that a pencil of parallel rays of the same refrangibility will consist of parallel rays after reflexion or refraction at a plane surface.

Two plane vertical mirrors intersect at right angles and a person looks into the angle formed by them. Prove that, supposing no light can be reflected at the line of junction of the mirrors, he will see only one eye in the mirrors and that if he shut either eye the image seen will be that of a closed eye.

Let OA, OB (fig. 21) be the mirrors; E, F, the eyes; E', F' their images after two reflexions in the mirrors.

E will see F' only and F will see E' only, and since EF' and FE' are parallel, the two images will be seen in the same direction, and therefore give the impression to the brain of only one image.

If F be shut, E will see the image F' shut, and if E be shut, F will see the image E' shut.

viii. Determine the geometrical focus of a pencil of rays after direct refraction at a spherical refracting surface.

A hollow globe of glass has a speck on its interior surface, if this be observed from a point outside the sphere on the opposite side of the centre, prove that the speck will appear nearer than it really is by a distance $\frac{\mu-1}{3\mu-1}t$, provided that t the thickness of the glass is equal to the radius of the internal cavity and μ is the refractive index for the glass.

Let A be the speck (fig. 22); O the centre of the sphere; P_1, P_2 the foci after the first and second refractions; therefore

$$\frac{1}{OP_1} - \frac{\mu}{t} = \frac{\mu - 1}{t},$$

$$\frac{1}{OP_1} - \frac{\mu}{OP_2} = \frac{\mu - 1}{2t};$$

theefore
$$\frac{\mu}{OP_2} = \frac{3\mu - 1}{2t},$$

$$AP_2 = t - OP_2$$

$$= \frac{\mu - 1}{3\mu - 1} t.$$

ix. If a ray of light pass in a principal plane through a prism denser than the surrounding medium, the deviation is towards the thicker part of the prism.

Two triangular isosceles prisms are placed with two faces in contact and the refracting edges parallel, prove that the deviation of a ray which in passing through the combination in a principal plane is reflected at each base is independent of the refractive indices of the prisms and of the angle of incidence.

If i be the refracting angle of an isosceles prism and ϕ the angle of incidence of a ray measured positively towards the refracting edge of the prism; then if the ray be reflected at the base, the angle of emergence will be ϕ and the deviation towards the refracting edge of the prism will be $i + 2\phi$.

If another isosceles prism of refracting angle i' be placed with a face in contact with a face of the first prism, then if the prisms be related as in fig. 23, ϕ will be the angle of incidence on the second prism, and the total deviation of the ray towards the refracting edge of the first prism will be $i+2\phi-i'-2\phi=i-i'$.

But if the prisms be related as in fig. 24, $-\phi$ will be the angle of incidence on the second prism and the total deviation of the ray will be $i+2\phi+i'-2\phi=i+i'$.

x. Define the illumination at any point of a surface and prove that the illumination due to rays proceeding from a bright point varies directly as the cosine of the angle of incidence and inversely as the square of the distance from the bright point.

Explain how units of brightness and illumination could be selected and defined.

The unit of brightness is defined by a light which consumes a certain amount of oil or gas in the unit of time; for instance, the light of a wax candle of certain weight to burn a certain time.

The unit illumination would be that produced at unit distance from the light of unit brightness.

xi. Shew how to find the focal length of a system of lenses of known focal lengths whose axes are coincident and which are separated by given intervals.

If the lenses be all concave, each of focal length f, and such that the interval between the rth and $(r+1)$th lenses is equal to the distance of the focus after the rth refraction from the rth lens, and if the original pencil be parallel, prove that the distance of the nth focus from the nth lens is $\dfrac{2^{n-1}}{2^n-1}f$.

Let $v_1, v_2, v_3, \ldots v_n$ be the distances of the foci from their respective lenses, then

$$\frac{1}{v_1} \qquad\qquad = \frac{1}{f},$$

$$\frac{1}{v_2} - \frac{1}{2v_1} = \frac{1}{f},$$

$$\frac{1}{v_3} - \frac{1}{2v_2} = \frac{1}{f},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{v_n} - \frac{1}{2v_{n-1}} = \frac{1}{f};$$

therefore $$\frac{1}{v_n} = \frac{1}{f} + \frac{1}{2f} + \frac{1}{2^2 f} + \ldots + \frac{1}{2^{n-1} f}.$$

$$= \frac{1}{f}\,\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}};$$

therefore $$v_n = \frac{2^{n-1} f}{2^n - 1}.$$

xii. Describe the Newtonian telescope and find an expression for the magnifying power.

If the focal length of the reflector be 2 feet and the focal length of the eye-glass 1 inch, and if the instrument be in focus for a star to a person who sees most distinctly at a distance of 6 feet, prove that it requires no readjustment for a person who sees most distinctly at a distance of 2 feet and is viewing an object whose distance is 609 yards.

Let F, F' (fig. 25) be the images of the star and object formed by the first mirror; f, f' the images formed by the small mirror; g, g' the images formed by the eye-piece.

$$AF = 2 \text{ and } \frac{1}{AF'} + \tfrac{1}{1827} = \tfrac{1}{2};$$

therefore $$AF' = \tfrac{3654}{1825}, \quad FF' = \tfrac{4}{1825}.$$

Also $$\frac{1}{Ef} - \frac{1}{Eg} = 12, \text{ and } Eg = 6;$$

therefore $$Ff = \tfrac{6}{73}, \quad Ef' = \tfrac{146}{1825},$$

and $$\frac{1}{Eg'} = \frac{1}{Ef'} - 12 = \tfrac{1}{2};$$

therefore $$Eg' = 2.$$

WEDNESDAY, *Jan.* 6, 1875. 9 to 12.

MR. COCKSHOTT. Roman numbers.
MR. WRIGHT. Arabic numbers.

1. UPON the sides of a triangle ABC as bases are described three equilateral triangles aBC, bCA, and cAB, all upon the same side of their bases as the triangle ABC. Prove that Aa, Bb, Cc are all equal and pass through a point which lies on all the three circles circumscribing the equilateral triangles.

If Aa be produced (fig. 26) to meet the circle described round aBC in P, then $\angle aPC = 120°$ and $\angle aPB = 60°$; therefore $\angle BPC = 60°$.

Since $\angle APc = 120°$, therefore the circle described round bCA will pass through P, and $\angle bPC = \angle bAC = 60°$. Therefore BbP is a straight line.

Since $\angle APB = 60°$; therefore the circle described round cAB will pass through P, and $\angle cPB = \angle cAB = 60°$. Therefore cCP is a straight line.

In the triangles AaC and BbC, $aC = BC$, $CA = Cb$ and $\angle AaC = \angle bBC$; therefore $Aa = Bb$.

In the triangles BaA and BCc, $Ba = BC$, $BA = Bc$ and $\angle BaA = \angle BCc$; therefore $Aa = Cc$.

ii. Given the circumscribed and inscribed circles of a triangle, prove that the centres of the escribed circles lie on a fixed circle.

The circle circumscribing the triangle ABC is the nine-pointic circle of the triangle $A'B'C'$ formed by the centres of the escribed circles.

If D be the centre of the inscribed circle, and O of the circumscribed circle of the triangle ABC, then A', B', C' lie on a circle of radius double the radius of the circumscribed circle and with centre at O' on DO produced, such that $DO' = 2DO$.

3. Out of m persons who are sitting in a circle three are selected at random; prove that the chance that no two of those selected are sitting next one another is

$$\frac{(m-4)(m-5)}{(m-1)(m-2)}.$$

If A be the first person selected, the chance B is not next him is $\dfrac{m-3}{m-1}$.

If A and B be next but one, the chance of which is $\dfrac{2}{m-3}$, C must not be next either, the chance of which is $1 - \dfrac{3}{m-2} = \dfrac{m-5}{m-2}$.

If A and B have at least two people between them, the chance of which is $\dfrac{m-5}{m-3}$, C must not be next either, the chance of which is

$$1 - \frac{4}{m-2} = \frac{m-6}{m-2}.$$

Therefore the whole chance is

$$\frac{m-3}{m-1}\left(\frac{2}{m-3}\cdot\frac{m-5}{m-2} + \frac{m-5}{m-3}\cdot\frac{m-6}{m-2}\right) = \frac{(m-4)(m-5)}{(m-1)(m-2)}.$$

4. A person has n sewing-machines, each of which requires one worker and will yield each day it is at work q times the worker's wages as nett profits; the machines are never all in working order at once, and at any time it is equally likely that 1, 2, 3 or any other number of them are out of repair. The worker's wages must be paid whether there is a machine for him to work or not. Prove that the most profitable number of workers to be permanently engaged will be the integer nearest to $\dfrac{nq}{q+1} - \dfrac{1}{2}$.

Let x be the number of workers.

If 1 machine is out of order, the profits are qx,

2 .... qx,

....

$n-x$ .... qx,

$n-x+1$ .... $q(x-1)-1$,

$n-x+2$ .... $q(x-2)-2$,

....

$n-1$ .... $q-(x-1)$,

n .... $-x$.

Therefore the average profit is

$$\frac{1}{n}\left\{(n-x)\,qx+\frac{qx\,(x-1)}{2}-\frac{x\,(x+1)}{2}\right\}$$

$$=\frac{1}{n}\left\{\left(nq-\frac{q+1}{2}\right)x-\frac{q+1}{2}\,.\,x^2\right\}$$

$$=\frac{q+1}{2n}\left\{\left(\frac{nq}{q+1}-\frac{1}{2}\right)^2-\left(x-\frac{nq}{q+1}+\frac{1}{2}\right)^2\right\},$$

which is a maximum when $x=\dfrac{nq}{q+1}-\dfrac{1}{2}$, or the integer nearest to this.

5. The unique solution of the equations

$$\frac{x-2\dfrac{xy-z^2}{x+y}}{a}=\frac{z}{c}=\frac{y-2\dfrac{xy-z^2}{x+y}}{b}=\frac{z^2-xy}{c^2-ab}$$

is $\dfrac{x}{a\,(a-b)+2c^2}=\dfrac{y}{b\,(b-a)+2c^2}=\dfrac{z}{c\,(a+b)}=-\dfrac{a+b}{(a-b)^2+4c^2}$.

From the equations

$$\frac{z}{c}=\frac{x-y}{a-b}\ \text{ and }\ \frac{x\,(x-y)+2z^2}{a}=\frac{y\,(y-x)+2z^2}{b};$$

therefore

$$\frac{x(x-y)+2c^2\left(\frac{x-y}{a-b}\right)^2}{a}=\frac{y(y-x)+2c^2\left(\frac{x-y}{a-b}\right)^2}{b},$$

$x-y=0$ would require $a-b=0$; therefore

$$\frac{x(a-b)^2+2c^2(x-y)}{a}=\frac{-y(a-b)^2+2c^2(x-y)}{b};$$

therefore

$$\frac{x}{a(a-b)+2c^2}=\frac{y}{b(b-a)+2c^2}=\frac{x-y}{a^2-b^2}=\frac{z}{c(a+b)},$$

and $$\frac{z}{c}=\frac{z^2-xy}{c^2-ab}$$

$$=\frac{z^2}{c^2}\cdot\frac{c^2(a+b)^2+ab(a-b)^2-2c^2(a-b)^2-4c^4}{(a+b)^2(c^2-ab)}$$

$$=-\frac{z^2}{c^2}\cdot\frac{(a-b)^2+4c^2}{(a+b)^2};$$

therefore $$\frac{z}{c(a+b)}=-\frac{a+b}{(a-b)^2+4c^2}.$$

6. If A', B', C' be any points on the sides of the triangle ABC, prove that $AB'.BC'.CA'+B'C.C'A.=A'B$ the area of the triangle $A'B'C'\times$ twice the diameter of the circle circumscribing the triangle ABC.

$$\begin{aligned}&AB'.BC'.CA'+B'C.C'A.A'B\\&=AB'.BC'.CA'+(CA-AB')(AB-BC')(BC-CA')\\&=BC.CA.AB-BC.C'A.AB'-CA.A'B.BC'-AB.B'C.CA'\\&=4R(\triangle ABC-\triangle AB'C'-\triangle BC'A'-\triangle CA'B')\\&=4R.\triangle A'B'C'.\end{aligned}$$

vii. If
$$\begin{aligned}x&=2\cos(\beta-\gamma)+\cos(\theta+\alpha)+\cos(\theta-\alpha)\\&=2\cos(\gamma-\alpha)+\cos(\theta+\beta)+\cos(\theta-\beta)\\&=-2\cos(\alpha-\beta)-\cos(\theta+\gamma)-\cos(\theta-\gamma),\end{aligned}$$

prove that $x=\sin^2\theta$, provided that the difference between any two of the angles α, β, γ, neither vanishes nor equals a multiple of π.

$$\cos\theta(\cos\alpha+\cos\gamma)=-\cos(\beta-\gamma)-\cos(\alpha-\beta);$$

therefore $$\cos\theta\cos\frac{\alpha+\gamma}{2}=-\cos\left(\beta-\frac{\alpha+\gamma}{2}\right),$$

so $$\cos\theta\cos\frac{\beta+\gamma}{2}=-\cos\left(\alpha-\frac{\beta+\gamma}{2}\right);$$

therefore

$$\cos\frac{\alpha+\gamma}{2}\cos\left(\alpha-\frac{\beta+\gamma}{2}\right)=\cos\frac{\beta+\gamma}{2}\cos\left(\beta-\frac{\alpha+\gamma}{2}\right);$$

therefore

$$\cos\left(\frac{\alpha-\beta}{2}-\gamma\right)+\cos\frac{3\alpha-\beta}{2}=\cos\left(\frac{\alpha-\beta}{2}+\gamma\right)+\cos\frac{\alpha-3\beta}{2};$$

therefore $$2\sin\frac{\alpha-\beta}{2}\sin\gamma=2\sin(\alpha-\beta)\sin\frac{\alpha+\beta}{2};$$

therefore, provided $\sin\dfrac{\alpha-\beta}{2}$ is not zero,

$$\sin\gamma=2\cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}=\sin\alpha+\sin\beta;$$

therefore $$\cos\theta=-\cos\beta-\sin\beta\tan\frac{\alpha+\gamma}{2}$$

$$=-\cos\beta-(\sin\alpha-\sin\gamma)\tan\frac{\alpha+\gamma}{2}$$

$$=-\cos\alpha-\cos\beta+\cos\gamma;$$

therefore $$x=2\cos(\beta-\gamma)+2\cos\alpha(-\cos\alpha-\cos\beta+\cos\gamma)$$

$$=2\cos\beta\cos\gamma+\sin^2\beta+\sin^2\gamma-\sin\alpha$$

$$-2\cos^2\alpha-2\cos\alpha\cos\beta+2\cos\alpha\cos\gamma$$

$$=1-(-\cos\alpha-\cos\beta+\cos\gamma)^2$$

$$=1-\cos^2\theta=\sin^2\theta.$$

viii. A circle and parabola touch one another at both ends of a double ordinate to the parabola, prove that the latus rectum is a third proportional to the parts into which the abscissa of the points of contact is divided by the circle either internally or externally.

$$\text{(fig. 27) } PN^2 = MN.NM'$$

$$= MN(2NG + MN) = MN(4AS + MN);$$

therefore $$4AS.AN = MN(4AS + MN);$$

therefore $$4AS.AM = MN^2.$$

So also $$4AS.AN = NM'(NM' - 4AS);$$

therefore $$4AS.AM' = NM'^2.$$

9. Inscribe in a given parabola a triangle having its sides parallel to those of a given triangle.

LEMMA. If from a point A (fig. 28) two straight lines AbB, AcC be drawn meeting a conic in b, B; c, C; and BB', CC' be drawn parallel respectively to AcC, AbB intersecting in P and meeting the conic in B', C'; then $B'C'$ is parallel to bc.

For $$\frac{Ab.AB}{Ac.AC} = \frac{PC.PC'}{PB.PB'},$$

and $$AB = PC, \ AC = PB;$$

therefore $$\frac{Ab}{Ac} = \frac{PC'}{PB'};$$

and therefore bc is parallel to $B'C'$.

If A lie on the conic, then $B'C'$ is parallel to the tangent at A.

Let A, B, C (fig. 29) be the points of contact of the tangents to *any* conic parallel to the sides of the given triangle, and let AA' be drawn parallel to BC to meet the conic again in A', BB' parallel to CA to meet the conic in B', and CC' parallel to AB to meet the conic in C'.

Then $A'B'C'$ will be the required triangle.

(In the same manner a polygon may be inscribed in any conic having its sides parallel to the sides of a given polygon).

x. Two given ellipses in the same plane have a common focus, and one revolves about the common focus, while the other remains fixed; prove that the locus of the point of intersection of their common tangents is a circle.

Let S be the common focus of the two ellipses (fig. 30), H the other focus of the fixed ellipse, K of the revolving ellipse, b, b' the semi-minor axes.

Let T be the point of intersection of the common tangents, and draw the perpendiculars SY, SY', HZ, HZ', KW, KW' on the common tangents.

Then $$SY.HZ = b^2 \text{ and } SY.KW = b'^2,$$

and therefore $$\frac{HZ}{KW} = \frac{b^2}{b'^2}.$$

So also $$\frac{HZ'}{KW'} = \frac{b^2}{b'^2} = \frac{HZ}{KW},$$

and therefore HKT is a straight line, and

$$\frac{HT}{KT} = \frac{b^2}{b'^2}.$$

But the locus of K is a circle, therefore the locus of T is a circle.

11. Two elastic strings are fastened at a fixed point P and pass through fixed smooth rings A and B such that PA, PB are the natural lengths of the respective strings; the other ends of the strings are fastened to C and D two points of a rigid lamina which is moveable in its plane about a fixed point O. If A and B are in the same plane as the lamina and if the angles COA, DOB are supplementary, and the system is in equilibrium, prove that the equilibrium will be neutral.

Let λ, λ' be the moduli of elasticity of the strings; then the condition of equilibrium is (fig. 31)

$$\lambda . \frac{AC}{AP} . OH = \lambda' \frac{BD}{BP} . OK,$$

$$\frac{\lambda}{\lambda'} = \frac{AP}{BP} \cdot \frac{BD}{AC} \cdot \frac{OK}{OH} = \frac{AP}{BP} \cdot \frac{OB}{OA} \cdot \frac{DN}{CM}$$

$$= \frac{AP}{BP} \cdot \frac{OB}{OA} \cdot \frac{OD}{OC},$$

which is independent of the position of the body, and therefore if there is equilibrium in one position there is equilibrium in all positions.

xii. A beam AB lies horizontally upon two others at points A and C, prove that the least horizontal force applied at B in a direction perpendicular to BA which is able to move the beam is the less of the two forces $\mu W \frac{b-a}{2a-b}$ and $\frac{\mu W}{2}$, where $AB = 2a$, $AC = b$, W is the weight of the beam, and μ is the coefficient of friction.

The maximum friction at C is $\frac{\mu Wa}{b}$, and at A is $\frac{\mu W}{b}(b-a)$.

(i) If C is fixed, the friction at A is perpendicular to BA, and if P be the force applied at B,

$$P = \mu W \frac{b-a}{2a-b},$$

and force at C $$= \mu W \left(\frac{b-a}{2a-b} + \frac{b-a}{b} \right),$$

which is less than the maximum friction if

$$\frac{a}{b} > \frac{2a(b-a)}{b(2a-b)}, \quad b < \frac{4a}{3}.$$

(ii) If A is fixed, $P = \frac{\mu W}{2}$, and force at $A = \mu W \frac{2a-b}{2b}$, which is less than the maximum friction if $\frac{b-a}{b} > \frac{2a-b}{2b}$, $b > \frac{4a}{3}$.

(iii) If $b = \frac{4a}{3}$, slipping begins at both points and $P = \frac{\mu W}{2}$.

P must obviously be applied perpendicularly to the rod, to be a minimum.

xiii. A bucket and a counterpoise, connected by a string passing over a pulley, just balance one another, and an elastic ball is dropped into the centre of the bucket from a distance h above it; find the time that elapses before the ball ceases to rebound, and prove that the whole descent of the bucket during this interval is $\frac{4mh}{2M+m}\frac{e}{(1-e)^2}$, where m, M are the masses of the ball and the bucket, and e is the coefficient of restitution.

Let v be the velocity of the ball just before the first impact.

The relative velocity after the first impact is ev, and the relative acceleration is g, since the acceleration of the bucket is zero.

Therefore the time during which the ball rebounds is

$$\frac{2v}{g}(e+e^2+e^3+\ldots) = \frac{2v}{g}\cdot\frac{e}{1-e} = 2\sqrt{\left(\frac{2h}{g}\right)}\cdot\frac{e}{1-e}.$$

Let V_1, V_2, V_3... be the velocities of the bucket during the intervals between the first, second, third, ... impacts.

Then
$$V_1 = \frac{m(1+e)}{2M+m}.v,$$

$$V_2 = V_1 + \frac{m(1+e)}{2M+m}.ev,$$

$$V_3 = V_2 + \frac{m(1+e)}{2M+m}.e^2v,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

and the space described by the bucket is

$$\frac{2v}{g}(eV_1+e^2V_2+e^3V_3+\ldots)$$

$$=\frac{2me}{(2M+m)(1-e)^2}\cdot\frac{v^2}{g}=\frac{4mh}{2M+m}\cdot\frac{e}{(1-e)^2}.$$

xiv. A particle is projected from the foot of an inclined plane and returns to the point of projection after several rebounds one of which is perpendicular to the inclined plane: if it takes r more leaps in coming down than in going up, prove that $\cot\alpha\cot\theta=\dfrac{2\sqrt{(1-e^r)}-2(1-e^r)}{(1-e)e^r}$, where α is the inclination of the plane, θ the angle between the direction of projection and the plane, and e the coefficient of restitution.

What is the condition that it may be possible to project the particle so that one of its impacts may be perpendicular to the plane?

Let n be the number of leaps going up and $n+r$ the number coming down the plane, and v the velocity of projection.

Considering the motion perpendicular to the plane, the times occupied by the leaps are $\dfrac{2v\sin\theta}{g\cos\alpha}$, $\dfrac{2ev\sin\theta}{g\cos\alpha}$, ..., and considering the motion parallel to the plane, since the n^{th} impact is perpendicular to the plane, the time occupied by the first n leaps is $\dfrac{v\cos\theta}{g\sin\alpha}$, and the time of coming down is equal to the time of going up; therefore

$$\frac{v\cos\theta}{g\sin\alpha}=\frac{2v\sin\theta}{g\cos\alpha}(1+e+e^2+\ldots+e^{n-1})$$

$$=\frac{2v\sin\theta}{g\cos\alpha}(e^n+e^{n+1}+\ldots+e^{2n+r-1});$$

therefore $$1-2e^n+e^{2n+r}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

therefore $$e^n=\frac{1-\sqrt{(1-e^r)}}{e^r},$$

the negative sign being taken to the radical because $e<1$; therefore

$$\cot\alpha \cot\theta = \frac{2\sqrt{(1-e^r)} - 2(1-e^r)}{(1-e)e^r}.$$

The condition that it may be possible to project the particle so that one of its impacts may be perpendicular to the plane is that the equation (1) should give an integral value of n for an integral value of r.

15. If in BA, CA two sides of a triangle ABC two points D, E be taken respectively, such that $BA : AC :: EA : AD$ and G the middle point of DE be joined to A, and if BH, CK be constructed in the same way as AG, shew that AG, BH, CK intersect in a fixed point O.

Prove also that if from O perpendiculars be drawn to the sides of the triangle the sum of their squares is less than the sum of the squares of the perpendiculars from any other point.

If GQ, GR (fig. 32) be drawn perpendicular to CA, AB,

$$GQ : GR :: AD : AE :: CA : AB,$$

and therefore at O

$$OL : OM : ON :: BC : CA : AB,$$

if OL, OM, ON be the perpendiculars on the sides of the triangle ABC.

(DE is parallel to the tangent at A to the circumscribing circle and may be called an anti-parallel to BC, and O may be called the centre of anti-parallels.)

If x, y, z be the perpendiculars on the sides of the triangle ABC and a, b, c the sides,

$$(a^2+b^2+c^2)(x^2+y^2+z^2) = (ax+by+cz)^2$$
$$+ (bz-cy)^2 + (cx-az)^2 + (ay-bx)^2,$$

a minimum when

$$x : y : z :: a : b : c,$$

since $ax+by+cz$ is constant, being double the area of the triangle ABC.

xvi. A body is describing an ellipse about a centre of force in the focus, and when its radius vector is half the latus rectum it receives a blow which causes it to move towards the other focus with a momentum equal to that of the blow; find the position of the axis of the new orbit and shew that its eccentricity is $\frac{1-e^2}{2e}$, where e is the eccentricity of the original orbit.

Let LQ, LR (fig. 33) represent the velocities of the body before and after the blow, then QR represents the velocity generated by the blow, and therefore $QR = LR$, and QR is parallel to LS.

The blow is therefore towards the centre of force and the latus-rectum is unaltered in magnitude.

Hence SL being a radius vector of the new orbit equal to half the latus-rectum coincides with the latus-rectum in position.

LS' being the tangent to the new orbit, S' is the foot of the directrix, and the eccentricity is

$$\frac{SL}{SS'} = \frac{1-e^2}{2e}.$$

17. A cone floats in liquid which fills a fixed conical shell: both the cone and the shell have their axes vertical and vertices downwards: the vertical angles of the cone and shell are equal and the axis of the shell is twice that of the cone. If the cone be pressed down until its vertex very nearly reaches the vertex of the shell, so that some of the liquid overflows, and then released, it is found that the cone rises until it is just wholly out of the liquid and then begins to fall. Prove that the densities of the cone and the liquid are in the ratio $45 - 21\sqrt[3]{(7)} : 4\sqrt[3]{(7)}$, the free surface of the liquid being supposed to remain horizontal throughout the motion.

In the two positions in which the velocity of the cone is zero, the heights of the centre of gravity of the cone and liquid are equal.

Let ρ be the density of the liquid, σ of the cone, h the height of the cone; in the second position the height of the surface of the liquid is $\sqrt[3]{(7)}\,h$; therefore

$$\tfrac{3}{4}.2h.\frac{8\pi h^3}{3}\tan^2\alpha.\rho-\tfrac{3}{4}.h.\frac{\pi h^3}{3}\tan^2\alpha\,(\rho-\sigma)$$

$$=\tfrac{3}{4}.\sqrt[3]{(7)}\,h.\frac{7\pi h^3}{3}\tan^2\alpha.\rho+\{\sqrt[3]{(7)}+\tfrac{3}{4}\}\,h\,\frac{\pi h^3}{3}\tan^2\alpha.\sigma;$$

therefore $$\{45-21\sqrt[3]{(7)}\}\,\rho=4\sqrt[3]{(7)}\,\sigma.$$

xviii. A conical cup of uniform thickness floats in water, with its axis inclined to the vertical at an angle θ and the fraction m of the axis below the level of the surface of the water; prove that $\cos^2\theta=\dfrac{8\sin^2\alpha}{8-9m}$, where 2α is the vertical angle of the cone.

Let G (fig. 34) be the centre of inertia of the conical cup, H of the water displaced; then GH is vertical. Let $AD=h$, then $AE=mh$, $AG=\frac{2}{3}h$.

$$FN=\tfrac{2}{3}h\sin\theta=\tfrac{3}{4}FR=\tfrac{3}{8}(FP+FQ)$$

$$=\frac{3mh}{8}\cos\theta\,\{\tan(\theta-\alpha)+\tan(\theta+\alpha)\};$$

therefore $$\frac{16}{9m}\sin\theta=\frac{\cos\theta\sin2\theta}{\cos^2\theta-\sin^2\alpha};$$

therefore $$8(\cos^2\theta-\sin^2\alpha)=9m\cos^2\theta.$$

19. B stands in front of a plane vertical mirror, find the position that A must take in order to see B's profile directly and B's full face by reflection at the mirror; determine in what cases no such position exists. If B without changing his station turn on a vertical axis, prove that the locus of A will be a rectangular hyperbola whose vertices are B and the image of B in the mirror.

If B' be the image of B (fig. 35), then if C be the point of the mirror towards which B is looking, the angle ABC

must be a right angle. The angle CBB' must therefore be less than half a right angle.

Draw AN perpendicular to BB', then the triangles ABN and $AB'N$ are similar; therefore $AN^2 = BN.B'N$, and the locus of A is a rectangular hyperbola with its vertices at B and B'.

xx. A hollow acute-angled triangular prism, whose ends are perpendicular to its axis, is capable of reflecting light at its inner surfaces; if it is placed with one face on a horizontal table and a small pencil of light is admitted through a hole in one face immediately opposite an edge so as to be incident upon the bottom face directly under the top edge, prove that the axis of the pencil will emerge at the hole after five reflections at the faces and one at each end of the prism if the direction of first incidence makes with the axis of the prism an angle

$$\tan^{-1} \frac{a \cos A + b \cos B + c \cos C}{l},$$

where ABC is a transverse section and l the length of the prism.

The course of the light is the same as that of a perfectly elastic ball.

The projections of the path on a transverse section ABC perpendicular to the axis of the prism must be the triangle DEF, where D, E, F are the feet of the perpendiculars from A, B, C on BC, CA, AB.

The projection of the ball will describe the perimeter of DEF twice while the ball moves once up and down the prism.

Therefore the ratio of the velocities perpendicular and parallel to the axis of the prism

$$= \frac{2 \text{ perimeter of the triangle } DEF}{2 \text{ length of the prism}}$$

$$= \frac{a \cos A + b \cos B + c \cos C}{l}.$$

WEDNESDAY, *Jan.* 6, 1875. 1 to 4.

MR. GREENHILL. Arabic numbers.
MR. FREEMAN. Roman numbers.

1. STATE and prove Newton's first lemma.

Prove that the quadrilateral of maximum area that can be formed with four straight lines AB, BC, CD, DA of given lengths is such that a circle can be described about it. Hence, prove that the curve of given length which on a given chord encloses a maximum area is an arc of a circle.

Let AD, BC (fig. 36) intersect in O. Keeping AB fixed displace C to C' and D to D'; O is the instantaneous centre of CD.

The area being a maximum $ABCD = ABC'D'$ to the first order and $ODC = OD'C'$ to the same order.

Therefore $$\triangle ABO = ABC'OD',$$

$$lt\frac{\triangle AOD'}{\triangle BOC'} = 1, \text{ and since } \angle AOD' = \angle BOC',$$

$$lt\frac{OA.OD'}{OB.OC'} = 1,\ \ OA.OD = OB.OC,$$

and therefore a circle can be described about $ABCD$.

If BC, CD, DA be replaced by any number of straight lines BC, CD, ... KA, a successive application of the method will prove that the area enclosed will be a maximum when the angular points of the polygon lie on a circle; and proceeding to the limit when the number of the sides is indefinitely increased, the curve of given length, which encloses the maximum area, will be the arc of a circle.

2. State and prove the eleventh lemma.

If OP, OQ be the tangents to a small arc PQ of continued curvature, prove by Newton's method that the ratio $OP+OQ-$ arc PQ : arc $PQ-$ chord PQ tends to the limit 2 : 1 as the arc PQ is indefinitely diminished.

Let the involute PB (fig. 37) of the arc PQ be drawn meeting OQ in B, and PN be drawn perpendicular to OQ to meet it in N, and let the tangent PT to the involute meet OQ in T.

Then $lt\,\dfrac{TP^2}{TB.TQ}=2$, and therefore $lt\,\dfrac{TP^3}{TB^2}=4lt\,\dfrac{TQ^2}{TP}=4$ diameter of curvature at P of arc PQ.

Since IB varies as $TP^{\frac{3}{2}}$, therefore $lt\,\dfrac{TB}{TN}=\frac{2}{3}$, or $lt\,\dfrac{TB}{BN}=2$.

If the circular arc PA be described with centre O, and the arc PC with centre Q, then $lt\,\dfrac{TN}{TA}=2$; $lt\,\dfrac{AN}{CN}=2$; and therefore $lt\,\dfrac{TA}{CN}=2$; therefore $lt\,\dfrac{AB}{BC}=2$; or

$$lt\; OP+OQ-\text{arc } PQ : \text{arc } PQ-\text{chord } PQ=2:1.$$

3. A body moves in a plane curve and the radius drawn to it from a point in the plane which is either fixed or moves uniformly in a straight line describes areas about the point proportional to the time. Prove that the body is acted on by a force tending to the point.

If a body moves in a conic section so that the resolved part of the velocity perpendicular to the focal distance is constant, the force tends to the centre of the conic section.

Let v be the velocity at the point P (figs. 38, 39).

The resolved part of the velocity perpendicular to the focal distance

$$=v.\frac{SZ}{SP}=v\frac{S'Z'}{S'P}=v\frac{S'Z'\pm SZ}{S'P\pm SP}=v.\frac{CY}{CA}.$$

Therefore $v.CY$ is constant, and the force tends to C.

4. Find an expression for the law of force under which a body will describe a given orbit about a centre of force.

If S be the centre of force and SY the perpendicular on the tangent at a point P of the orbit, prove that the acceleration at P is equal to the product of the velocities of P and Y divided by SY.

Let SV represent the velocity at P (fig. 40), $SV.SY$ is constant.

Then if T be the time the body takes to move from P to P', the acceleration at P

$$= lt\,\frac{VV'}{T} = lt\,\frac{VV'}{YY'}\cdot\frac{YY'}{T} = \frac{SV}{SY}\cdot lt\,\frac{YY'}{T}$$

= product of the velocities of P and Y divided by SY.

5. A body describes an orbit under the action of a force tending to and varying as the distance from a fixed point, prove that the orbit is an ellipse, and shew from elementary considerations, that the periodic time is the same for all circumstances of projection.

A number of bodies which describe ellipses about the centre of force as centre in the same periodic time, are projected from a given point with a given velocity in different directions in a plane. Prove that their paths will all touch a fixed ellipse with the given point as focus.

If C (fig. 41) be the centre of force, P the point of projection, since the velocity at P and the periodic time are constant, therefore CD the diameter conjugate to CP is constant.

If the tangent at Q be at right angles to the tangent at P and intersect it in Y, then $CY^2 = CP^2 + CD^2$ is constant.

QV the normal at Q is an ordinate of the diameter PP', and therefore bisects the angle PQP'.

Therefore $PQ + QP' = 2CY$, and therefore the locus of Q is an ellipse with foci P, P', and all the orbits will touch this ellipse.

Compare the Rider to ix, Monday morning, Jan. 4.

6. A body describes an ellipse about a centre of force in the focus, find the law of force.

If S be the centre of force, A the nearer apse, P the body, and a small impulse which generates the velocity T act on the body at right angles to SP, prove that the change in the direction of the apse line is given approximately by

$$\frac{T}{h}\left(\frac{2}{e}+\cos ASP\right) SP \sin ASP,$$

where e is the excentricity of the orbit and h twice the rate of description of area about S.

Let the normal at P (fig. 42) meet the apse line in G, and let GL be drawn perpendicular to SP.

The radial velocity being unaltered by the impulse, if PHg be the new normal and GH be drawn parallel to PL to meet PHg in H, then

$$\frac{GH}{PL} \doteq \frac{T}{\text{transverse velocity}} = \frac{T.SP}{h}.$$

If Sg be the new apse line, and gl be drawn perpendicular to SP, then PL being the old and Pl the new semi-latus-rectum

$$\frac{Pl}{PL} = \left(\frac{h+T.SP}{h}\right)^2, \quad \frac{Ll}{PL} = \frac{2T.SP}{h},$$

neglecting T^2; therefore $Ll = 2GH$.

Let $\angle ASP = \theta$, $\angle SPG = \psi$,

$$\frac{SG}{SP} = \frac{\sin\psi}{\sin(\theta-\psi)} = e, \quad \tan\psi = \frac{e\sin\theta}{1+e\cos\theta}, \quad gH = \frac{Ll}{2\cos\psi} = \frac{GH}{\cos\psi}.$$

The change of direction of the apse line

$$= \angle GSg = \angle GSH + \angle HSg$$

$$= \frac{GH}{SG}\left\{\sin\theta + \frac{\sin(\theta-\psi)}{\cos\psi}\right\} \text{ ult.}$$

$$= \frac{T.PL}{eh} \cdot \frac{2\sin\theta + e\sin\theta\cos\theta}{1+e\cos\theta}$$

$$= \frac{T}{h}\left(\frac{2}{e}+\cos\theta\right) SP \sin\theta.$$

vii. Shew how to determine the meridian and latitude of a place by observations made with an altazimuth on a circumpolar star.

In north latitude 45° the greatest azimuth attained by one of the circumpolar stars is 45° from the north point of the horizon. Prove that the star's polar distance is 30°.

Let σ be the star (fig. 43) when its azimuth $NZ\mu$ is greatest and equal to 45°.

Let $P\sigma$ produced meet the prime vertical, horizon, and meridian in η, ζ, p respectively. Then ζ is the pole of $\sigma\mu$, hence $\zeta\sigma = 90° = \zeta\mu$; but $\mu N = 45°$, hence $S\zeta = 45°$, and $Sp = 45°$, therefore ζSp is isosceles and angle $\zeta Sp = 90°$, therefore if $S\nu$ be drawn perpendicular to ζp it will bisect it and the angle at S.

Hence the three right-angled triangles $PZ\sigma$, $\zeta S\nu$, $pS\nu$ have the same hypothenuse, and one angle the same (45°), and are therefore in other respects equal.

Therefore $$P\sigma = \zeta\nu = \nu p.$$

But $$Pp = 180° \text{ and } \zeta\sigma = 90°.$$

Therefore $$P\sigma = 30°.$$

It may be proved also, that in this position the star's altitude $\sigma\mu$ is equal to its hour angle σPZ.

viii. Describe the arrangement of the axes of motion and graduated circles of an equatorial telescope, and state the errors of adjustment to which it is liable.

If the telescope be fitted with a divided object-glass, shew how to measure by it the distance between the cusps of the partially eclipsed sun and the rate at which that distance increases.

Determine previously the reading of the micrometer head which moves one-half of the object glass, for the position when the two images coincide, and the value of one revolution of the screw.

Open out the two halves of the object glass, and turn the cell which contains them about the collimation axis of the telescope, until the four cusps of the two images are in one line; superpose the upper cusp of one image on the lower cusp of the other, and note the reading of the micrometer head and also the time. The difference of the reading from the reading for coincidence of images will give the distance between the cusps.

Next open out the two halves still further through any arbitrary number of revolutions of the screw, and, if the eclipse is not central, continue to turn the object glass cell so as to keep the four cusps in one line, and note the instant when the two cusps formerly superposed again coincide. Divide the arbitrary number of revolutions by the number of seconds between the observations, and multiply by the value of one revolution to obtain the rate of increase of the cuspidal distance.

ix. Explain the annual course of changes in the length of the day at places in mean latitude, on the arctic circle, and on the equator.

Prove that at a place on the arctic circle the daily displacement of the point of sunset is equal to the sun's change in longitude during the same interval.

In fig. 44, $NESW$ are the cardinal points of the horizon of a place on the arctic circle.

$E\,♎\,W\,♈$ is the position of the equator, and $♈\odot♎$ the position of the ecliptic when the sun $\odot$ is setting.

Now, the co-latitude of a plane on the arctic circle is equal to the obliquity of the ecliptic.

Hence the angle $\odot ♈ W$ is equal to the angle $♈ W \odot$.

Therefore $W\odot = ♈\odot$, or the sun's distance from the west point of the horizon at sunset is equal to the sun's longitude, and therefore the daily displacement of the point of sunset is equal to the sun's change in longitude since the sunset.

x. Explain how mean time and apparent time are reckoned. Define equation of time, and prove that it vanishes four times a-year.

A clock at Cambridge keeps Greenwich mean time; what time did it indicate when the sun's preceding limb arrived at our meridian to-day? Longitude $22^{s}\cdot75$ E.; sun's semi-diameter passed meridian in $1^{m}\ 10^{s}\cdot62$; equation of time $6^{m}\ 2^{s}\cdot88$.

On the 6th of January, the equation of time must be added to apparent time to obtain mean time.

When the sun's preceding limb crossed the meridian, the apparent time at Cambridge was

$$12^{h} - 1^{m}\ 10^{s}\cdot62 = 11^{h}\ 58^{m}\ 49^{s}3.8.$$

Add to this $6^{m}\ 2^{s}\cdot88$ to obtain Cambridge mean time, and subtract $22^{s}\cdot75$ to obtain Greenwich mean time.

Result $12^{h}\ 4^{m}\ 29^{s}\cdot51$.

xi. Account for the error of aberration in the observed position (1) of a star, (2) of a planet; and prove that all stars are displaced by aberration towards the same point on the ecliptic. When has a planet no aberration?

The velocity of Venus is to that of the Earth as 47 : 40. Determine the aberration of Venus at inferior and superior conjunction, the constant of aberration for a star being $20''\cdot44$.

A planet has no aberration when it is stationary.

Assuming Venus to move in the plane of the ecliptic, the velocity of Venus relative to the Earth at superior conjunction is $\frac{87}{40}$, and at inferior conjunction $\frac{7}{40}$ of the Earth's velocity. Hence the aberration of Venus at those times is $+44''\cdot46$ and $-3''\cdot58$ respectively.

xii. Explain the phenomena presented by a satellite of Jupiter to an observer on the Earth. When will the eclipses be best seen? State the influence of the phase of Jupiter on the times of beginning and ending of the transits and shadow-passages of a satellite. Shew that the effect is small.

On May 19th, 1874, at 1^{h} P.M. Jupiter was stationary. Near that time the following succession of phenomena occurred with the first satellite.

Transit Ingress	$18^{d}\ 6^{h}\ 34^{m}$	$20^{d}\ 1^{h}\ \ 1^{m}$
Shadow Ingress	$18^{d}\ 7^{h}\ 43^{m}$	$20^{d}\ 2^{h}\ 12^{m}$
Transit Egress	$18^{d}\ 8^{h}\ 51^{m}$	$20^{d}\ 3^{h}\ 18^{m}$
Shadow Egress	$18^{d}\ 9^{h}\ 59^{m}$	$20^{d}\ 4^{h}\ 28^{m}$.

Find the periods of the sidereal and synodic revolutions of this satellite, and the Jovicentric elongation of the Earth from the Sun when Jupiter was stationary.

When Jupiter is in quadrature with the Sun, the points of immersion and emersion of the satellite in the shadow of the planet are least likely to be concealed by the disc of the planet.

Transits of satellites are referred to the line joining the centres of the Earth and Jupiter.

When part of Jupiter's disc visible to the Earth is not illuminated by the Sun, the exact time of the beginning or ending of a transit cannot be noted, but only the contacts with the boundaries of the visible illuminated portion. The ingress may be retarded or the egress accelerated.

Passages of the shadow are referred to the line joining the centres of the Sun and Jupiter.

When part of Jupiter's illuminated hemisphere is not visible to the Earth, the whole shadow passage cannot be observed, the ingress may be retarded or the egress accelerated.

On account of the great distance of Jupiter from the Earth and Sun, the phase is small and its effect inconsiderable.

Middle of first transit	$18^d\ 7^h\ 42^m{\cdot}5$
" second "	$20^d\ 2^h\ 9^m{\cdot}5$
Difference .	$1^d\ 18_h\ 27^m$

This is the *sidereal* revolution of the satellite, for Jupiter is stationary.

Middle of first shadow passage	$18^d\ 8^h\ 51^m$
" second "	$20^d\ 3^h\ 20^m$
Difference .	$1^d\ 18^h\ 29^m$

This is the *synodical* revolution of the satellite with respect to the Sun.

Interval between middle of first transit and first shadow passage	$1^h\ 8^m{\cdot}5$
Interval between middle of second transit and second shadow passage...	$1^h\ 10^m{\cdot}5$
Mean interval .	$1^h\ 9^m{\cdot}5$

Hence the angle of elongation of the Earth and Sun, seen from Jupiter when he was stationary, is equal to

$$\frac{1^{h}\ 9^{m}\cdot 5}{1^{d}\ 18^{h}\ 29^{m}}\ 360^{\circ} = 9^{\circ}49' \text{ nearly.}$$

THURSDAY, *Jan.* 7, 1875. 9 to 12.

PROF. TAIT.	Roman numbers.
MR. FREEMAN.	Arabic numbers.

1. IF P_r be the numerator of the r^{th} convergent to the continued fraction whose quotients are q_1, $q_2 \ldots q_n$, and if P_r' be the numerator of the r^{th} convergent to the fraction whose quotients are q_n, $q_{n-1} \ldots q_1$; prove that

$$P_n = P_{n-r}P_r' + P_{n-r-1}P'_{r-1} = P_n'.$$

Assuming that

$$\sqrt{N} = A + \frac{1}{q_1 +}\,\frac{1}{q_2 +}\cdots\cdots\frac{1}{q_2 +}\,\frac{1}{q_1 +}\,\frac{1}{2A +}\cdots\cdots,$$

and that n is the number of the recurring quotients q_1, $q_2 \ldots 2A$, if $\frac{P_n}{Q_n}$, $\frac{P_{2n}}{Q_{2n}}$ be the n^{th} and $2n^{\text{th}}$ convergents to $\sqrt{N}$, prove that $Q_{2n} = 2P_nQ_n$, and $P_{2n} = 2P_n^2 + (-1)^{n+1}$.

For the rider, it is convenient to assume the following notation, which is employed in a tract on *The Expression of a Quadratic Surd as a Continued Fraction*, by Thomas Muir, M.A., Glasgow, 1874.

Let $K\,(abc \ldots l)$ denote the numerator of the last convergent to the continued fraction $a + \frac{1}{b+}\,\frac{1}{c+}\cdots\frac{1}{l}$ regarded as the result of an operation on the quotients $a, b, c \ldots l$.

Then if $\frac{P_r}{Q_r}$ be the r^{th} convergent to the continued fraction which expresses $\sqrt{(N)}$,

$$\frac{P_n}{Q_n} = \frac{K(Aq_1, q_2..q_1)}{K(q_1..q_1)}$$

$$\frac{P_{2n}}{Q_{2n}} = \frac{K(Aq_1, q_2..2Aq_1..q_1)}{K(q_1..2A..q_1)}$$

$$= \frac{K(A, q_1q_2..q_12A).K(q_1..q_1) + K(Aq_1q_2..q_1).K(q_1..q_2)}{K(q_1q_2..q_12A).K(q_1..q_1) + K(q_1..q_1).K(q_1..q_2)} \text{ by } (\alpha).$$

$$= \frac{\{2A.K(Aq_1..q_1) + K(Aq_1..q_2)\}\, K(q_1..q_1) + K(Aq_1..q_1).K(q_1..q_2)}{K(q_1..q_1)\,\{2A.K(q_1..q_1) + 2K(q_1..q_2)\}}$$

$$= \frac{\{A.K(Aq_1..q_1)+K(Aq_1..q_2)\}K(q_1..q_1)+K(Aq_1..q_1)\{A.K(q_1..q_1)+K(q_1..q_2)\}}{2K(q_1..q_1).K(Aq_1..q_1)}$$

$$= \frac{K(Aq_1..q_1A).K(q_1..q_1) + K(Aq_1..q_1)^2}{2K(q_1..q_1).K(Aq_1..q_1)}.$$

Therefore,

$$\frac{P_n.Q_{2n} - P_{2n}.Q_n}{Q_n} = K(Aq_1\cdots q_1)^2 - K(Aq_1\cdots q_1A).K(q_1\cdots q_1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

Now, if we consider the continued fraction whose quotients are A, q_1, $q_2 \cdots q_2$, q_1, A, in number $n+1$, and take the difference of the two last convergents,

$$\frac{K(Aq_1\cdots q_1)}{K(q_1\cdots q_1)} - \frac{K(Aq_1\cdots q_1A)}{K(q_1\cdots q_1A)} = \frac{(-1)^n}{K(q_1\cdots q_1).K(q_1\cdots q_1A)},$$

we see that (γ) becomes

$$P_n.Q_{2n} - P_{2n}Q_n = (-1)^n Q_n.$$

2. Prove that one of the values of

$$\log\{1 + \cos 2\theta + \sqrt{(-1)}\sin 2\theta\} \text{ is } \log(2\cos\theta) + \theta\sqrt{(-1)},$$

when θ is between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

Deduce Gregory's series for the expansion of θ in terms of $\tan\theta$.

Prove also that one of the values of

$$\sin^{-1}\{\cos\theta+\sqrt{(-1)}\sin\theta\}$$

is $\qquad \cos^{-1}\sqrt{(\sin\theta)}+\sqrt{(-1)}\log\{\sqrt{(\sin\theta)}+\sqrt{(1+\sin\theta)}\}$

when θ is between 0 and $\frac{\pi}{2}$.

If $\qquad i^2=-1,$

$$\log(1+\cos 2\theta+i\sin 2\theta)=\tfrac{1}{2}\log(4\cos^2\theta)+(n\pi+\theta)\,i,$$

and
$$\log(1+\cos 2\theta+i\sin 2\theta)$$
$$=\log(2\cos^2\theta)+\log(1+i\tan\theta),$$

expanding $\log(1+i\tan\theta)$, and equating the coefficients of i, we have Gregory's Series

$$n\pi+\theta=\tan\theta-\tfrac{1}{3}\tan^3\theta+\ldots,$$

where $n\pi+\theta$ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

If we assume

$$\sin^{-1}(\cos\theta+i\sin\theta)=\alpha+\beta i,$$

$$\cos\theta+i\sin\theta=\sin\alpha\,\frac{e^{\beta}+e^{-\beta}}{2}+i\cos\alpha\,\frac{e^{\beta}-e^{-\beta}}{2},$$

$$\cos\theta=\sin\alpha\,\frac{e^{\beta}+e^{-\beta}}{2},\quad \sin\theta=\cos\alpha\,\frac{e^{\beta}-e^{-\beta}}{2},$$

$$e^{\beta}=\frac{\cos(\theta-\alpha)}{\cos\alpha\,\sin\alpha},\quad e^{-\beta}=\frac{\cos(\theta+\alpha)}{\cos\alpha\,\sin\alpha},$$

$$\cos^2\alpha\,\sin^2\alpha=\cos^2\theta-\sin^2\alpha,$$

$$\sin^2\alpha=1\pm\sin\theta.$$

If θ lies between 0 and $\frac{\pi}{2}$,

$$\sin\alpha=\sqrt{(1-\sin\theta)},\quad \cos\alpha=\sqrt{(\sin\theta)},\quad e^{\beta}=\sqrt{(\sin\theta)}+\sqrt{(1+\sin\theta)}.$$

3. Prove that the equation to the circle which cuts at right angles three circles whose equations are given in the form $(x-a)^2+(y-b^2)=c^2$ is

$$\begin{vmatrix} x^2+y^2, & x, & y, & 1 \\ a_1^2+b_1^2-c_1^2, & a_1, & b_1, & 1 \\ a_2^2+b_2^2-c_2^2, & a_2, & b_2, & 1 \\ a_3^2+b_3^2-c_3^2, & a_3, & b_3, & 1 \end{vmatrix}=0.$$

Prove that the diameter of the circle which cuts at right angles the three escribed circles of the triangle ABC is

$$\frac{a}{\sin A}(1+\cos B\cos C+\cos C\cos A+\cos A\cos B)^{\frac{1}{2}}.$$

The condition that the circle

$$x^2+y^2-2a_1x-2b_1y+a_1^2+b_1^2-c_1^2=0,$$

should cut at right angles the circle

$$x^2+y^2-2ax-2by+a^2+b^2-c^2=0$$

may be written

$$a_1^2+b_1^2-c_1^2-2aa_1-2bb_1+a^2+b^2-c^2=0,$$

eliminating a, b, and $a^2+b^2-c^2$, we have the required result.

In the rider take as axes of x and y the exterior and interior bisectors of the angle A; let α, β, γ be the centres of the escribed circles, and r_1, r_2, r_3 their radii, and D the diameter of the circumscribing circle

$$r_1=2D\sin\frac{A}{2}\cos\frac{B}{2}\cdot\cos\frac{C}{2},\ldots$$

$$A\beta=r_2\sec\frac{A}{2}=2D\sin\frac{B}{2}\cos\frac{C}{2};\quad A\gamma=2D\cos\frac{B}{2}\sin\frac{C}{2}.$$

$$A\alpha=A\beta\cot\frac{B}{2}=2D\cos\frac{B}{2}\cos\frac{C}{2}.$$

Hence

$$\left.\begin{aligned} &a_1 = 0,\ b_1 = 2D\cos\frac{B}{2}\cos\frac{C}{2} \\ &a_2 = -2D\sin\frac{B}{2}\cos\frac{C}{2},\ b_2 = 0 \\ &a_3 = 2D\cos\frac{B}{2}\sin\frac{C}{2},\ b_3 = 0 \end{aligned}\right\} \text{are the coordinates of} \left\{\begin{aligned} &\alpha \\ &\beta, \\ &\gamma \end{aligned}\right.$$

and

$$a_1^2 + b_1^2 - r_1^2 = 4D^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2},$$

$$a_2^2 + b_2^2 - r_2^2 = 4D^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\cos\frac{C}{2},$$

$$a_3^2 + b_3^2 - r_3^2 = 4D^2\sin^2\frac{A}{2}\cos^2\frac{B}{2}\sin^2\frac{C}{2},$$

if then the equation to the orthotomic circle be

$$K(x^2 + y^2) = Px + Qy + R,$$

the square of the diameter $= \left(\frac{P}{K}\right)^2 + \left(\frac{Q}{K}\right)^2 + \frac{4R}{K}$, and

$$K = 4D^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

$$\frac{P}{K} = 2D\sin^2\frac{A}{2}\sin\frac{B-C}{2},$$

$$\frac{Q}{K} = 2D\left(\sin^2\frac{A}{2}\cos\frac{B-C}{2} - \cos\frac{B}{2}\cos\frac{C}{2}\right),$$

$$\frac{R}{K} = D^2\sin B\sin C\sin^2\frac{A}{2},$$

whence the result.

4. Prove the relation $\phi(x) - \phi(0) = x\phi'(\theta x)$, where θ is a proper fraction, stating the conditions subject to which it is true.

Hence deduce Maclaurin's Theorem for the expansion of $f(x)$ in ascending powers of x.

Prove that, when x is between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$,

$$\frac{1}{1^3}\cos x - \frac{1}{3^3}\cos 3x + \frac{1}{5^3}\cos 5x - \ldots \text{ to infinity} = \frac{\pi}{8}\left(\frac{\pi^2}{4} - x^2\right).$$

Assuming the form of the expansion of Maclaurin's series

$$f(x) = f(0) + xf'(0) + \ldots + \frac{x^n}{\lfloor n} f^n(x) + \frac{x^{n+1}}{\lfloor n+1}.R,$$

where $R = f^{n+1}(x) + \frac{x}{n+2} f^{n+2}(x) + \ldots$ is a function of x.

Let $\phi(z) = f(x) - f(z) - (x-z)f'(z) - \ldots - \frac{(x-z)^{n+1}}{\lfloor n+1}.R,$

then $$\phi'(z) = \frac{(x-z)^n}{\lfloor n}\{R - f^{x+1}(z)\},$$

and $\phi(x) = 0$, $\phi(0) = 0$; therefore $\phi'(\theta x) = 0$, and therefore $R = f^{n+1}(\theta x)$.

If $$f(x) = \frac{\cos x}{1^3} - \frac{\cos 3x}{3^3} + \ldots,$$

$$f''(x) = -\frac{\cos x}{1} + \frac{\cos 3x}{3} - \ldots = -\frac{\pi}{4},$$

for all values of x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and therefore $f''(\theta x) = -\frac{\pi}{4}$; also $f'(0) = 0$; therefore $f(x) = f(0) - \frac{\pi x^2}{8}$, and $f\left(\frac{\pi}{2}\right) = 0$, therefore $f(0) = \frac{\pi^3}{32}$, and $f(x) = \frac{\pi}{8}\left(\frac{\pi^2}{4} - x^2\right)$.

v. Solve the equation

$$\frac{d^2x}{dt} + 2k\frac{dx}{dt} + n^2x = 0,$$

k being essentially positive. Point out the effect of the relative magnitudes of k and n upon the form of the expression for x.

Give a graphical representation of the relation between x and t when $k < n$.

[Maxwell's *Electricity*, § 731. Thomson and Tait's *Elements of Natural Philosophy*, § 295].

vi. Form the equations for the equilibrium of a flexible elastic string, of uniform material, under the influence of any system of forces; supposing Hooke's Law to hold for all amounts of extension.

Let one end be fixed to the rim of a wheel, sufficiently rough to prevent slipping, and let the other be attached to a mass resting on the ground, so that when the string (of length a) is just taut it shall be vertical. Show that the work which must be spent in turning the wheel so as just to lift the mass off the ground is

$$Mga + Ea \log \frac{E}{E+Mg},$$

where E is the tension which would double the length of the string, neglecting the weight of the string.

At any stage of the operation let x be the unstretched length of the part already wound on the wheel.

Then the tension, by Hooke's law, is

$$E\frac{a-(a-x)}{a-x}.$$

Under this tension the stretched length of dx is $\dfrac{adx}{a-x}$, and the work done in winding it on is the product of these quantities, or

$$dW = Ea\frac{a-(a-x)}{(a-x)^2}\,.\,dx.$$

This is to be integrated from $T=0$ to $T=Mg$, or from $x=0$ to $x=\dfrac{Mga}{E+Mg}$, whence the result.

vii. Explain how Carnot's idea of reversible cycles of operation has rendered an absolute definition of temperature possible.

Define the intrinsic energy of a substance; and calling it E, explain fully the physical signification of the quantity ϕ defined by the equation

$$dE = -pdv + Jtd\phi.$$

By the help of this equation find the value of ϕ in terms of v and t for the ideal perfect gas.

[Tait's *Thermodynamics*, § 26, 204, 186, 198].

8. Two parallel plane conducting surfaces infinite in extent are at a given distance, and contain a known quantity of electricity on each unit area of their opposed surfaces, one of the surfaces being maintained always at zero potential. Determine the electric energy due to the distribution of electricity on opposite unit areas.

If each surface of a second pair of infinite plane conducting surfaces separated by the same distance and unelectrified is connected by a conducting wire with one of the surfaces of the first pair, determine the change in the electric energy of opposite unit areas on the first pair, and account for the apparent disappearance of energy.

Let e be the electricity on unit area of one of the plane surfaces, and let its potential be A, the potential of the other plane surface being zero and the distance between them c. It may be proved by the method given in Maxwell's *Electricity*, § 124, that

$$4\pi e = \frac{A}{c},$$

and if Q be the electric energy due to the distribution of electricity on opposite unit areas,

$$Q = \tfrac{1}{2}eA = 2\pi e^2 c.$$

When the second pair of planes is connected with the first pair the surface density becomes $\frac{1}{2}e$, and the electric energy due to the electricity on opposite unit areas is given by

$$Q' = 2\pi \frac{e^2 c}{4}.$$

Corresponding to each unit area of the first pair of planes there is an apparent loss of energy given by

$$Q - 2Q' = \pi e^2 c.$$

Half the energy has disappeared, and has been transformed into work done by electric forces, generally in the form of light, sound, and heat of a spark, and of heat in the discharging conductor, whilst the electricity was being distributed over a larger surface.

ix. Define the Action of a particle moving under given forces.

Jets of water escape horizontally from orifices along a generating line of a vertical cylinder kept always full. Show that (to axes inclined 45° to the vertical) the equation of the lines of equal Action for unit mass of water is of the form

$$x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}.$$

Show also that the line of equal time for particles of water issuing simultaneously from the orifices is the free path of the water which leaves the vessel by an orifice at a depth below the surface due to that time.

Measuring x vertically downwards and y horizontally in the direction of issue, if ξ be the depth of an orifice, for the water thence escaping (fig. 45),

$$\dot{y} = \sqrt{(2g\xi)},\ y = \sqrt{(2g\xi)}\,t,\ \dot{x} = \sqrt{\{2g(x-\xi)\}},\ x - \xi = \tfrac{1}{2}gt^2.$$

Action of unit mass $= \int v^2 dt$,

$$= \int \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}}\,dx = \sqrt{(2g)} \int \frac{x\,dx}{\sqrt{(x-\xi)}}$$

$$= \sqrt{(2g)}\,\{\tfrac{2}{3}(x-\xi)^{\frac{3}{2}} + 2\xi(x-\xi)^{\frac{1}{2}}\}.$$

By the values of x and y, eliminating t, $x-\xi=\dfrac{y^2}{4\xi}$; therefore $2\xi=x+\sqrt{(x^2-y^2)}$, where either sign may be given to the radical.

Therefore $$\xi=\left\{\frac{(x+y)^{\frac{1}{2}}-(x-y)^{\frac{1}{2}}}{2}\right\}^2,$$

$$x-\xi=\left\{\frac{(x+y)^{\frac{1}{2}}+(x-y)^{\frac{1}{2}}}{2}\right\}^2,$$

and the Action $=\dfrac{\sqrt{(2g)}}{6}\{(x+y)^{\frac{3}{2}}+(x-y)^{\frac{3}{2}}\}$.

If we change to axes inclined 45° to the vertical we must change $x+y$ into $x\sqrt{(2)}$, and $x-y$ into $y\sqrt{(2)}$, and the Action $=\dfrac{2^{\frac{1}{4}}\sqrt{(g)}}{3}(x^{\frac{3}{2}}+y^{\frac{3}{2}})$.

Were there no gravity the line of equal time would be $y=\sqrt{(2gx)}\,t$, a parabola with vertex at the origin and parameter $\frac{1}{2}gt^2$.

But gravity takes all particles down through $\frac{1}{2}gt^2$ in the time t, and the velocity depends on the height only, not on the particular orifice from which the particle started.

Therefore all are on the path of the particle which left the cylinder at a depth $\frac{1}{2}gt^2$.

x. What is meant by the equilibrium theory of the tides?

Give from it, without formulæ, a general explanation of the lunar semidiurnal, diurnal, and fortnightly tides.

Point out, also without details, the mode of taking account, in this theory, of the distribution of land and water.

In the equilibrium theory the sea-level is at every instant a level surface for the attraction of the Earth and Moon.

To treat the problem of the tides statically, the Earth is fixed and the Moon is divided into two halves, moon and anti-moon, which are supposed to revolve round the Earth's axis once in the lunar twenty-four hours with the line joining them inclined to the equator at an angle equal to the Moon's declination.

There will be a rise and fall twice in every lunar day, which will be the lunar semi-diurnal tide.

When the Moon is not on the equator and also in consequence of the unsymmetrical distribution of land and water the two tides in the lunar day will not be equal, so that to produce the resultant effect a tide of period a lunar day must be superimposed on the lunar semi-diurnal tide, and this is the lunar diurnal tide.

In consequence of the variation of the Moon's declination there will be a variation with a period of a fortnight of the average height of water at any place, and this effect can be produced by the superposition of a lunar fortnightly tide.

To take account of the distribution of land and water, the level surface of the sea must be such that the part of it which bounds the sea encloses the same volume of water.

[Thomson and Tait's *Natural Philosophy*, § 807, 808, 809.]

THURSDAY, *Jan.* 7, 1875. $1\frac{1}{2}$ to 4.

PROF. TAIT. Roman numbers.
MR. GREENHILL. Arabic numbers.

1. FIND the polar equation of a conic section referred to the focus as pole, and the polar equation of a chord and a tangent.

A hyperbola is described similar to the given hyperbola $\frac{l}{r} = 1 + e\cos\theta$ having the same focus and touching it at the point $\theta = \alpha$, prove that the length of its latus rectum will be

$$2l\frac{e^2-1}{e^2+2e\cos\alpha+1},$$

and the equation of the common chord of the hyperbolas will be

$$\frac{l}{r} = e\cos\theta - e\frac{e+\cos\alpha}{1+e\cos\alpha}\cos(\theta-\alpha).$$

Let the equation of the hyperbola be

$$\frac{l'}{r} = 1 + e\cos(\theta-\beta),$$

then the equation of a pair of common chords of the hyperperbolas is

$$\frac{l \pm l'}{r} = e\cos\theta \pm e\cos(\theta-\beta).$$

If
$$\frac{l+l'}{r} = e\cos\theta + e\cos(\theta-\beta)$$

be a tangent where $\theta = \alpha$,

$$\frac{l+l'}{l} = \frac{e+e\cos\beta}{e+\cos\alpha} = \frac{e\sin\beta}{\sin\alpha};$$

therefore $\tan\frac{\beta}{2} = \frac{\sin\alpha}{e+\cos\alpha}$ and $\frac{l'}{l} = \frac{e^2-1}{e^2+2e\cos\alpha+1}$,

and the equation

$$\frac{l-l'}{r} = e\cos\theta - \cos(\theta-\beta)$$

reduces to

$$\frac{l}{r} = e\cos\theta - e\,\frac{e+\cos\alpha}{1+e\cos\alpha}\cos(\theta-\alpha).$$

2. If a, b, c be the sides, A, B, C the angles, and E the spherical excess of a spherical triangle, prove that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{1}{2}\cdot\frac{\sin\frac{E}{2}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}.$$

If $C = A + B$, prove that the chord triangle is right-angled, the angular radius of the circumscribing small circle is $\frac{c}{2}$ and $\sin\frac{E}{2} = \tan\frac{a}{2}\tan\frac{b}{2}$.

If the arc CD be drawn making the angle $ACD = A$, then the angle $BCD = B$ and $AD = DC = DB$.

Hence D, the middle point of the arc B, is the centre of the circumscribing small circle, and if the radius to D meet the chord AB in d, d is the centre of the plane circle circumscribing the chord triangle ABC, and therefore the chord triangle is right angled.

Therefore $$\sin^2\frac{c}{2} = \sin^2\frac{a}{2} + \sin^2\frac{b}{2},$$

$$\cos c = \cos a + \cos b - 1,$$

and $$\sin\frac{E}{2} = -\cos C = \frac{\cos a\cos b - \cos c}{\sin a\sin b}$$

$$= \frac{1-\cos a - \cos b + \cos a\cos b}{\sin a\sin b} = \tan\frac{a}{2}\tan\frac{b}{2}.$$

(This spherical triangle in which one angle is equal to the sum of the other two is more analogous to the right-angled plane triangle than the spherical triangle which has one angle a right angle.)

3. Explain Horner's method of approximating to the real roots of an equation.

Find to two places of decimals the roots of the equation

$$x^3 - 6x^2 + 9x - 3 = 0.$$

The roots are 0·47, 1·65, 3·88.

4. If x, y be the rectangular, r, θ the polar coordinates of a point which moves once round the perimeter of a closed curve in a certain direction, prove that the area of the curve is expressed by $\int xdy$ or $-\int ydx$ or $\frac{1}{2}\int r^2 d\theta$.

Interpret these expressions when the perimeter cuts itself.

Prove that the areas of the two loops of the curve

$$r^2 - 2ar\cos\theta - 8ar + 9a^2 = 0,$$

are $$\{32\pi + 24\sqrt{(3)}\}\, a^2 \text{ and } \{16\pi - 24\sqrt{(3)}\}\, a^2.$$

As the point travels round the perimeter in the direction so as to have the area of the curve on the left hand, the sum of all the elements xdy for the same value of y (fig. 46) will be the area of the curve cut off by two straight lines parallel to the axis of x at distances y and $y + dy$, and therefore the whole area is $\int xdy$.

Similarly the area is $-\int ydx$ (fig. 47), and $\frac{1}{2}\int r^2 d\theta$ (fig. 48).

(Since $xdy + ydx = d.xy$ is a complete differential, therefore $\int(xdy + ydx) = 0$ round a closed curve; but $xdy - ydx = r^2 d\theta$ is not a complete differential, therefore $\int(xdy - ydx) = \int r^2 d\theta$ depends on all the intermediate value of x and y, and is equal to twice the area of the curve.

An integrating factor of $xdy - ydx$ is $\dfrac{1}{x^2 + y^2}$, and then $\displaystyle\int \frac{xdy - ydx}{x^2 + y^2} = \int d\theta = 0$ or 2π, according as the origin is outside or inside the curve).

If the perimeter cuts itself (fig. 49) the value of any one of the integrals taken round a loop will be numerically equal to the area of the loop, and positive or negative, according as the area of the loop is to the left or the right of the point as it travels round the perimeter of the loop.

Therefore, if the point travel once round the perimeter the value of any one of the integrals will be the sum of the areas of the loops which were on the left hand diminished by the sum of the areas of the loops which were on the right hand.

The curve is a limaçon or nodal Cartesian oval (fig. 50)

$$\frac{r}{a} = 4 + \cos\theta \pm \sqrt{\{(1+\cos\theta)(7+\cos\theta)\}},$$

$$\frac{r^2}{a^2} = 24 + 16\cos\theta + \cos 2\theta \pm 2(4+\cos\theta)\sqrt{\{(1+\cos\theta)(7+\cos\theta)\}}.$$

The area of a loop is $\int_0^\pi r^2 d\theta$, the upper sign being taken with the radical for the outer loop, the lower sign for the inner loop.

Now $\int_0^\pi (24 + 16\cos\theta + \cos 2\theta)\, d\theta = 24\pi,$

and if $\sin\frac{\theta}{2} = 2\sin\frac{\phi}{2},$

$$\int_0^\pi (4+\cos\theta)\sqrt{\{(1+\cos\theta)(7+\cos\theta)\}}\, d\theta$$

$$= 8\int_0^{\frac{\pi}{3}} \left(5 - 8\sin^2\frac{\phi}{2}\right)\cos^2\frac{\phi}{2}\, d\phi$$

$$= 4\int_0^{\frac{\pi}{3}} (3 + 5\cos\phi + 2\cos 2\phi)\, d\phi$$

$$= 4\{\pi + 3\sqrt{(3)}\}, \text{ whence the result.}$$

5. Prove the differential equation $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$ for the motion of a particle under the action of a central force P.

If $P = \mu u^2 (1 + k^2 \sin^2\theta)^{-\frac{3}{2}}$ find the orbit and interpret the result geometrically.

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} (1 + k^2 \sin^2\theta)^{-\frac{3}{2}}.$$

Multiplying by $\cos\theta$ and integrating

$$\frac{du}{d\theta} \cos\theta + u \sin\theta = \frac{\mu}{h^2} \int \frac{\cos\theta d\theta}{(1 + k^2 \sin^2\theta)^{\frac{3}{2}}}$$

$$= \frac{\mu}{h^2} \cdot \frac{\sin\theta}{\sqrt{(1 + k^2 \sin^2\theta)}} + A.$$

Multiplying by $\sin\theta$ and integrating

$$\frac{du}{d\theta} \sin\theta - u \cos\theta = \frac{\mu}{h^2} \int \frac{\sin\theta d\theta}{(1 + k^2 - k^2 \cos^2\theta)^{\frac{3}{2}}}$$

$$= -\frac{\mu}{h^2 (1 + k^2)} \frac{\cos\theta}{\sqrt{(1 + k^2 \sin^2\theta)}} + B;$$

therefore $u = \dfrac{\mu}{h^2 (1 + k^2)} \sqrt{(1 + k^2 \sin^2\theta)} + A \sin\theta - B \cos\theta,$

the equation of a conic section, the projection of the orbit of an undisturbed planet inclined at an angle $\tan^{-1}k$ to the plane of reference.

vi. If ξ, η, ζ be continuous functions of the coordinates, and represent the components, parallel to three rectangular axes, of the velocity of a point at x, y, z; shew that the rate at which the dilatation, thus produced, takes place in the group of points near x, y, z is

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}.$$

Shew, by actual integration, that

$$\iiint \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) dx\,dy\,dz = \iint (l\xi + m\eta + n\zeta)\, ds;$$

where the integrations extend, respectively, through the volume and over the surface of a closed space S; l, m, n being the direction-cosines of the outward-drawn normal to the surface-element ds.

Shew that this equation expresses that no points come into or go out of existence during the motion. Does it matter whether S is simply-connected or no? Give your reasons.

If x', y', z' be the coordinates after an infinitesimal time dt of the particle originally at xyz,

$$x' = x + \xi dt, \quad y' = y + \eta dt, \quad z' = z + \zeta dt,$$

and if the group of points which filled the parallelepiped $dxdydz$ now fill the parallelepiped $dx'dy'dz'$,

$$\frac{dx'}{dx} = 1 + \frac{d\xi}{dx}dt, \quad \frac{dy'}{dy} = 1 + \frac{d\eta}{dy}dt, \quad \frac{dz'}{dz} = 1 + \frac{d\zeta}{dz}dt,$$

and to the first order the volume of the parallelepiped $dx'dy'dz'$

$$= \left\{1 + \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)dt\right\} dxdydz.$$

Therefore the rate of dilatation is $\dfrac{d\xi}{dx} + \dfrac{d\eta}{dy} + \dfrac{d\zeta}{dz}$.

If the base $dydz$ generate a prism by moving from $-\infty$ to $+\infty$ parallel to the axis of x and cutting out from the surface S successively at entrance, and exit the elements of surface ds_1, ds_2, ds_3, ... (fig. 51),

$$\iiint \frac{d\xi}{dx} dxdydz = \iint (-\xi_1 + \xi_2 - \xi_3 + \ldots)\, dydz$$

$$= \iint (l_1\xi_1 ds_1 + l_2\xi_2 ds_2 + l_3\xi_3 ds_3 + \ldots) = \iint l\xi ds,$$

since $\qquad dydz = -l_1 ds_1 = l_2 ds_2 = -l_3 ds_3 \ldots$.

Similarly

$$\iiint \frac{d\eta}{dy} dxdydz = \iint m\eta ds \text{ and } \iiint \frac{d\zeta}{dz} dxdydz = \iint n\zeta ds,$$

and therefore

$$\iiint\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)dx\,dy\,dz=\iint(l\xi+m\eta+n\zeta)\,ds.$$

The left-hand side is the rate at which the contents of S diminish. The right-hand side is the excess of the rate of escape from S over that of entrance into it.

If these are equal no points can come into or go out of existence, since S is *any* closed surface whatever.

If S be multiply connected, draw diaphragms so as to make it simply connected.

Then the points that pass these diaphragms are counted both at entrance and escape, hence it does not matter whether S is simply or multiply connected.

vii. Define the terms quantity and potential as applied to a charge of electricity, and the term capacity as applied to the conductor on which it is distributed. Point out the numerical relation between these quantities, and shew how to find the energy of a given charge on a conductor of given capacity.

Two spherical soap-bubbles are caused to unite into a single spherical one. Shew that a diminution of surface takes place, and calculate the charge of electricity which must be given to the single bubble in order to draw out the film to its former superficial extent.

The quantity of a charge is measured in electrostatic units, the unit being that quantity of electricity which, when placed at unit distance from an equal quantity, repels it with unit of force.

The potential at any point is the work required to convey unit of negative electricity from that point to an infinite distance, supposing the distribution of electricity not to be disturbed by induction.

The force varying inversely as the square of the distance, the potential of a small quantity m of electricity at distance r is $\int_r^\infty \frac{mdr}{r^2}=\frac{m}{r}$, and therefore the potential of an electrified body is $\Sigma\frac{m}{r}$.

The electric force at any point in any direction is equal to the rate of decrease of the potential in that direction, and therefore the surface of a conductor in electrical equilibrium is an equi-potential surface.

If one conductor be insulated while all the surrounding conductors are kept at zero potential by being put in communication with the earth, and if the conductor when charged with a quantity E of electricity has the potential V, the ratio of E to V is called the capacity of the conductor.

The capacity is therefore the charge required to produce unit potential on the conductor.

The energy of the charge

$$W = \tfrac{1}{2}EV = \tfrac{1}{2}\cdot\frac{E^2}{C}$$

where C denotes the capacity.

[Maxwell's *Electricity*, § 41–50, 85].

Let r, r' be the radii of the soap-bubbles, p, p' the pressure of the air inside them, ϖ the atmospheric pressure, T the superficial tension of the soap-bubble film,

$$p - \varpi = \frac{2T}{r}, \quad p' - \varpi = \frac{2T}{r'}.$$

If the two bubbles condense into a single bubble of radius R, and if P be the pressure of the air inside,

$$P - \varpi = \frac{2T}{R} \text{ and } pr^3 + p'r'^3 = PR^3 \text{ by Boyle's law,}$$

therefore $\quad \varpi(R^3 - r^3 - r'^3) + 2T(R^2 - r^2 - r'^2) = 0.$

Therefore if δV express the increase of volume, δS the increase of surface,

$$3\varpi\delta V + 2T\delta S = 0.$$

Now physical circumstances shew that the tendency is to an increase of volume and a diminution of surface; both can thus be satisfied.

If ρ be the surface density of the charge required, and if R' be the radius of the bubble, P' the pressure of the air inside, the electric repulsion on any surface element ds is along the radius and of magnitude $2\pi\rho^2 ds$, and the action of the charge is equivalent to an additional pressure $2\pi\rho^2$ in the interior.

Therefore $$P' + 2\pi\rho^2 - \varpi = \frac{2T}{R'},$$

and $$P'R'^3 = pr^3 + p'r'^3,$$

therefore $$2\pi\rho^2 R'^3 = \varpi(R'^3 - r^3 - r'^3),$$

since $R'^2 = r^2 + r'^2$, and the required charge is $4\pi\rho R'^2$.

viii. Give a general explanation of the mode in which a sound-wave is propagated, and point out why its velocity in air depends upon the temperature but not upon the pressure.

Investigate, on thermo-dynamical principles, the velocity of propagation of plane waves of sound.

Consider plane waves of longitudinal displacement perpendicular to the front propagated in an unlimited medium.

Take the axis of x in the direction of displacement, and suppose arbitrary velocities $u = f(x)$, and arbitrary dilatations $s = \phi(x)$ to be given to the medium between $x = 0$ and $x = l$, the remainder of this medium being supposed undisturbed.

The motion being supposed small, we may consider the resultant motion by superposing two states of motion defined by

$$u_1 = -as_1 = \tfrac{1}{2}\{f(x) - a\phi(x)\},$$

$$u_2 = as_2 = \tfrac{1}{2}\{f(x) + a\phi(x)\}.$$

Considering the initial state of motion in which $u = -as$, let the curve OVL (fig. 52) represent the velocities of the particles between O and L, such that PV represents the velocity of the particles which at rest were at distance OP from the initial plane $x = 0$; then $-\dfrac{PV}{a}$ represents the dilatation.

After a time dt, taking $PP' = adt$, the increment of velocity of these particles

$$= (\text{pressure at } P - \text{pressure at } P')\, dt$$
$$\div \text{ mass per unit of area between } P \text{ and } P'$$

$$= (\text{pressure at } P - \text{pressure at } P')\, \frac{1}{a\rho_0}$$

$$= -\frac{1}{a} \cdot \frac{dp}{d\rho} \cdot \frac{\delta\rho}{\rho_0} = \frac{1}{a} \cdot \frac{dp}{d\rho} \cdot \delta s = \frac{1}{a^2} \cdot \frac{dp}{d\rho}\,(P'V' - PV),$$

neglecting squares of u and s, which is equivalent to considering the velocities and dilatations in the actual position the same as in the mean position.

Therefore if we take $a^2 = \dfrac{dp}{d\rho}$, then after a time dt the velocity at P will be represented by $P'V'$, and therefore the motion will be represented by sliding the curve OVL with velocity a in the positive direction.

Therefore the initial state of motion in which $u_1 = -as_1$ gives rise to a wave propagated without change of form in the positive direction, with velocity $a = \sqrt{\left(\dfrac{dp}{d\rho}\right)}$.

Similarly it may be shewn that the initial state in which $u_2 = as_2$ gives rise to a negative wave.

If we increase ρ_0 in any ratio, p, by Boyle's law, will be increased in the same ratio, and the ratio of the moving forces to the masses moved will be unaltered, and therefore the motion will be unaltered. Hence the velocity of propagation is independent of the density and pressure.

For the second part we have the exact equations

$$\rho_0 \frac{d^2\xi}{dt^2} = -\frac{dp}{dx}, \quad \frac{\rho_0}{\rho} = 1 + \frac{d\xi}{dx}.$$

Also because the compressions and dilatations are rapid, neglecting conduction, radiation, &c. by Thermodynamics,

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\beta = \frac{1}{\left(1 + \dfrac{d\xi}{dx}\right)^\beta},$$

where β is the ratio of the specific heat at constant volume to the specific heat at constant pressure; therefore

$$\frac{d^2\xi}{dt^2} = \beta \frac{p_0}{\rho_0} \cdot \frac{\dfrac{d^2\xi}{dx^2}}{\left(1 + \dfrac{d\xi}{dx}\right)^{\beta+1}},$$

the *exact* differential equation of motion for finite displacements.

With the usual approximation this reduces to

$$\frac{d^2\xi}{dt^2} = \beta \frac{p_0}{\rho_0} \cdot \frac{d^2\xi}{dx^2},$$

and therefore the velocity of propagation is $\sqrt{\left(\beta \dfrac{p_0}{\rho_0}\right)}$.

ix. Investigate an expression for the mutual potential energy of two bar magnets which are placed, in any positions, at a distance from one another great compared with the length of either.

Employ it to find the positions of equilibrium when the magnets are free to turn about their middle points in given planes, distinguishing between stable and unstable positions.

Calculate the disturbances produced on each other's indications by a variation-compass and a dip-needle whose relative position is given.

If (xyz), $(x'y'z')$ be the centres of the magnets, M, M' their moments, and lmn, $l'm'n'$ their direction cosines, then the mutual potential energy

$$W = MM' \frac{d^2}{dh\,dh'} \left(\frac{1}{r}\right)$$

$$= MM' \left(l \frac{d}{dx} + m \frac{d}{dy} + n \frac{d}{dz}\right) \left(l' \frac{d}{dx} + m' \frac{d}{dy} + n' \frac{d}{dz}\right) \frac{1}{r}$$

$$= \frac{MM'}{r^5} \left[ll'\{3(x-x')^2 - r^2\} + \ldots + 3(mn' + m'n)(y-y')(z-z') + \ldots\right].$$

[Tait, *Thermodynamics*, Note C; Maxwell, *Electricity*, §387].

For the second part taking the line joining the centres of the magnets equally inclined to the coordinate axes

$$W = \frac{MM'}{r^3}(mn' + m'n + nl' + n'l + lm' + l'm),$$

this is to be a maximum or minimum in the positions of equilibrium subject to the conditions

$$l\lambda + m\mu + n\nu = 0, \quad l^2 + m^2 + n^2 = 1,$$
$$l'\lambda' + m'\mu' + n'\nu' = 0, \quad l'^2 + m'^2 + n'^2 = 1,$$

where $\lambda\mu\nu$, $\lambda'\mu'\nu$ are the direction cosines of the axes about which the magnets are free to turn.

Therefore $\lambda dl + \mu dm + \nu dn = 0$, $ldl + mdm + ndn = 0$,

$$\frac{dl}{m\nu - n\mu} = \frac{dm}{n\lambda - l\nu} = \frac{dn}{l\mu - m\lambda} = d\psi \text{ suppose.}$$

$$dW = \frac{MM'}{r^3}\{(m' + n')\,dl + (n' + l')\,dm + (l' + m')\,dn$$
$$+ (m + n)\,dl' + (n + l)\,dm' + (l + m)\,dn'\}$$

$$= \frac{MM'}{r^3}\left\{\begin{vmatrix} m'+n', & n'+l', & l'+m' \\ l, & m, & n \\ \lambda, & \mu, & \nu \end{vmatrix} d\psi + \begin{vmatrix} m+n, & n+l, & l+m \\ l', & m', & n' \\ \lambda', & \mu', & \nu' \end{vmatrix} d\psi'\right\}.$$

Therefore the condition of equilibrium of the first magnet for a given direction of the second magnet is

$$\begin{vmatrix} m' + n', & n' + l', & l' + m' \\ l, & m, & n \\ \lambda, & \mu, & \nu \end{vmatrix} = 0,$$

or the lines whose direction cosines are

$$(lmn), \quad (\lambda\mu\nu), \quad (m' + n', \; n' + l', \; l' + m'),$$

lie in a plane.

Refer all the directions to the surface of a sphere, and let O (fig. 53) denote the direction of the line joining the centres of the magnets, and let CA, CB denote the planes in which the magnets are free to move.

Then if B denote the direction in which the second magnet is held, and BO be produced to B' such that $\tan OB' = \frac{1}{2}\tan BO$, it may be easily proved that B' denotes the direction of the line $(m'+n', n'+l', l'+m')$; and, therefore, if $B'A$ be drawn perpendicular to CA, A will be the direction of the first magnet in the position of equilibrium relative to B. B will also be in equilibrium relatively to A, if when AO is produced to A' such that $\tan OA' = \frac{1}{2}\tan AO$, $A'B$ is perpendicular to CB.

If we eliminate $l'm'n'$ between the equations we get an equation of the second degree in lmn, which combined with $l\lambda + m\mu + n\nu = 0$, gives two values of $l : m : n$, to each of which corresponds a value of $l' : m' : n'$.

If $CA = \theta$, $CB = \phi$; $\angle ACO = \alpha$, $\angle BCO = \beta$, $CO = \delta$,

$$W = \frac{MM'}{r^3}\{(l+m+n)(l'+m'+n') - (ll'+mm'+nn')\}$$

$$= \frac{MM'}{r^3}(3\cos OA\cos OB - \cos AB)$$

$$= \frac{MM'}{r^3}\{3(\cos\theta\cos\delta + \sin\theta\sin\delta\cos\alpha)(\cos\phi\cos\delta + \sin\phi\sin\delta\cos\beta)$$

$$- (\cos\theta\cos\phi + \sin\theta\sin\phi\cos\overline{\alpha+\beta})\}$$

$$= a\cos\theta\cos\phi + b\sin\theta\sin\phi + c\sin\theta\cos\phi + d\cos\theta\sin\phi,$$

if $$a = \frac{MM'}{r^3}(3\cos^2\delta - 1),\ b = \ldots.$$

In the position of equilibrium $\frac{dW}{d\theta} = 0$, $\frac{dW}{d\phi} = 0$; or putting $x = \tan\theta$, $y = \tan\phi$,

$$-ax + by + c - dxy = 0,\quad -ay + bx - cxy + d = 0,$$

two homographic relations between x and y.

Therefore $$(x^2 - 1)(ac + bd) + x(a^2 - b^2 - c^2 + d^2) = 0,$$

$$(y^2 - 1)(ad + bc) + y(a^2 - b^2 + c^2 - d^2) = 0.$$

If x_1, y_1; x_2, y_2 be the roots of these equations, since

$x_1x_2 = y_1y_2 = -1$, for each magnet the directions of equilibrium are at right angles, and

$$\frac{d^2W_1}{d\theta^2} = \frac{d^2W_1}{d\phi^2} = -W_1, \quad \frac{d^2W_1}{d\theta\,d\phi} = W_2,$$

$$\frac{d^2W_1}{d\theta^2}\,\frac{d^2W_1}{d\phi^2} - \left(\frac{d^2W_1}{d\theta\,d\phi}\right)^2 = W_1^2 - W_2^2.$$

Now

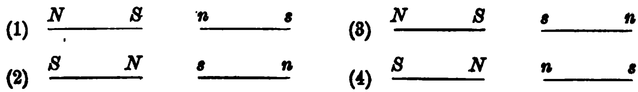
$$W_1^2 = \frac{(cx_1+a)^2 + (bx_1+d)^2}{1+x_1^2} = \frac{(dy_1+a)^2 + (by_1+c)^2}{1+y_1^2},$$

$$W_1^2 - W_2^2 = (ac+bd)(x_1-x_2) = (ad+bc)(y_1-y_2);$$

therefore if $ac+bd$ and x_1 be of the same sign, so also will $ad+bc$ and y_1, W_1 is a maximum or minimum, and the directions given by x_1 and y_1 give four positions of equilibrium, two stable when W_1 is negative, two unstable when W_1 is positive.

W_2 is then a maximum-minimum, and the directions given by x_2 and y_2 give four positions of equilibrium which are stable-unstable, that is, stable for some displacements, unstable for others.

In the simplest case when the planes in which the magnets move are parallel, the stable and unstable positions of equilibrium are in the plane through the line joining the centres of the magnets perpendicular to the planes in which the magnets move, and the stable-unstable positions are perpendicular to this plane.



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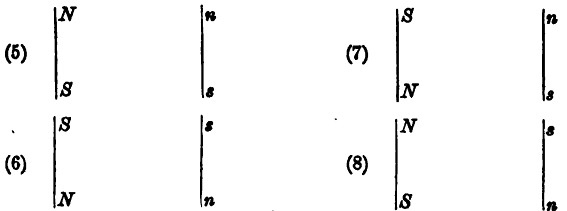

stable-unstable.

For the third part of the question, in the variation-compass and the dip-needle, the planes are at right angles, and $\frac{dW}{d\theta}$, $\frac{dW}{d\phi}$ are the couples interfering with the effect of the earth's directive force.

MONDAY, *Jan.* 18, 1875. 9 to 12.

PROF. TAIT.	Roman numbers.
MR. WRIGHT.	Arabic numbers.

1. DEMONSTRATE formulae involving polar coordinates for the position of the centre of inertia of a plane lamina and of a solid.

If the density at any point of a circular disc whose radius is a vary directly as the distance from the centre, and a circle described on a radius as diameter be cut out, prove that the centre of inertia of the remainder will be at a distance $\dfrac{6a}{15\pi - 10}$ from the centre.

The mass of the large circle if complete

$$=\int_0^a \mu r \,.\, 2\pi r dr = \tfrac{2}{3}\mu\pi a^3.$$

The mass cut away

$$=2\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \mu r \,.\, r d\theta \,.\, dr = \frac{2\mu a^3}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta d\theta = \frac{4\mu a^3}{9}.$$

Therefore, if C is the centre of the large circle, and B the centre of inertia of the circle cut away,

$$CB = \frac{2\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \mu r \,.\, r\cos\theta \,.\, r d\theta \,.\, dr}{\tfrac{4}{9}\mu a^3} = \frac{9}{8}\int_0^{\frac{\pi}{2}} \cos^5\theta d\theta = \frac{3a}{5}.$$

Therefore the required distance is

$$\frac{\dfrac{3a}{5} \,.\, \tfrac{4}{9}\mu a^3}{\tfrac{2}{3}\mu\pi a^3 - \tfrac{4}{9}\mu a^3} = \frac{6a}{15\pi - 10}.$$

2. Find expressions for the accelerations of a moving point estimated (α) along and perpendicular to the radius vector (β) along the tangent and normal.

If a curve be described under a force P tending to the pole and a normal force N, prove that

$$p^2 \frac{d}{dr}\left(Nr\frac{dr}{dp}\right) + \frac{d}{dr}\left(Pp^3\frac{dr}{dp}\right) = 0.$$

Resolving along the normal

$$N + P\frac{p}{r} = \frac{v^2}{\rho},$$

resolving along the tangent

$$P\frac{dr}{ds} = -\frac{1}{2}\cdot\frac{dv^2}{ds}, \text{ or } P = -\frac{1}{2}\cdot\frac{dv^2}{dr};$$

therefore, eliminating v^2,

$$\frac{d}{dr}\left(\rho N + \rho P\frac{p}{r}\right) + 2P = 0,$$

$$p^2 \frac{d}{dr}\left(Nr\frac{dr}{dp}\right) + p^2\frac{d}{dr}\left(Pp\frac{dr}{dp}\right) + 2Pp^2 = 0,$$

whence the result.

3. A particle moves under the action of given forces on a given smooth surface, shew how to determine the motion and the pressure on the surface.

Given the resultant impressed force and the velocity of the particle at any point, determine by a geometrical construction the osculating plane and the centre of curvature of the path on the surface.

Let S be the surface (fig. 54), M the particle of mass m projected in the direction MT with velocity v, and let F be the resultant impressed force represented by MF.

If N be the normal reaction of the surface represented by MN, we can consider the particle as moving freely under the action of the resultant K, represented by MK, of F and N; KMT will then be the osculating plane of the path.

N is unknown, but is determined from the condition that if MI be taken equal to $\frac{mv^2}{R}$, where R is the radius of curvature of the normal section made through MT; then MI is equal to the algebraical sum of MN and the projection of MF on MN.

The osculating plane being then determined, if a line be drawn through I at right angles to the plane IMT, it will, by Meunier's theorem, meet the osculating plane in H, the centre of curvature of the path.

iv. Form the equation of motion of a rigid plate of any form consequent upon one of its points being constrained to move in a given manner in the plane of the plate. Integrate it for the special cases of uniform rectilinear, and uniform circular, motion of the point.

Apply your results to explain the action of a flail, gravity being neglected.

Let ξ, η be the coordinates of the point given in terms of t; $x = a\cos\theta$, $y = a\sin\theta$ the relative coordinates of the centre of inertia of the plate; X, Y the component forces applied by the constraint. Then

$$M(\ddot{\xi} + \ddot{x}) = X,\quad M(\ddot{\eta} + \ddot{y}) = Y,\quad \text{and } Mk^2\ddot{\theta} = Xy - Yx.$$

Therefore $k^2\ddot{\theta} = (\ddot{\xi} + \ddot{x})\,y - (\ddot{\eta} + \ddot{y})\,x = \ddot{\xi}y - \ddot{\eta}x - a^2\ddot{\theta}$,

$$(k^2 + a^2)\,\ddot{\theta} = \ddot{\xi}y - \ddot{\eta}x.$$

(1) In the case of uniform rectilinear motion of the point $\ddot{\xi} = 0$, $\ddot{\eta} = 0$; therefore $\ddot{\theta} = 0$, and the plate rotates uniformly.

(2) In the case of uniform circular motion of the point

$$\ddot{\xi} = -bn^2\cos nt,\quad \ddot{\eta} = -bn^2\sin nt;$$

therefore $$\left(\frac{d}{dt}\right)^2(\theta - nt) = -\frac{abn^2}{k^2 + a^2}\sin(\theta - nt),$$

which represents pendulum motion about the uniformly revolving radius.

These two cases explain the flail. The handle is worked so that the joint has the circular motion until the striker gains its maximum angular velocity; the motion of the joint is then changed into the rectilinear motion, when the striker continues to move with the same angular velocity until the blow is delivered.

v. Write down Euler's equations which give the angular velocities of a rigid body about its principal axes, and interpret the various terms.

If a constant couple be applied about the axis of symmetry of a body supported at its centre of inertia, and initially rotating about an axis perpendicular to that of symmetry, determine the motion completely; and shew that the cone described in the body by the instantaneous axis has the equation

$$\tan^{-1}\frac{x}{y}=\frac{A-C}{A}\cdot\frac{\Omega^2 C}{2N}\cdot\frac{z^2}{x^2+y^2};$$

where N is the couple, Ω the initial angular velocity.

The equations of motion are

$$C\dot{\omega}_3=N;\ \text{ therefore }\ \omega_3=\frac{N}{C}t,$$

and

$$A\dot{\omega}_1+(C-A)\,\omega_2\omega_3=0,$$

$$A\dot{\omega}_2+(A-C)\,\omega_3\omega_1=0;$$

therefore

$$\omega_1\dot{\omega}_1+\omega_2\dot{\omega}_2=0,\quad \omega_1^2+\omega_2^2=\Omega^2,$$

and

$$\frac{\omega_2\dot{\omega}_1-\omega_1\dot{\omega}_2}{\omega_1^2+\omega_2^2}=\frac{A-C}{A}\omega_3=\frac{A-C}{A}\cdot\frac{N}{C}.t;$$

therefore

$$\tan^{-1}\frac{\omega_1}{\omega_2}=\frac{A-C}{A}\cdot\frac{Nt^2}{2C}=\frac{A-C}{A}\cdot\frac{C\omega_3^2}{2N}$$

$$=\frac{A-C}{A}\cdot\frac{\Omega^2 C}{2N}\cdot\frac{\omega_3^2}{\omega_1^2+\omega_2^2},$$

hence the equation of the cone described by the instantaneous axis.

6. Shew how to find the coordinates of the centre of pressure of a plane area immersed in homogeneous liquid.

If a straight line be taken, in the plane of the area, parallel to the surface of the liquid and as far below the centre of inertia of the area as the surface of the liquid is above, the pole of this straight line with respect to the momental ellipse at the centre of inertia whose semi-axes are equal to the principal radii of gyration at that point will be the centre of pressure of the area.

Taking the principal axes at the centre of inertia as coordinate axes, if the equation to the momental ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $a^2=\frac{\iint x^2 dxdy}{\iint dxdy}$, $b^2=\frac{\iint y^2 dxdy}{\iint dxdy}$.

If $x\cos\alpha+y\sin\alpha=p$ be the equation of the line in the surface of the liquid and $(\bar{x}\bar{y})$ the centre of pressure,

$$\bar{x}=\frac{\iint(p-x\cos\alpha-y\sin\alpha)\,xdxdy}{\iint(p-x\cos\alpha-y\sin\alpha)\,dxdy}=-\frac{a^2}{p}\cos\alpha,$$

$$\bar{y}=\frac{\iint(p-x\cos\alpha-y\sin\alpha)\,ydxdy}{\iint(p-x\cos\alpha-y\sin\alpha)\,dxdy}=-\frac{b^2}{p}\sin\alpha.$$

Therefore $(\bar{x}\bar{y})$ is the pole of $x\cos\alpha+y\sin\alpha=-p$ with respect to the momental ellipse.

vii. Shew that a cloud of small particles or of fine dust, if only deep enough, however far the particles may be separated in comparison with their diameters, can give a brightness equal to half that of a slab of the same material similarly illuminated by a distant source of light.

Hence shew that the brightness of a comet, and the visibility of a star through the head of a comet, are consistent with the comet's being a mere swarm of meteorites.

Shew how to compare, approximately, the utmost brightness of a cloud nearly opposite to the sun, and consisting of small spheres of water, with the brightness of the sun's image in a pool.

Suppose the cloud arranged in layers, parallel or not, each allowing $1-e$ of the incident light to pass, and sending back ef of it.

Then 1st layer gets 1 sends back ef,

2nd............ $1-e$ .... $(1-e)^2ef$,

3rd .... $(1-e)^2$ .... $(1-e)^4ef$,

....

the factors $(1-e)$, $(1-e)^2$, ..., being squared, because of the loss in coming out.

Therefore, altogether there is sent back

$$ef\{1+(1-e)^2+(1-e)^4+\dots \textit{ad inf.}$$

$$=\frac{ef}{1-(1-e)^2}=\frac{f}{2-e}=\frac{f}{2},$$

when e is small. But, if $e=1$, no light gets through the first layer, which is then virtually a slab; and it sends back f, double the light of the cloud.

Comets (except occasionally their nuclei, which are, probably, to a great extent self-luminous) are not nearly of half the brightness of planets equally distant from the sun.

Through the cloud, above spoken of, the brightness of a star would be reduced from 1 to $(1-e)^n$, where n is the number of layers. The individual particles of the comet are usually too far apart to eclipse more than a small fraction of the disc of a fixed star, even though the disc is invisible in our best telescopes.

For the last rider using the same process, and taking account of the size of the images in the reflecting spheres; if 2ω be the angular diameter of the sun, and r the radius of a raindrop, the area of the image of the sun formed by reflection at the convex surface of the raindrop will be $\frac{1}{4}\pi r^2 \sin^2\omega$, and therefore the ratio of the apparent size of the image of the sun to the apparent size of the raindrop will be $\frac{1}{4}\sin^2\omega$.

Therefore the cloud will give at most a brightness which will bear to the brightness of the image of the sun in a pool the ratio $\frac{1}{8}\sin^2\omega$, which is $\frac{1}{4}$ the ratio of the apparent area of the sun to the apparent area of the hemispherical celestial vault.

The light reflected from the interior of the rain-drops, by which the rainbows are formed, is insensible nearly opposite the sun.

viii. If θ, ϕ, be the angles of incidence and emergence of two parallel rays passing through a prism in a plane perpendicular to the edge; d_1, d_2 the distances between these rays before incidence and after emergence; shew that

$$\frac{d_1}{d_2} = -\frac{\delta\phi}{\delta\theta},$$

where $\delta\theta$ is any small change of θ, and $\delta\phi$ the corresponding change of ϕ.

Shew from this that the position of minimum deviation is that of most distinct vision through a thin prism.

If d be the distance between the rays inside the prism,

$$d_1 \sec\theta = d \sec\theta', \quad d_2 \sec\phi = d \sec\phi';$$

therefore $$\frac{d_1}{d_2} = \frac{\cos\theta \cos\phi'}{\cos\theta' \cos\phi} = -\frac{d\phi}{d\theta}.$$

In the position of minimum deviation $d_1 = d_2$, and therefore the divergence of the rays is unaltered by the prism, which is the condition for the most distinct vision.

ix. Discuss separately, and without formulæ, the effects of annual parallax and aberration on the apparent position of a fixed star as it would be seen from a comet of a year period, moving in the ecliptic, in a path of great excentricity. Compare these with the corresponding effects as seen from the earth.

Trace the curves representing, from each of these points of view, the apparent annual path of a star, without proper motion, situated near the pole of the ecliptic.

The effect of annual parallax is to make the star describe an equal and parallel orbit turned through 180°, and the projection of this on the celestial sphere will be the apparent path of the star due to annual parallax.

The effect of aberration is to make the star describe the hodograph of the orbit, which is a circle parallel to the plane of the orbit, and the projection of this on the celestial sphere will be the apparent path of the star due to aberration.

If the star be near the pole of the ecliptic, the orbits due to annual parallax and aberration will be unprojected on the celestial sphere.

The orbits seen from the comet will be an ellipse of great excentricity, and a circle passing very nearly through the mean position of the star (figs. 55, 56, 57) and the apparent annual path will be the resultant of these two orbits; seen from the earth the orbits will be approximately concentric circles.

10. Explain the construction of charts on the gnomonic, the stereographic and Mercator's projections. Examine what curve in each case will represent (α) a rhumb line, (β) a great circle.

Shew how to draw the trace in the two first projections of the great circle passing through any two given places.

Prove that the equation of the trace on a Mercator's chart of a great circle will be always of the form

$$2 \sin\left(\frac{x}{a}+\alpha\right)=k\left(\varepsilon^{\frac{y}{a}}-\varepsilon^{-\frac{y}{a}}\right),$$

where a is the radius of the sphere.

The traces of (α) a rhumb-line are respectively a transcendental spiral of the form $\dfrac{2a}{r}=e^{\frac{\theta}{c}}-e^{-\frac{\theta}{c}}$, an equiangular spiral and a straight line; and the traces of (β) a great circle are respectively a straight line, a circle, and the transcendental curve of the third part of the question.

To draw the trace of a great circle joining two given points in the gnomonic projection, draw the straight line joining the projections of the points; in the stereographic, draw the circle passing through the projections of the points and their antipodes.

Let PM be the great circle (fig. 58), OX, OY the equator and initial meridian which project into the coordinate axes in Mercator's projection, and let l, λ be the longitude and latitude of P, and x, y its coordinates on the chart.

Then $$\frac{dy}{dx} = \frac{d\lambda}{\cos\lambda\, dl}, \text{ and } x = al;$$

therefore $$y = a\int \sec\lambda\, d\lambda$$
$$= a \log(\sec\lambda + \tan\lambda),$$

since λ and y vanish together.

Therefore $$e^{\frac{y}{a}} = \sec\lambda + \tan\lambda,$$

and $$e^{\frac{y}{a}} - e^{-\frac{y}{a}} = 2\tan\lambda = 2\sin MN \tan PMN$$
$$= 2\tan PMN \sin\left(\frac{x}{a} + MO\right),$$

which reduces to the given form, putting
$$k = \cot PMN, \quad \alpha = MO.$$

Monday, *Jan.* 18, 1875. 1½ to 4.

Mr. Cookshott. Arabic numbers.
Mr. Freeman. Roman numbers.

1. Expand a^x in powers of x.

Shew that, if n is greater than 3,

$$n^3 + \frac{n(n-1)}{1.2}(n-2)^3 + \frac{n(n-1)(n-2(n-3)}{1.2.3.4}(n-4)^3 + \&c.$$

$$= n^2(n+3)\,2^{n-4}.$$

The series is $\frac{\lfloor 3}{2}$ of the coefficient of x^3 in $(e^x+1)^n + (e^x-1)^n$,

that is, in $\left(2 + x + \frac{x^2}{\lfloor 2} + \dots\right)^n + \left(x + \frac{x^2}{\lfloor 2} + \dots\right)^n.$

If $n > 3$, x^3 appears only in the first of these two terms, and its coefficient is $\frac{n^2(n+3)\,2^{n-3}}{\lfloor 3}$, and therefore the required sum is $n^2(n+3)\,2^{n-4}$.

2. Prove that there are only five kinds of regular polyhedrons.

If one of each kind be inscribed in the same sphere, prove that their edges will be in the ratio of

$$2\sqrt{(2)} : 2 : \sqrt{(6)} : \sqrt{(5)} - 1 : \sqrt{[\tfrac{6}{5}\{5 - \sqrt{(5)}\}]}.$$

If m be the number of sides in each face of a regular polyhedron, n the number of plane angles in each solid angle,

a the length of an edge, and D the diameter of the circumscribing sphere

$$\frac{a}{D} = \sqrt{\left(1 - \frac{\cos^2 \frac{\pi}{m}}{\sin^2 \frac{\pi}{n}}\right)}.$$

In the (i) tetrahedron $m=3$, $n=3$, therefore $\frac{a}{D} = \sqrt{(\frac{2}{3})}$,

(ii) cube $m=4$, $n=3$, $\frac{a}{D} = \sqrt{(\frac{1}{3})}$;

(iii) octahedron $m=3$, $n=4$, $\frac{a}{D} = \sqrt{(\frac{1}{2})}$;

(iv) dodecahedron $m=5$, $n=3$, $\frac{a}{D} = \frac{\sqrt{(5)}-1}{2\sqrt{(3)}}$;

(v) icosahedron $m=3$, $n=5$, $\frac{a}{D} = \sqrt{\left\{\frac{5-\sqrt{(5)}}{10}\right\}}$.

Therefore the required ratios are

$$\sqrt{\left(\frac{2}{3}\right)} : \sqrt{\left(\frac{1}{3}\right)} : \sqrt{\left(\frac{1}{2}\right)} : \frac{\sqrt{(5)}-1}{2\sqrt{(3)}} : \sqrt{\left\{\frac{5-\sqrt{(5)}}{10}\right\}}$$

$$= 2\sqrt{(2)} : 2 : \sqrt{(6)} : \sqrt{(5)}-1 : \sqrt{[\tfrac{6}{5}\{5-\sqrt{(5)}\}]}.$$

3. Find the equation of the chord joining the points of contact of two tangents drawn to the parabola $y^2 = 4ax$ from the point (h, k).

If $\phi(x, y) \equiv (\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$ be the equation of a parabola, prove that the equation of its axis is

$$\alpha \frac{d\phi}{dx} + \beta \frac{d\phi}{dy} = 0.$$

The direction-cosines of a diameter are proportional to α and β, and of the polar of (xy) to $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$.

If (xy) is on the axis these lines are at right angles, and therefore

$$\alpha \frac{d\phi}{dx} + \beta \frac{d\phi}{dy} = 0.$$

iv. Find the length of the straight line drawn from the point (α, β, γ) parallel to the straight line $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$ to meet the plane $lx + my + nz = p$.

Prove that the four planes,

$$my + nz = 0, \quad nz + lx = 0, \quad lx + my = 0, \quad lx + my + nz = p,$$

form a tetrahedron whose volume is

$$\frac{2p^3}{3lmn}.$$

If the first three planes meet the coordinate planes of yz, zx, xy in BC, CA, AB respectively (fig. 59), and the fourth plane meet the coordinate planes in bc, ca, ab; then BC passes through a and is parallel to bc, CA passes through b and is parallel to ca, and AB passes through c and is parallel to ab.

Hence the triangle abc is $\frac{1}{4}$ the triangle ABC.

The volume of the tetrahedron

$$Oabc = \tfrac{1}{6}\, Oa.Ob.Oc = \tfrac{1}{6}\, \frac{p^3}{lmn}.$$

Therefore the volume of the tetrahedron

$$OABC = 4 \text{ tetrahedron } Oabc = \tfrac{2}{3}\, \frac{p^3}{lmn}.$$

5. Find the equations of the tangent plane and the normal at a point of the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

If the plane $lx + my + nz = p$ cut this surface in a parabola, prove that $l^2a^2 + m^2b^2 = n^2c^2$, and the coordinates of the vertex of the parabola satisfy the equation

$$\frac{x}{la^2}(b^2 + c^2) - \frac{y}{mb^2}(a^2 + c^2) + \frac{z}{nc^2}(b^2 - a^2) = 0.$$

The equations of the line conjugate to the plane $lx+my+nz=p$ are

$$\frac{x}{la^2}=\frac{y}{mb^2}=-\frac{z}{nc^2},$$

and if the curve of section be a parabola, this line must be parallel to its conjugate plane, therefore

$$l^2a^2+m^2b^2-n^2c^2=0.$$

If $(\alpha\beta\gamma)$ be the coordinates of the vertex, the equations of the diameter are

$$\frac{x-\alpha}{la^2}=\frac{y-\beta}{mb^2}=-\frac{z-\gamma}{nc^2},$$

and of the tangent line are

$$\left.\begin{array}{r}\dfrac{x\alpha}{a^2}+\dfrac{y\beta}{b^2}-\dfrac{z\gamma}{c^2}=1\\ lx+my+nz=p\end{array}\right\},$$

and these are at right angles.

Therefore

$$la^2\left(\frac{n\beta}{b^2}+\frac{m\gamma}{c^2}\right)-mb^2\left(\frac{l\gamma}{c^2}+\frac{n\alpha}{a^2}\right)-nc^2\left(\frac{m\alpha}{a^2}-\frac{l\beta}{b^2}\right)=0,$$

or $$\frac{\alpha}{la^2}(b^2+c^2)-\frac{\beta}{mb^2}(c^2+a^2)-\frac{\gamma}{nc^2}(a^2-b^2)=0.$$

6. Shew how to determine the maximum and minimum values of a function of two or more variables connected by a given equation.

A framework crossed or uncrossed is formed of two unequal rods joined together at their ends by two equal rods; prove that the distance between the middle points of either pair of rods is a maximum when the unequal rods are parallel and a minimum when the equal rods are parallel; unless the two unequal rods are together less than the two equal rods, in which case the unequal rods are parallel in both the maximum and minimum positions.

Let $ACDB$ be the framework; $AB=a$, $AC=BD=b$, $CD=c$, $\angle CAB=\theta$, $\angle DBA=\phi$; and let x be the distance between the middle points of the unequal rods.

$$x^2=\frac{b^2}{4}(\sin\theta+\sin\phi)^2+\frac{b^2}{4}(\cos\theta-\cos\phi)^2=b^2\sin^2\frac{\theta+\phi}{2}.$$

This is to be a maximum or minimum subject to the condition

$$c^2=a^2-2ab(\cos\theta+\cos\phi)+2b^2\{1+\cos(\theta+\phi)\}.$$

Hence,

$$2ab\sin\theta-2b^2\sin(\theta+\phi)=2ab\sin\phi-2b^2\sin(\theta+\phi),$$

$\sin\theta=\sin\phi$ and $\theta=\phi$; for $\theta+\phi=\pi$ would require the quadrilateral to be a parallellogram.

If $a>c$, θ has its greatest numerical value when BDC is a straight line, and ϕ when ACD is a straight line.

If $a+c>2b$, θ and ϕ can vanish, and it is possible for the equal rods to become parallel, the unequal rods being crossed.

If $a+c<2b$, θ has its least numerical value when BCD is a straight line, and ϕ when ADC is a straight line, and it is impossible for the equal rods to become parallel.

From the figures 60 and 61 we see that x^2 is a maximum when the unequal rods are parallel; and if $a+c>2b$, x^2 is a minimum, and zero when the equal rods are parallel; but if $a+c<2b$, x^2 is a minimum when the unequal rods are parallel and the equal rods crossed.

If y be the distance between the middle points of the equal rods,

$$y^2=a^2-ab(\cos\theta+\cos\phi)+\frac{b^2}{2}\{1+\cos(\theta+\phi)\};$$

therefore $y^2-x^2=\dfrac{a^2+c^2}{2}-b^2$, and y^2 is a maximum or minimum when x^2 is a maximum or minimum.

vii. Prove that there are two ways of generating the same hypocycloid by the trace of a point on a circle rolling on a fixed circle which encloses it. And that in any position of the two rolling circles, which roll in opposite directions, a circle may be drawn concentric with the fixed circle so as to touch both rolling circles at points such that the line joining them is the tangent to the hypocycloid.

If any three of the tangents to a three-cusped hypocycloid form an equilateral triangle, prove that the angular points of the triangle will lie on a curve whose polar equation is $r = a\cos 3\theta$.

Let AQ (fig. 62) be the hypocycloid traced out by the point Q on the rolling circle PQp.

Produce PQ to meet the fixed circle in P', and let $P'O$ meet Qp in p', where O is the centre of the fixed circle.

The circle on $P'p'$ as diameter will pass through Q; and if C, C' be the centres of the circles PQp, $P'Qp'$, then, since the angles at P and P' are equal, $CQC'O$ is a parallelogram and $C'Q$ equal to OC.

Hence the arcs PQ, $P'Q$, PP' are similar, and as the radii PC, CO, PO.

But $$PC + CO = PO;$$

therefore arc PQ + arc $P'Q$ = arc PP', and the arc PQ being equal to the arc PA, therefore the arc $P'Q$ is equal to the arc $P'A$.

Thus, if the circle $P'Qp'$ roll on the fixed circle, it will generate the same hypocycloid AQ.

Also since $Op = Op'$, a circle whose centre is O will touch the two rolling circles in p, p', points on the tangent at Q.

[For further developments see Wolstenholme, *Proceedings of the London Mathematical Society*, vol. IV., p. 321.]

In the three-cusped hypocycloid, if $OC = 2a$, $CP = a$, and the angle $POA = \theta$,

$$x = 2a\cos\theta + a\cos 2\theta,$$

$$y = 2a\sin\theta - a\sin 2\theta,$$

and the equation of the tangent is

$$x\sin\frac{\theta}{2} + y\cos\frac{\theta}{2} = a\sin\frac{3\theta}{2};$$

therefore the perpendicular on the tangent being $a \sin \frac{3\theta}{2}$, three tangents forming an equilateral triangle are equidistant from the centre; and if r, ϕ be the polar coordinates of an angular point of the equilateral triangle

$$\phi = \frac{\pi}{2} - \frac{\theta}{2} - \frac{\pi}{3} = \frac{\pi}{6} - \frac{\theta}{2}, \quad \frac{3\theta}{2} = \frac{\pi}{2} - 3\phi,$$

and $$r = 2a \sin \frac{3\theta}{2} = 2a \cos 3\phi,$$

the required equation of the locus.

viii. Integrate the differential equation

$$\frac{dy}{dx} + Py = Q,$$

in the case where P and Q are functions of x only.

Prove that the variables in the differential equation

$$\frac{dy}{dx} = \frac{y(x+y) + b^2}{x(x+y) + a^2}$$

may be separated by the substitutions, $x = u + v$, $y = ku - v$, provided the constant k be suitably chosen, and integrate the equation.

Making the substitution and reducing, the equation becomes

$$\{(1+k)^2 uv + ka^2 - b^2\} du = \{(1+k)^2 u^2 + a^2 + b^2\} dv.$$

If then $ka^2 = b^2$, the variables can be separated, and

$$\frac{(1+k)^2 udu}{(1+k)^2 u^2 + a^2 + b^2} = \frac{dv}{v};$$

therefore $$(1+k)^2 u^2 + a^2 + b^2 = Cv^2,$$

or $$(x+y)^2 + a^2 + b^2 = C'(b^2x - a^2y)^2.$$

ix. Prove that in the expansion of $\dfrac{t}{e^t-1}$ in ascending powers of t no odd powers except the first appear, and if $(-1)^{n+1}\dfrac{B_{2n-1}}{\lfloor\underline{2n}}$ be the coefficient of t^{2n},

$$B_{2n-1} = (-1)^{n+1}\left(\frac{\Delta}{1^2} - \frac{\Delta^2}{2^2} + \frac{\Delta^3}{3^2} - \text{\&c.}\right)0^{2n+1}.$$

Prove that

$$B_1\frac{\pi^2}{\lfloor\underline{3}} + B_3\frac{\pi^4}{\lfloor\underline{5}} + B_5\frac{\pi^6}{\lfloor\underline{7}} + \text{\&c.} = 1 - \log 2.$$

Since $$\frac{t}{e^t-1} = 1 - \frac{t}{2} + B_1\frac{t^2}{\lfloor\underline{2}} - B_3\frac{t^4}{\lfloor\underline{4}} + \ldots;$$

therefore $$\frac{i\theta}{\cos\theta + i\sin\theta - 1} = 1 - \frac{i\theta}{2} - B_1\frac{\theta^2}{\lfloor\underline{2}} - B_3\frac{\theta^4}{\lfloor\underline{4}} - \ldots,$$

equating the real parts

$$\frac{\theta}{2}\cot\frac{\theta}{2} = 1 - B_1\frac{\theta^2}{\lfloor\underline{2}} - B_3\frac{\theta_4}{\lfloor\underline{4}} - \ldots,$$

$$\int_0^\pi \frac{\theta}{2}\cot\frac{\theta}{2}\,d\theta = \pi - B_1\frac{\pi^3}{\lfloor\underline{3}} - B_3\frac{\pi^5}{\lfloor\underline{5}} - \ldots,$$

and $$\int_0^\pi \frac{\theta}{2}\cot\frac{\theta}{2}\,d\theta = 2\int_0^{\frac{\pi}{2}} \phi\cot\phi\,d\phi$$

$$= 2\,(\phi\log\sin\phi)_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}}\log\sin\phi\,d\phi = \pi\log 2;$$

therefore $$B_1\frac{\pi^2}{\lfloor\underline{3}} + B_3\frac{\pi^4}{\lfloor\underline{5}} + \ldots = 1 - \log 2.$$

TUESDAY, *Jan.* 19, 1875. 9 to 12.

MR. COCKSHOTT.

1. IF A, B be two fixed points and any plane be drawn through AB meeting a fixed plane conic in Q and R, prove that the locus of the point of intersection of AQ and BR will be another fixed plane conic.

The quadric cones with vertices A and B having as common plane section the given conic will intersect in another plane conic, on which AQ and BR will intersect, as also AR and BQ.

2. Prove that, if α, β, γ, δ are all different,

$$\frac{\cos 2\alpha}{\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}}+\frac{\cos 2\beta}{\sin\frac{\beta-\alpha}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\beta-\delta}{2}}$$

$$+\frac{\cos 2\gamma}{\sin\frac{\gamma-\alpha}{2}\sin\frac{\gamma-\beta}{2}\sin\frac{\gamma-\delta}{2}}+\frac{\cos 2\delta}{\sin\frac{\delta-\alpha}{2}\sin\frac{\delta-\beta}{2}\sin\frac{\delta-\gamma}{2}}$$

$$=8\sin\frac{\alpha+\beta+\gamma+\delta}{2}.$$

In the algebraical identity

$$\frac{a^3}{(a-b)(a-c)(a-d)}+\frac{b^3}{(b-c)(b-d)(b-a)}$$

$$+\frac{c^3}{(c-d)(c-a)(c-b)}+\frac{d^3}{(d-a)(d-b)(d-c)}=1,$$

putting $a=\cos\alpha+i\sin\alpha$, $b=\ldots$,

$$a-b=2i\sin\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha+\beta}{2}+i\sin\frac{\alpha+\beta}{2}\right),$$

and the first term of the identity becomes

$$\frac{i}{8\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}}\cdot\frac{\cos3\alpha+i\sin3\alpha}{\cos\frac{3\alpha+\beta+\gamma+\delta}{2}+i\sin\frac{3\alpha+\beta+\gamma+\delta}{2}}$$

$$=\frac{i}{8\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}}\cdot\frac{\cos2\alpha+i\sin2\alpha}{\cos\frac{\alpha+\beta+\gamma+\delta}{2}+i\sin\frac{\alpha+\beta+\gamma+\delta}{2}};$$

therefore

$$\frac{\cos2\alpha}{\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}}+\ldots=\ 8\sin\frac{\alpha+\beta+\gamma+\delta}{2},$$

$$\frac{\sin2\alpha}{\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}}+\ldots=-8\cos\frac{\alpha+\beta+\gamma+\delta}{2}.$$

3. If $AB'CA'BC'$ be a regular hexagon, prove that three rectangular hyperbolas can be described, the first touching AB, AC at B, C and touching $A'B'$, $A'C'$ at B', C', the second touching BC, BA at C, A, and $B'C'$, $B'A'$ at C', A', and the third touching CA, CB at A, B, and $C'A'$, $C'B'$ at A', B'; and that any one of the three is the polar reciprocal of the second with respect to the third.

Prove also that an infinite number of triangles can be described, each of which is self-conjugate to one hyperbola, whose sides touch the second, and whose angular points lie on the third.

Taking ABC or $A'B'C'$ as the triangle of reference, the equations of the three rectangular hyperbolas will be

$$x^2+2yz=0\ldots(1),\quad y^2+2zx=0\ldots(2),\quad z^2+2xy=0\ldots(3).$$

If the line (lmn) touch (1), $l^2+2mn=0$; and if (xyz) be its pole with respect to (2), $\frac{l}{z}=\frac{m}{y}=\frac{n}{x}$; therefore $z^2+2xy=0$.

Let $(l_1m_1n_1)$, $(l_2m_2n_2)$, $(l_3m_3n_3)$ be the sides of a triangle inscribed in (1); therefore

$$l_1^2+2m_1n_1=0,\quad l_2^2+2m_2n_2=0,\quad l_3^2+2m_3n_3=0.$$

If possible, let the triangle be self-conjugate to (2); the conditions for this are

$$m_2m_3+n_2l_3+n_3l_2=0,$$
$$m_3m_1+n_3l_1+n_1l_3=0,$$
$$m_1m_2+n_1l_2+n_2l_1=0,$$

and eliminating n_1, n_2, n_3,

$$m_2m_3-\frac{l_2^2l_3}{2m_2}-\frac{l_2l_3^2}{2m_3}=0,\quad\text{or}\quad \frac{l_2l_3}{m_2m_3}\left(\frac{l_2}{m_2}+\frac{l_3}{m_3}\right)=2,$$

with two similar equations.

But the three equations

$$\mu\nu(\mu+\nu)=\nu\lambda(\nu+\lambda)=\lambda\mu(\lambda+\mu)=2$$

are only equivalent to the two independent equations $\lambda+\mu+\nu=0$, $\lambda\mu\nu=2$; hence, there is an infinite number of triangles circumscribing (1), and self-conjugate to (2).

Also, any such triangle is inscribed in (3); for the conditions for this are

$$(l_2m_3-l_3m_2)^2+2(m_2n_3-m_3n_2)(n_2l_3-n_3l_2)=0,\ \ldots,$$

and eliminating n_1, n_2, n_3,

$$(l_2m_3-l_3m_2)^2-\tfrac{1}{2}(l_2^2m_3^2-l_3^2m_2^2)(l_2m_3-l_3m_2)\frac{l_2l_3}{m_2^2m_3^2}=0,\ \ldots,$$

or $$2=(l_2m_3+l_3m_2)\frac{l_2l_3}{m_2^2m_3^2}=\left(\frac{l_2}{m_2}+\frac{l_3}{m_3}\right)\frac{l_2l_3}{m_2m_3}=\ldots,$$

which have already been obtained.

Hence there is a singly infinite series of triangle whose sides touch (1), which are self-conjugate to (2), and whose angular points lie on (3); and by taking the hyperbolas in different order six such series of triangles can be formed.

4. Prove that

$$2\begin{vmatrix} \alpha^6, & \alpha^3, & \alpha^2, & \alpha, & 1 \\ \beta^6, & \beta^3, & \beta^2, & \beta, & 1 \\ \gamma^6, & \gamma^3, & \gamma^2, & \gamma, & 1 \\ \delta^6, & \delta^3, & \delta^2, & \delta, & 1 \\ \varepsilon^6, & \varepsilon^3, & \varepsilon^2, & \varepsilon, & 1 \end{vmatrix} - 3\begin{vmatrix} \alpha^5, & \alpha^4, & \alpha^2, & \alpha, & 1 \\ \beta^5, & \beta^4, & \beta^2, & \beta, & 1 \\ \gamma^5, & \gamma^4, & \gamma^2, & \gamma, & 1 \\ \delta^5, & \delta^4, & \delta^2, & \delta, & 1 \\ \varepsilon^5, & \varepsilon^4, & \varepsilon^2, & \varepsilon, & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)(\beta-\gamma)\ldots\{(\alpha-\beta)^2+(\alpha-\gamma)^2 \text{ \&c.}\},$$

that is half the product of all the differences of any two $\times$ the sum of the squares of the differences of any two.

Calling the determinants A and B,

$$A = \alpha^6(\beta-\gamma)(\beta-\delta)(\beta-\varepsilon)(\gamma-\delta)(\gamma-\varepsilon)(\delta-\varepsilon)+\ldots,$$

$$B = \alpha^5(\beta+\gamma+\delta+\varepsilon)(\beta-\gamma)(\beta-\delta)(\beta-\varepsilon)(\gamma-\delta)(\gamma-\varepsilon)(\delta-\varepsilon)+\ldots;$$

therefore $\quad A+B = (\alpha+\beta+\gamma+\delta+\varepsilon)\{\alpha^5(\beta-\gamma)(\beta-\delta)\ldots\}$

$$= (\alpha+\beta+\gamma+\delta+\varepsilon)^2 \text{ (product of differences)},$$

and since B does not contain $\alpha^6, \beta^6, \ldots$;

therefore $\quad B = (\alpha\beta+\alpha\gamma+\ldots)$ (product of the differences);

therefore $\quad A = (\alpha^2+\ldots+\alpha\beta+\ldots)$ (product of the differences);

$$2A-3B = (2\alpha^2+\ldots-\alpha\beta-\ldots) \text{ (product of the differences)}$$

$$= \tfrac{1}{2}(\alpha-\beta)(\alpha-\gamma)\ldots\{(\alpha-\beta)^2+(\alpha-\gamma)^2+\ldots\}.$$

5. Prove that all tangent planes to an anchor-ring which pass through the centre of the ring cut the surface in two circles.

Also if a surface be generated by the revolution of any conic section about an axis in its own plane, prove that a double tangent plane cuts the surface in two conic sections.

The equation of the anchor ring in cylindrical coordinates r, θ, z, is

$$(r-a)^2+z^2=b^2,$$

which rationalized in rectangular coordinates x, y, z, becomes

$$(x^2+y^2+z^2+a^2-b^2)^2=4a^2(x^2+y^2),$$

if the axis of the surface be taken as the axis of z, and if b be the radius of the generating circle, a the distance of its centre from the axis.

To find the section made by a tangent plane through the centre, change x into

$$x\cos\alpha-z\sin\alpha \text{ and } z \text{ into } x\sin\alpha+z\cos\alpha,$$

where $\sin\alpha=\dfrac{b}{a}$, and then put $z=0$.

Therefore the equation of the section is

$$(x^2+y^2+a^2-b^2)^2=4a^2(x^2\cos^2\alpha+y^2)$$

$$=4(a^2-b^2)x^2+4a^2y^2=4(a^2-b^2)(x^2+y^2)+4b^2y^2,$$

or

$$(x^2+y^2-a^2+b^2)^2=4b^2y^2,$$

$$x^2+(y\pm b)^2=a^2,$$

the equation of two circles.

If we give the surface a homogeneous strain, so that x is changed into kx, y into ky, z into $k'z$, the surface is deformed into the surface generated by the revolution of a conic section about an axis parallel to a principal axis of the conic section, and the double tangent plane cuts the new surface in two conic sections, intersecting at the points of contact.

In the general case if $x^2\sin^2\alpha-z^2\cos^2\alpha=0$ be the equation of the double tangent planes OT, OT' (fig. 64), the equation of the generating conic in the plane of xz will be of the form

$$(ax+bz-c^2)^2-d^2(x^2\sin^2\alpha-z^2\cos^2\alpha)=0,$$

where $ax+bz-c^2=0$ is the equation of TT', the chord of contact.

Therefore the equation in cylindrical coordinates of the ring generated by the revolution of the conic will be

$$(ar + bz - c)^2 - d^2 (r^2 \sin^2\alpha - z^2 \cos^2\alpha) = 0,$$

which rationalized in rectangular coordinates becomes

$$[a^2 (x^2 + y^2) + (bz - c^2)^2 - d^2 \{(x^2 + y^2) \sin^2\alpha - z^2 \cos^2\alpha\}]^2$$
$$= 4a^2 (x^2 + y^2) (bz - c^2)^2.$$

To find the section made by the plane OT, change x into $x \cos\alpha$, z into $x \sin\alpha$ as before; therefore the equation of the section is

$$\{a^2x^2 \cos^2\alpha + (bx \sin\alpha - c^2)^2 - (d^2 \sin^2\alpha - a^2) y^2\}^2$$
$$= 4a^2 (x^2 \cos^2\alpha + y^2) (bx \sin\alpha - c^2)^2.$$

Solving this quadratic in y^2,

$$y^2 (d^2 \sin^2\alpha - a^2)^2 =$$
$$[a \sqrt{\{(d^2 \sin^2\alpha - a^2)x^2 \cos^2\alpha + (bx \sin\alpha - c^2)^2\}} \pm d \sin\alpha (bx \sin\alpha - c^2)]^2,$$

or $\{y (d^2 \sin^2\alpha - a^2) \pm d \sin\alpha (bx \sin\alpha - c^2)\}^2$

$$= a^2 \{(d^2 \sin^2\alpha - a^2) x^2 \cos^2\alpha + (bx \sin\alpha - c^2)^2\},$$

the equation of two conics.

6. If ABC is an acute-angled triangle and D the intersection of the perpendiculars from the angles upon the opposite sides, shew that the four conic sections which can be described touching the sides of the triangles DBC, DCA, DBC, ABC respectively, and having one of their foci at A, B, C, D respectively, will have their transverse axes equal, their centres coincident, and the squares of their eccentricities equal to

$$1 + 8 \cos A \sin B \sin C,\quad 1 + 8 \cos B \sin C \sin A,$$

$$1 + 8 \cos C \sin A \sin B,\quad 1 - 8 \cos A \cos B \cos C,$$

respectively.

If E, F, G (fig. 63) be the feet of the perpendiculars, the circle circumscribing the triangle EFG, which is the nine-pointic circle of the triangle ABC, will be the common auxiliary circle of the conics, which will therefore have a common centre and transverse axes equal to R, the radius of the circumscribing circle of the triangle ABC.

If P be the centre of the nine-pointic circle, the eccentricities of the conics will be

$$\frac{2AP}{R}, \frac{2BP}{R}, \frac{2CP}{R}, \frac{2DP}{R}.$$

The $$\angle OAD = \frac{\pi}{2} - B - \frac{\pi}{2} + C = C - B,$$

and $$AD = 2R\cos A,$$

therefore $$4DP^2 = OD^2 = R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C-B)$$
$$= R^2(1 - 8\cos A\cos B\cos C),$$
$$4AP^2 = 2OA^2 + 2DA^2 - OD^2$$
$$= R^2(1 + 8\cos^2 A + 8\cos A\cos B\cos C)$$
$$= R^2(1 + 8\cos A\sin B\sin C),$$

similarly $$4BP^2 = R^2(1 + 8\cos B\sin C\sin A),$$
$$4CP^2 = R^2(1 + 8\cos C\sin A\sin B).$$

7. Along the normal at a point P of an ellipsoid is measured PQ of a length inversely proportional to the perpendicular from the centre on the tangent plane at P, prove that the locus of Q is another ellipsoid, and that the envelope of all such ellipsoids is the "surface of centres," that is the locus of the centres of principal curvature.

If $(x'y'z')$ be the coordinates of P, (xyz) of Q, $(\alpha\beta\gamma)$ the direction cosines of the normal at P, p the length of the perpendicular on the tangent plane at P,

$$\cos\alpha = \frac{px'}{a^2}, \quad \cos\beta = \frac{py'}{b^2}, \quad \cos\gamma = \frac{pz'}{c^2};$$

and if
$$PQ=\frac{k^2}{p},$$
$$x=x'\left(1+\frac{k^2}{a^2}\right),\ \frac{x'}{a}=\frac{ax}{a^2+k^2},\ \ldots;$$
therefore
$$\frac{a^2x^2}{(a^2+k^2)^2}+\frac{b^2y^2}{(b^2+k^2)^2}+\frac{c^2z^2}{(c^2+k^2)^2}=1.$$

Considering the line of intersection of two such surfaces, each point lies on two normals at consecutive points of the ellipsoid, and therefore the envelope is the locus of the points of ultimate intersection of the normals, which is the surface of centres.

8. Trace the curve
$$y=\frac{2}{\pi}\int_0^\infty \frac{\sin x\theta \sin^2\theta}{\theta^3}.d\theta.$$

$$\frac{dy}{dx}=\frac{2}{\pi}\int_0^\infty \frac{\cos x\theta \sin^2\theta d\theta}{\theta^2}$$
$$=\frac{1}{2\pi}\int_0^\infty \frac{2\cos x\theta-\cos(x+2)\theta-\cos(x-2)\theta}{\theta^2}d\theta.$$

Now $\displaystyle\int_0^\infty \frac{\cos r\theta}{\theta^2}d\theta=-\frac{\pi r}{2}$ if r is positive,

$$=\ \frac{\pi r}{2} \text{ if } r \text{ is negative.}$$

Therefore, if $2>x>0$,
$$\frac{dy}{dx}=\tfrac{1}{4}(-2x+2+x+2-x)=1-\frac{x}{2},$$
and therefore $y=x-\dfrac{x^2}{4}$, the curve OL (fig. 65).

If $x>2$, $\dfrac{dy}{dx}=\tfrac{1}{4}(-2x+x+2+x-2)=0,$

and therefore y is constant $=1$.

The origin is a centre of the curve.

9. The value of a diamond being proportional to the square of its weight, prove that if a diamond be broken into three pieces, the mean value of the three pieces together is $\frac{1}{2}$ of the value of the whole diamond.

If a be the value of the whole diamond, the mean value of the three pieces is

$$a\frac{\int_0^1\int_0^{1-y}\{x^2+y^2+(1-x-y)^2\}\,dx\,dy}{\int_0^1\int_0^{1-y}dx\,dy}=\frac{a}{2}.$$

10. Find the orbit described by a particle moving under the action of a central force $\mu\{2(a^2+b^2)u^5-3a^2b^2u^7\}$, if it be projected at a distance a with velocity $\frac{\sqrt{\mu}}{a}$ in a direction at right angles to the radius vector.

$$\frac{d^2u}{d\theta^2}+u=2(a^2+b^2)u^3-3a^2b^2u^5,$$

$$\left(\frac{du}{d\theta}\right)^2+u^2=(a^2+b^2)u^4-a^2b^2u^6,$$

$$\left(\frac{du}{d\theta}\right)^2=u^2(a^2u^2-1)(1-b^2u^2),$$

$$\theta=\int_{\frac{1}{a}}^{u}\frac{du}{u\sqrt{\{(a^2u^2-1)(1-b^2u^2)\}}},$$

$$2\theta=\int_r^a\frac{dr^2}{\sqrt{\{(a^2-r^2)(r^2-b^2)\}}}$$

$$=\int_r^a\frac{dr^2}{\sqrt{\left\{\left(\frac{a^2-b^2}{2}\right)^2-\left(r^2-\frac{a^2+b^2}{2}\right)^2\right\}}}$$

$$=\cos^{-1}\frac{r^2-\frac{a^2+b^2}{2}}{\frac{a^2-b^2}{2}},$$

$$r^2=\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\cos 2\theta=a^2\cos^2\theta+b^2\sin^2\theta,$$

the pedal of an ellipse.

11. If a river is flowing due north, prove that the pressure on the eastern bank at a depth z is increased by the change of latitude of the running water in the ratio $gz + bv\omega \sin l : gz$, where b is the breadth of the stream, v its velocity, l the latitude, and ω the angular velocity of the earth about its axis.

The time of describing δx being $\frac{\delta x}{v}$, and the change of velocity $\delta (R\omega \cos l) = - R\omega \sin l \delta l$, the acceleration westward is

$$Rv\omega \sin l \frac{dl}{dx} = v\omega \sin l.$$

Therefore the additional pressure on the eastern bank is $\rho b v\omega \sin l$.

12. If the moon is seen in the form of a bright semicircle with its diameter vertical at a place in latitude 45°, when the sun is in the summer solstice, and the obliquity of the ecliptic be considered to be $22\frac{1}{2}°$, prove that the time of night is $6 + \frac{12}{\pi} \sin^{-1} \{\sqrt{(2)} - 1\}$ hours before or after midnight, and the azimuth of the moon is $\cos^{-1} \frac{1}{\sqrt[4]{(2)}}$: the moon being supposed to move in the ecliptic.

Let P be the pole, Z the zenith, S the sun, M the moon.

Then since $SM = \angle SMZ = 90°$; therefore $SZ = 90°$, and the sun is on the horizon.

If λ be the latitude, ω the obliquity of the ecliptic, then $PS = 90° - \omega$, $PZ = 90° - \lambda$; $\cos SPZ = - \tan\omega \tan\lambda$, and the time of night is $6 + \frac{12}{\pi} \sin^{-1} (\tan\omega \tan\lambda)$ hours before or after midnight.

If α be the moon's azimuth measured from the south point, $\cos\alpha = - \cos PZM = \tan\lambda \cot ZM$; but $ZM = \angle ZSM = \angle PSN$, therefore $\sin ZM = \frac{\sin\lambda}{\cos\omega}$, and $\cos\alpha = \sqrt{(\cos^2\omega - \sin^2\omega \tan^2\lambda)}$.

If $\lambda = 45°$, $\omega = 22\frac{1}{2}°$, then $\tan\lambda = 1$, $\tan\omega = \sqrt{(2)} - 1$, and $\cos\alpha = \sqrt{(\cos 2\omega)} = \frac{1}{\sqrt[4]{(2)}}$.

13. Two pieces of similar uniform rigid wire in the shape of a semicircle and its diameter are joined together at their ends by two pins, and the whole is set in rotation in its own plane, which is vertical, about an axis through one pin; find the stress at the other pin, and shew that it attains its maximum and minimum values when the inclination of the diameter to the vertical is $\tan^{-1}\frac{\pi}{4}$, and the semicircle meets the vertical through the fixed axis, supposing the rotations about the axis are complete. If the rotations are not complete, when does the stress attain its maximum and minimum values?

If m be the mass of the diameter, M of the semicirole, θ the inclination of the diameter to the vertical, and F the stress at the pin, the equations of motion are

$$m\frac{4a^2}{3}\cdot\frac{d^2\theta}{dt^2} = -mga\sin\theta + F.2a,$$

$$M.2a^2\frac{d^2\theta}{dt^2} = -Mg\left(a\sin\theta - \frac{2a}{\pi}\cos\theta\right) - F.2a,$$

and $m = \mu.2a$, $M = \mu.\pi a$; eliminating $\frac{d^2\theta}{dt^2}$,

$$F = \frac{\mu ga}{3\pi + 4}(\pi\sin\theta + 4\cos\theta) = \frac{\mu ga\sqrt{(\pi^2 + 16)}}{3\pi + 4}\cos(\theta - \alpha),$$

where $\tan\alpha = \frac{1}{4}\pi$; therefore F is a maximum when $\theta = \alpha$, a minimum when $\theta = \alpha + \pi$; but if the angle swung through does not attain this value, the maximum value of F will be at an instant of rest, because $\frac{dF}{d\theta}$ is positive.

14. A sphere moves on the concave side of a rough cylindrical surface of which the transverse section perpendicular to the generating lines is a hypocycloid.

The sphere is rolling initially with a velocity V along the generating line at which the curvatures of the sphere and cylinder are equal, and the sphere is acted upon during the motion solely by a force tending to and equal to μ times the distance of its centre from the nearest point of the central axis of the cylinder.

Prove that the motion of the sphere will be comprised within a length

$$\frac{14V}{a-2b}\sqrt{\left\{\frac{2b}{5\mu}(a-b)\right\}}$$

of the cylinder and that the time between successive instants of the sphere reaching the original generating line on which it was projected is

$$\frac{4\pi}{a}\sqrt{\left\{\frac{7b}{5\mu}(a-b)\right\}},$$

where b is the radius of the rolling circle which generates the hypocycloid by rolling on a circle of radius a.

If $NOM=\theta$, $XOM=\phi$ (fig. 66), then

$$x = OM = (a-b)\cos\phi - b\cos\left(\frac{a}{b}-1\right)\phi,$$

$$y = MP = (a-b)\sin\phi + b\sin\left(\frac{a}{b}-1\right)\phi,$$

$$\frac{dy}{dx} = \cot\theta = \frac{\cos\phi + \cos\left(\frac{a}{b}-1\right)\phi}{-\sin\phi + \sin\left(\frac{a}{b}-1\right)\phi} = \cot\left(\frac{a}{2b}-1\right)\phi;$$

therefore $\theta = \left(\frac{a}{2b}-1\right)\phi$ and $XON = \theta + \phi = \dfrac{a\theta}{a-2b}$;

$$PX = ON\sin(\theta+\phi) = a\sin\frac{a\theta}{a-2b};$$

also
$$\frac{ds}{d\phi} = 2(a-b)\cos\frac{a\phi}{2b},$$

$$s = 4\frac{b}{a}(a-b)\sin\frac{a\phi}{2b} = 4\frac{b}{a}(a-b)\sin\frac{a\theta}{a-2b}.$$

Therefore, if $s = l \sin n\theta$ be the intrinsic equation of the hypocycloid, then

$$n = \frac{a}{a-2b}, \text{ and } l = 4\frac{b}{a}(a-b).$$

With the usual notation (fig. 67), $\theta_1 = 0$, $\theta_2 = -\frac{d\theta}{dt}$, $\theta_3 = 0$; and the equations of motion of the sphere are, c being the radius of the sphere

$$\frac{d\omega_1}{dt} - \omega_3 \frac{d\theta}{dt} = \frac{F'c}{k^2} \quad \text{.................(1)},$$

$$\frac{d\omega_2}{dt} = -\frac{Fc}{k^2} \quad \text{.................(2)},$$

$$\frac{d\omega_3}{dt} + \omega_1 \frac{d\theta}{dt} = 0 \quad \text{.................(3)},$$

$$\frac{du}{dt} - w\frac{d\theta}{dt} = F - \mu.PX \quad \text{.........(4)},$$

$$\frac{dv}{dt} = F' \quad \text{.................(5)},$$

$$\frac{dw}{dt} + u\frac{d\theta}{dt} = R - \mu.X'0 \quad \text{.........(6)}.$$

Since the point of the sphere which is in contact with the hypocycloid is instantaneously at rest,

$$u - c\omega_2 = 0, \quad v + c\omega_1 = 0, \quad w = 0;$$

also $u = (\rho - c)\frac{d\theta}{dt}$, where ρ is the radius of curvature of the hypocycloid.

Therefore, by (2) and (4),

$$\frac{du}{dt} = -\frac{Fc^2}{k^2} = -\frac{c^2}{k^2}\left(\frac{du}{dt} + \mu.PX\right),$$

$$\left(1 + \frac{k^2}{c^2}\right)\frac{du}{dt} = -\mu.PX = -\mu a \sin n\theta.$$

If initially $\theta = \alpha$, then $c = nl \cos n\alpha$, and

$$u = (\rho - c)\frac{d\theta}{dt} = nl(\cos n\theta - \cos n\alpha)\frac{d\theta}{dt}.$$

Therefore

$$nl\left(1 + \frac{k^2}{c^2}\right)\frac{d}{dt}\left\{(\cos n\theta - \cos n\alpha)\frac{d\theta}{dt}\right\} = -\mu a \sin n\theta.$$

Multiplying by $2(\cos n\theta - \cos n\alpha)\dfrac{d\theta}{dt}$ and integrating,

$$nl\left(1 + \frac{k^2}{c^2}\right)\left\{(\cos n\theta - \cos n\alpha)\frac{d\theta}{dt}\right\}^2 = \frac{\mu a}{n}(\cos n\theta - \cos n\alpha)^2;$$

therefore $$\left(\frac{d\theta}{dt}\right)^2 = \frac{\mu a}{n^2 l}\cdot\frac{c^2}{c^2 + k^2} = \tfrac{5}{7}\mu\,\frac{(a-2b)^2}{4b(a-b)}.$$

Now $s = l \sin n\theta$; therefore

$$\frac{d^2 s}{dt^2} = -n^2 l \sin n\theta\left(\frac{d\theta}{dt}\right)^2 = -\tfrac{5}{7}\mu\,\frac{a^2 s}{4b(a-b)},$$

and therefore the point P on the hypocycloid oscillates harmonically with a period

$$\frac{4\pi}{a}\sqrt{\left\{\frac{7b}{5\mu}(a-b)\right\}}.$$

Also from equations (1) and (5),

$$-\frac{dv}{dt} - c\omega_3\frac{d\theta}{dt} = \frac{F'c^2}{k^2} = \frac{c^2}{k^2}\cdot\frac{dv}{dt},$$

$$\left(1 + \frac{c^2}{k^2}\right)\frac{dv}{dt} + c\omega_3\frac{d\theta}{dt} = 0,$$

and from equation (3) $\dfrac{d\omega_3}{dt} - \dfrac{v}{c}\dfrac{d\theta}{dt} = 0$; therefore eliminating ω_3

$$\left(1 + \frac{c^2}{k^2}\right)\frac{d^2 v}{d^2 t} + v\left(\frac{d\theta}{dt}\right)^2 = 0,$$

$$\frac{d^2 v}{dt^2} = -\tfrac{2}{7}\left(\frac{d\theta}{dt}\right)^2 v = -\tfrac{10}{49}\mu\,\frac{(a-2b)^2}{4b(a-b)}\,v = -\lambda^2 v \text{ suppose};$$

therefore $v = V\cos\lambda t$, and if x be the length of the cylinder transversed in the time t, $x = \frac{V}{\lambda}\sin\lambda t$, and the motion will be comprised within a length of the cylinder

$$\frac{2V}{\lambda} = \frac{14V}{a-2b}\sqrt{\left\{\frac{2b}{5\mu}(a-b)\right\}}.$$

15. If the velocity function denoting the motion of a homogeneous liquid be $\frac{Ax^2+By^2+Bz^2}{r^5}$, prove that the lines of flow are plane curves of the form $r^2 = \pm c^2\sin^2\theta\cos\theta$. If also the force function be $\frac{9A^2}{8r^8}(4\cos^4\theta+\sin^4\theta)$, prove that a sheet of fluid started from the origin will return to it without the use of a containing envelope.

If $\phi = \frac{Ax^2+By^2+Bz^2}{r^5}$ be the velocity function of a liquid, it must be a solid harmonic of degree -3, and $Ax^2+By^2+Bz^2$ will be a solid harmonic of degree 2; therefore $A+2B=0$; this result can also be obtained by substituting in the equation of continuity.

Changing to polar coordinates, $\phi = \frac{B}{r^3}(\sin^2\theta - 2\cos^2\theta)$, and the equation of a line of flow in a meridian plane is

$$\frac{dr}{\frac{d\phi}{dr}} = \frac{rd\theta}{\frac{1}{r}\frac{d\phi}{d\theta}};$$

therefore $$\frac{2}{r}\cdot\frac{dr}{d\theta} = 2\cot\theta - \tan\theta,$$

$$r^2 = C\sin^2\theta\cos\theta.$$

If V be the potential

$$\frac{p}{\rho} + V + \frac{v^2}{2} = C,$$

and $$\tfrac{1}{2}v^2 = \tfrac{1}{2}\left\{\left(\frac{d\phi}{dr}\right)^2 + \left(\frac{d\phi}{rd\theta}\right)^2\right\}$$

$$= \frac{9B^2}{2r^8}\{(2\cos^2\theta - \sin^2\theta)^2 + 4\sin^2\theta\cos^2\theta\}$$

$$= \frac{9A^2}{8r^8}(4\cos^4\theta + \sin^4\theta);$$

therefore the given potential makes p constant and equal to the surrounding pressure, and no containing envelope is required; also the lines of flow passing through the origin, the liquid started from the origin will return to the origin.

16. A uniform elastic beam lies unstrained on a smooth horizontal table; prove that if one end of the beam be moved in the direction of the length of the beam with uniform acceleration f, the length of the beam will oscillate between $l\left(1 + \frac{lf}{l'g}\right)$ and l, where l is the unstretched length of the beam, and l' the unstretched length of a similar beam whose weight is the modulus of elasticity.

If s be the distance at the time t of the section of the beam originally at the distance x from the end which is moved with acceleration f, and T be the tension, λ the modulus of elasticity of the substance, and ρ the mass per unit of length of the beam, the equations of motion are

$$\rho\left(\frac{d^2s}{dt^2} - f\right) = \frac{dT}{dx}, \quad \frac{T}{\lambda} = \frac{ds}{dx} - 1;$$

therefore $$\frac{d^2s}{dt^2} - f = a^2\frac{d^2s}{dx^2}, \text{ if } a^2 = \frac{\lambda}{\rho}.$$

The solution of this equation is

$$s = \phi(x + at) + \psi(x - at) + \tfrac{1}{2}ft^2.$$

(i) When $t=0$, $\frac{ds}{dt}=0$, $\frac{ds}{dx}=1$ for values of x from 0 to l; therefore $\phi'(z)-\psi'(z)=0$, $\phi'(z)+\psi'(z)=1$ for values of z from 0 to l.

(ii) When $x=0$, $\frac{ds}{dt}=0$ for all values of t; therefore $\phi'(z)-\psi'(-z)+\frac{fz}{a^2}=0$ for all positive values of z.

(iii) When $x=l$, $\frac{ds}{dx}=1$ for all values of t; therefore $\phi'(l+z)+\psi'(l-z)=1$ for all positive values of z.

Therefore $\phi''(z)=\psi''(z)=0$ for values of z from 0 to l, and

$$\phi''(z)+\psi''(-z)+\frac{f}{a^2}=0,\quad \phi\ (l+z)=\psi''(l-z)$$

for all positive values of z.

Therefore

$$\{\phi''(z)\}_l^{2l}=\{\phi''(l+z)\}_0^l=\{\psi''(l-z)\}_0^l=0,$$

and since $\quad\phi''(2l+z)=\psi''(-z)=-\phi''(z)-\frac{f}{a^2};$

therefore $\quad\{\phi''(z)\}_{2l}^{4l}=-\frac{f}{a^2},\ \{\phi''(z)\}_{4l}^{6l}=0,\ \ldots.$

The relative acceleration of the ends of the beam

$$=a^2\phi''(l+at)+a^2\psi''(l-at)+f=2a^2\phi''(l+at)+f,$$

and is therefore f from $t=0$ to $t=\frac{l}{a}$, $-f$ from $t=\frac{l}{a}$ to $t=\frac{3l}{a}$, f from $t=\frac{3l}{a}$ to $t=\frac{5l}{a}$, and so on.

The length of the beam therefore oscillates between $+\frac{l^2f}{a^2}$ and l; and since $l^2g\rho=\lambda$; therefore $a^2=l^2g$.

TUESDAY, *Jan.* 19, 1875. $1\frac{1}{2}$ to 4.

MR. FREEMAN. Roman numbers.
MR. GREENHILL. Arabic numbers.

1. PROVE that the normals to an ellipse drawn through a given point meet the ellipse at its points of intersection with a rectangular hyperbola which passes through the centre and the given point.

If the normals drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from any point on the normal at h, k, meet the ellipse in P, Q, R, prove that the sides of the triangle PQR will touch the parabola

$$\left(\frac{xh}{a^2}+\frac{yk}{b^2}+1\right)^2=\frac{4hkxy}{a^2b^2}.$$

The equation of the normal at (xy) is

$$a^2xY-b^2yX=(a^2-b^2)\,xy \;\ldots\ldots\ldots\ldots\; (1),$$

and if (XY) be considered constant and (xy) variable, this is the equation of a rectangular hyperbola passing through the centre of the ellipse, through (XY) and through the points where the normals drawn from (XY) meet the ellipse.

If $\frac{lx}{a}+\frac{my}{b}=1$ and $\frac{l'x}{a}+\frac{m'y}{b}=1$ be the equations of two of the lines joining the four points at which the normals meet the ellipse, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{lx}{a}+\frac{my}{b}-1\right)\left(\frac{l'x}{a}+\frac{m'y}{b}-1\right)=0$$

can be made to coincide with (1).

Therefore $\lambda=1$, $ll'+1=0$, $mm'+1=0$; and therefore the normals at the points where the ellipse is met by the straight lines

$$\frac{lx}{a}+\frac{my}{b}=1 \text{ and } \frac{x}{al}+\frac{y}{bm}=-1$$

meet in a point (Wolstenholme, *Mathematical Problems*, p. 119).

If the first pass through (hk), then the equation of the second becomes

$$l^2\frac{h}{a}+l\left(\frac{xh}{a^2}-\frac{yk}{b^2}-1\right)=\frac{x}{a}=0,$$

the envelope of which is

$$\left(\frac{xh}{a^2}-\frac{yk}{b^2}-1\right)^2+\frac{4xh}{a^2}=0,$$

which reduces to the given result.

(It is not necessary for h, k to lie on the ellipse).

2. Find the relations between the coordinates of the ends of three conjugate diameters of an ellipsoid.

Corresponding points on an ellipsoid of semi-axes a, b, c and a sphere of radius r being defined by

$$\frac{x}{a}=\frac{x'}{r},\ \frac{y}{b}=\frac{y'}{r},\ \frac{z}{c}=\frac{z'}{r},$$

then if OP and Op be corresponding radii of the ellipsoid and the sphere, Oq and Or any two radii of the sphere perpendicular to OP, prove that Op will be perpendicular to OQ and OR, the radii of the ellipsoid corresponding to Oq and Or.

Let

$$(x_1y_1z_1),\ (x_2y_2z_2),\ (x_3y_3z_3),\ (x_1'y_1'z_1'),\ (x_2'y_2'z_2'),\ (x_3'y_3'z_3'),$$

be the coordinates of the points P, Q, R, p, q, r respectively.

Then $\quad \frac{x_1}{a}=\frac{x_1'}{r}=\ldots$; and therefore $x_1x_2'=x_1'x_2, \ldots$.

If Oq and Or are perpendicular to OP, then

$$x_1x_2'+y_1y_2'+z_1z_2'=0,$$

$$x_1x_3'+y_1y_3'+z_1z_3'=0,$$

therefore

$$x_1'x_2+y_1'y_2+z_1'z_2=0,$$

$$x_1'x_3+y_1'y_3+z_1'z_3=0,$$

and therefore Op is perpendicular to OQ and OR.

(This is the geometrical interpretation of Hamilton's solution of the linear and vector equation when the function is self-conjugate).

3. If $\rho_1=\phi_1(x, y, z)$, $\rho_2=\phi_2(x, y, z)$, $\rho_3=\phi_3(x, y, z)$, shew how to change the independent variables in partial differential coefficients of any function from x, y, z, to ρ_1, ρ_2, ρ_3.

If ρ_1, ρ_2, ρ_3, denote a system of orthogonal surfaces, prove that

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}$$

$$=h_1h_2h_3\left\{\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}\right)+\frac{d}{d\rho_2}\left(\frac{h_2}{h_3h_1}\frac{dV}{d\rho_2}\right)+\frac{d}{d\rho_3}\left(\frac{h_3}{h_1h_2}\frac{dV}{d\rho_3}\right)\right\},$$

where $h_1^2=\left(\frac{d\rho_1}{dx}\right)^2+\left(\frac{d\rho_1}{dy}\right)^2+\left(\frac{d\rho_1}{dz}\right)^2$ and h_2, h_3 are similarly formed from ρ_2 and ρ_3.

Deduce the transformation for the usual polar coordinates r, θ, ϕ,

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}$$

$$=\frac{1}{r^2\sin\theta}\left\{\frac{d}{dr}\left(r^2\sin\theta\frac{dV}{dr}\right)+\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right)+\frac{d}{d\phi}\left(\frac{1}{\sin\theta}\frac{dV}{d\phi}\right)\right\}.$$

The concentration of V due to the pair of faces perpendicular to ds_1 in the element of space $ds_1ds_2ds_3$ formed by the orthogonal surfaces, is

$$-\frac{d}{ds_1}\left(ds_2ds_3\frac{dV}{ds_1}\right)=-d\rho_1d\rho_2d\rho_3\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}\right),$$

since $$ds_1 = \frac{d\rho_1}{h_1}, \quad ds_2 = \frac{d\rho_2}{h_2}, \quad ds_3 = \frac{d\rho_3}{h_3}.$$

If the axes of x, y, z be taken perpendicular to the surfaces ρ_1, ρ_2, ρ_3 respectively, then $ds_1 = dx$, $ds_2 = dy$, $ds_3 = dz$, and the same concentration of V due to the pair of faces perpendicular to dx is $-dx\,dy\,dz\,\dfrac{d^2V}{dx^2}$.

Therefore $$\frac{d^2V}{dx^2} = h_1h_2h_3\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}\right),$$

and $$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}$$

being an invariant for rectangular axes, by addition the required transformation is obtained.

If $d\sigma$ be the diagonal of the elementary parallelepiped $ds_1ds_2ds_3$,

$$d\sigma^2 = \frac{d\rho_1^2}{h_1^2} + \frac{d\rho_2^2}{h_2^2} + \frac{d\rho_3^2}{h_3^2}.$$

With the usual polar coordinates

$$d\sigma^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2;$$

therefore $h_1 = 1$, $h_2 = \dfrac{1}{r}$, $h_3 = \dfrac{1}{r\sin\theta}$, whence the result.

4. Prove that any system of forces can be reduced to a wrench, that is a force P along a certain axis and a couple Pa about that axis.

Prove that the wrench will not affect the equilibrium of a body moveable only about a screw of which the pitch for unit angle is b, if

$$(a + b)\cos\theta - d\sin\theta = 0,$$

where θ is the angle, and d is the shortest distance between the axes of the screw and the wrench.

If we take any arbitrary origin, the system of forces can be reduced to a single force acting at this origin and a couple.

Resolving the couple into two components, one parallel and the other perpendicular to the force, the resultant of the force and the couple of which the plane is parallel to the force is an equal and parallel force, and the system is thus equivalent to a single force, and a couple in a plane perpendicular to the force; such a force and couple constitute a wrench.

If the body be turned through a small angle β, the work done by the couple of the wrench is $\beta Pa\cos\theta$, and by the force of the wrench is $\beta Pb\cos\theta$ and $-\beta Pd\sin\theta$; hence the whole work done is

$$\beta P\{(a+b)\cos\theta - d\sin\theta\},$$

and the condition of equilibrium requires that this should vanish.

v. If x be eliminated between the equations

$$x^4+qx^2+rx+s=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

$$y=\lambda+\mu x+x^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$

prove that the coefficients of y^3 and y in the resulting biquadratic for y vanish when

$$2\lambda=q, \text{ and } r\mu^3+(4s-q^2)\mu^2-2qr\mu-r^2=0.$$

Apply this method to the solution of the equation $x^4+5x-6=0$, and explain why the four equations (β) corresponding to one value of μ give only one root of (α) for each solution of (β).

If s_1, s_2, ... denote the sums of the powers of the roots of the given equation

$$s_1=0,\ s_2=-2q,\ s_3=-3r,\ s_4=2q^2-4s,$$

$$s_5=qr \text{ and } s_6=3r^2+6qs-2q^3.$$

If σ_1, σ_2, ... denote the sums of the powers of the roots of the transformed equation in y, the coefficient of y^3 will vanish if $\sigma_1 = 0$, which gives $2\lambda = q$; and the coefficient of y will then become $-\frac{1}{3}\sigma_3$.

Now $\sigma_3 = \Sigma\{\mu x + (\lambda + x^2)\}^3 = \mu^3 s_3 + 3\mu^2(\lambda s_2 + s_4)$

$$+ 3\mu(\lambda^2 s_1 + 2\lambda s_3 + s_5) + 4\lambda^3 + 3\lambda^2 s_2 + 3\lambda s_4 + s_6,$$

and therefore substituting the values of $s_1, s_2, \ldots$, the coefficient of y in the transformed equation becomes

$$r\mu^3 + (4s - q^2)\mu^2 - 2qr\mu - r^2.$$

Applying the method to the given equation $x^4 + 5x - 6 = 0$, in this equation $\lambda = 0$, and the cubic in μ becomes

$$5\mu^3 - 24\mu^2 - 25 = 0,$$

which has one real root $\mu = 5$, which makes the biquadratic in y become

$$y^4 + 63y^2 - 3564 = 0.$$

The roots of this equation are ± 6 and $\pm 3\sqrt{(-11)}$, and the corresponding four pairs of values of x are

$$1, -6;\ -2, -3;\ \tfrac{1}{2}\{1 + \sqrt{(-11)}\},\ -\tfrac{1}{2}\{11 + \sqrt{(-11)}\};$$
$$\tfrac{1}{2}\{1 - \sqrt{(-11)}\},\ -\tfrac{1}{2}\{11 - \sqrt{(-11)}\};$$

and the first of each pair is a root of (α).

Each root of (α) gives one value of y; two different roots of (α) could not in general give the same value of y, without the other two also giving a value of y, equal and of opposite sign to the first y.

Hence, in general, one root of (β) is a root of (α); but if the quadratic in y^2 has equal roots, both roots of (β) are roots of (α).

vi. If a surface and a cone whose vertex is at the origin be referred to polar coordinates, the area of the cone between two of its generators and the curve in which it meets the surface is

$$\frac{1}{2}\int r^2\left\{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta.$$

The equations of a cylinder and cone are $r\sin\theta = a$, and $\cot\theta = \frac{1}{2}(e^{\phi} - e^{-\phi})$. If A_1, A_2, A_3 are the areas of the cone reckoned from $\phi = 0$ to $\phi = \beta - \alpha$, β, $\beta + \alpha$, respectively;

$$\frac{A_1 + A_3}{A_2} = e^{\alpha} + e^{-\alpha}.$$

In the rider $\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2 = 1$, and therefore the area

$$= \frac{a^2}{\sqrt{(2)}}\int \operatorname{cosec}^2\theta\, d\theta = \frac{a^2}{\sqrt{(2)}}\cot\theta = \frac{a^2}{2\sqrt{(2)}}(e^{\phi} - e^{-\phi});$$

between proper limits.

Therefore

$$A_1 = \frac{a^2}{2\sqrt{(2)}}(e^{\beta-\alpha} - e^{-\beta+\alpha}),\quad A_2 = \frac{a^2}{2\sqrt{(2)}}(e^{\beta} - e^{-\beta});$$

$$A_3 = \frac{a^2}{2\sqrt{(2)}}(e^{\beta+\alpha} - e^{-\beta-\alpha});$$

whence the result.

7. Explain the production of the focal lines of a small pencil of light which is obliquely reflected at a spherical surface, and calculate their positions.

If a pencil passing through two focal lines is incident at an angle ϕ on a small area of a reflecting surface of which r_1, r_2 are the principal radii of curvature, and the line of curvature corresponding to r_1 lies in the plane of incidence, and if u_1, u_2 are the distances of the focal lines from the surface; prove that the reflected pencil will pass through two focal lines at distances v_1, v_2 from the surface, where

$$\frac{1}{u_1} + \frac{1}{v_1} = \frac{2}{r_1\cos\phi},$$

$$\frac{1}{u_2} + \frac{1}{v_2} = \frac{2\cos\phi}{r_2},$$

provided that the focal lines at the distances u_1, v_1 from the surface be at right angles to the plane of incidence.

Let Q_1, Q_2 be the primary and secondary foci of the incident pencil, and first suppose the reflecting surface ABC plane (fig. 68).

The reflected pencil will be the image of the incident pencil seen by reflexion in the plane ABC, and if q_1, q_2 be the primary and secondary foci of the reflected pencil, then $q_1A = AQ_1 = u_1$, $q_2A = AQ_2 = v_2$.

Now let the plane ABC be bent into a cylinder of radius r_1, with generating lines perpendicular to the plane of incidence. The reflected rays in the primary plane q_1AB will pass through the primary focus q_1, and it may be proved, as in the bookwork, that $\frac{1}{u_1} + \frac{1}{v_1} = \frac{2}{r_1 \cos\phi}$; the reflected rays in the secondary plane will be unaltered.

Next let the plane ABC be bent into a cylinder of radius r_2 with generating lines parallel to the plane of incidence. The reflected rays in the primary plane will be unaltered, and the reflected rays in the secondary plane q_2AC will pass through the secondary focus q_2, and it may be proved, as in the bookwork, that $\frac{1}{u_2} + \frac{1}{v_2} = \frac{2\cos\phi}{r_2}$.

The superposition of these two cases will produce the general result.

viii. Find the position of the centre and focal centres of a lens whose refractive index, curvatures, and thickness are known.

If through the focal centres planes be drawn at right angles to the axis of the lens, and if a straight line parallel and near to the axis meet these planes in the points p and q; any incident ray which passes through p and is inclined at a small angle to the axis, though it may not meet it, will after emergence pass through q.

Let the centre of the lens be taken as the origin O, and the axis of the lens the axis of z (fig. 69), meeting the lens in

A, A', and let C, C' be the centres of the spherical surfaces AP, $A'P'$.

Let $OA=a$, $OC=c$, $OA'=a'$, $OC'=c'$; then since the centre of the lens is a centre of similitude of the spherical surfaces, $\frac{a}{c}=\frac{a'}{c'}$.

Let $SPP'Q'$ be the course of a ray which passes near the axis, meeting the planes through C, C' perpendicular to the axis in Q, R, Q', R'; and let the equations of PP', the refracted ray within the lens, be

$$x-h=\beta(z-a),\quad y-k=\gamma(z-a);$$

then to the second order, considering the deviations of the ray from the axis of the first order, the surface AP may be considered coincident with the tangent plane at A, and the equations of the incident ray SP will be of the form

$$x-h=\beta'(z-a),\quad y-k=\gamma'(z-a).$$

Since $\sin CPQ=\mu.\sin CPR$; therefore

$$\frac{CQ}{CR}=\mu\frac{\sin CRP}{\sin CQP}=\mu$$

to the second order; and, therefore, putting $z=c$,

$$h+\beta'(c-a)=\mu\{h+\beta(c-a)\},\quad k+\gamma'(c-a)=\mu\{k+\gamma(c-a)\},$$

or $$\beta'=\mu\beta+\frac{\mu-1}{c-a}h,\quad \gamma'=\mu\gamma+\frac{\mu-1}{c-a}k.$$

Similarly, if the equations of PP' be written

$$x-h'=\beta(z-a'),\quad y-k'=\gamma(z-a'),$$

and the equations of the emergent ray $P'Q'$

$$x-h'=\beta''(z-a'),\quad y-k'=\gamma''(z-a');$$

then $$\beta''=\mu\beta+\frac{\mu-1}{c'-a'}h',\quad \gamma''=\mu\gamma+\frac{\mu-1}{c'-a'}k'.$$

Putting $z=0$ in the equations of PP',

$$h-\beta a=h'-\beta a',\quad k-\gamma a=k'-\gamma a'$$

If m, n be the focal centres of the lens, and if $Om=u$, $On=u'$, then, since O and m are conjugate to A,

$$\frac{1}{c-a}+\frac{1}{a-u}=\frac{\mu}{a}+\frac{\mu}{c-a}, \text{ or } u=\frac{(\mu-1)\,ac}{\mu c-a};$$

and similarly $u'=\dfrac{(\mu-1)\,a'c'}{\mu c'-a'}$.

If the incident ray meet the focal plane through m in p, then at p

$$x=h+\left(\mu\beta+\frac{\mu-1}{c-a}.h\right)\left\{\frac{(\mu-1)\,ac}{\mu e-a}-a\right\}$$

$$=\frac{\mu c-\mu a}{\mu c-a}(h-\beta a).$$

Similarly, if the emergent ray meet the focal plane through n in q, then at q, $x=\dfrac{\mu c'-\mu a'}{\mu c'-a'}(h'-\beta a')$; and therefore the x's at p and q are equal; and in a similar way it may be proved that the y's are equal, which proves the proposition.

(Verdet *Œuvres*, tom. IY. p. 894; on Gauss' *Theory of Optical Instruments*).

ix. Establish the equation of vis viva for a system of bodies acting in any way on one another. What are the classes of mutual actions for which this equation does not hold in abstract dynamics?

An endless flexible and inextensible chain in which the mass for unit length is μ through one continuous half and μ' through the other half is stretched over two equal perfectly rough uniform circular discs (radius a and mass M) which can turn freely about their centres at a distance b in the same vertical line. Prove that the time of an oscillation of the chain under the action of gravity is

$$2\pi\sqrt{\left\{\frac{M+(\pi a+b)(\mu+\mu')}{2(\mu-\mu')g}\right\}}.$$

If the system be displaced from the position of stable equilibrium by turning the pullies through an angle θ, the equation of motion is

$$\left\{2M.\frac{a^2}{2} + (\pi a + b)(\mu + \mu')a^2\right\}\frac{d^2\theta}{dt^2} = -2g(\mu - \mu')a^2\theta,$$

or $$\{M + (\pi a + b)(\mu + \mu')\}\frac{d^2\theta}{dt^2} + 2g(\mu - \mu')\theta = 0,$$

an equation of harmonic motion, whence the time of oscillation.

x. Find the conditions of equilibrium of a fluid acted on by given forces, and prove that the resultant force at any point of a surface of equal pressure is normal to the surface and inversely as the density at the point and the distance to the consecutive surface of equal pressure.

A quantity of homogeneous fluid which completely fills a fixed rigid spherical shell (radius c) is under the action of such a system of forces that

$$\frac{p - \varpi}{\rho} = \frac{3(x^2 + y^2 + z^2)^2 - 5(x^4 + y^4 + z^4)}{a^2},$$

ϖ being the pressure at the centre, which is the origin of coordinates. Prove that the surfaces of equal pressure meet the four planes $x \pm y \pm z = 0$ in circles; and that the average pressure at the surface of the shell is equal to the pressure at the centre; determine also the least possible pressure at the centre.

If $$x \pm y \pm z = 0 \text{ .... } (1),$$

then $$x^4 + y^4 + z^4 = 2(y^2z^2 + z^2x^2 + x^2y^2)$$

or $$(x^2 + y^2 + z^2)^2 = 2(x^4 + y^4 + z^4),$$

therefore $$\frac{p - \varpi}{\rho} = \frac{1}{2a^2}(x^2 + y^2 + z^2)^2,$$

and the surfaces of equal pressure meet the planes (1) in circles.

The average value of $x^4+y^4+z^4$ over the surface of the sphere is three times the average value of x^4, and

$$\int x^4 dS = 4\pi c^6 \int_0^{\frac{1}{2}\pi} \cos^4\theta \sin\theta \, d\theta = \tfrac{4}{5}\pi c^6.$$

Therefore the average value of $3(x^2+y^2+z^2)^2 - 5(x^4+y^4+z^4)$ is zero, and the average value of p is ϖ.

The pressure is least at the points where the coordinate axes meet the sphere and $\dfrac{p-\varpi}{\rho} = -\dfrac{2c^4}{a^2}$; therefore ϖ cannot be less than $2\dfrac{c^4}{a^2}\rho$.

WEDNESDAY, *Jan.* 20, 1875. 9 to 12.

MR. WRIGHT.

1. IF PP', QQ' be diameters of an ellipse, and PR, PR' be let fall perpendiculars on $P'Q$, $P'Q'$, prove that the chord of the ellipse intercepted on the straight line RR' will subtend a right angle at P.

Let PR, PR' meet the conic in S, S'; then if SQ', QS' intersect in O, ROR' will be the Pascal line of the hexagon $PSQ'P'QS'P$ inscribed in the conic.

Since $PQP'Q'$ is a parallelogram, the chords SQ', QS' subtend a right angle at P, and therefore O is a fixed point on the normal at P.

Therefore the points where RR' meets the conic will subtend a right angle at P.

2. Prove that the determinants

$$\left\| \begin{matrix} 0 & 0 & 0 & a & b & c \\ 0 & 0 & z & a & b & 0 \\ 0 & y & 0 & a & 0 & c \\ x & 0 & 0 & 0 & b & c \\ x & y & z & 0 & 0 & 0 \end{matrix} \right\| = 0.$$

Adding the last row to the first row, and adding the second, third, and fourth rows to the last row, the determinants become

$$\begin{vmatrix} x, & y, & z, & a, & b, & c \\ 0, & 0, & z, & a, & b, & 0 \\ 0, & y, & 0, & a, & 0, & c \\ x, & 0, & 0, & 0, & b, & c \\ 2x, & 2y, & 2z, & 2a, & 2b, & 2c \end{vmatrix},$$

which vanish, because the first and last rows differ only by the factor 2.

3. If an ellipse U be described having the centre of a conic V for focus, and for axes the arithmetical and geometrical means of the axes of V, then of the common tangents to U and V, one is such that its point of contact with V lies on the auxiliary circle of U, and the line drawn through this point of contact parallel to the major-axis of V passes through one end of the major-axis of U, and the other three common tangents form a triangle whose angles are equidistant from the centre of V, and whose nine-pointic circle is the auxiliary circle of U.

If ACA', BCB' (fig. 70) be the axes of V, and LCL' the major-axis of U, C being a focus of U, then $CL = CB$, $CL' = CA$; and the auxiliary circle of U touches both the auxiliary circles of V.

If LP be drawn parallel to CA to meet the auxiliary circle of U in P and CB in N, then

$$PN : NL = L'C : CL = CA : CB,$$

and therefore P lies on V.

If the tangents to U at L, L' meet CB, CA respectively in T, T', then since

$$CT.CN = CL^2 = CB^2 \text{ and } CT'.NP = CL'^2 = CA^2;$$

therefore TT' is the tangent at P to V.

Since the angle TCT' is a right angle, therefore TT' is a tangent to U also, which proves the first part of the question.

If a circle be described with centre C and radius $CA+CB$, it is well known that there is an infinite series of triangles inscribed in the circle and touching V, and an infinite series of triangles inscribed in the circle and touching U.

Therefore the other three common tangents of U and V form a triangle which belongs to both series of triangles, and whose angular points are equidistant from C.

Since the feet of the perpendiculars from C on the sides of this triangle are the middle points of the sides, and also lie on the auxiliary circle of U; therefore the auxiliary circle of U is the nine-pointic circle of this triangle.

4. If one of the lines of curvature on a developable surface lie on a sphere, all the other lines of curvature, other than the rectilineal ones, lie on concentric spheres. If the common centre of these spheres lies on the surface, the surface must be a cone.

The lines of curvature on a developable are the generators and curves which cut them at right angles.

If $PQRS\ldots$, $P'Q'R'S'\ldots$ be lines of curvature, such that PP', $QQ'\ldots$ are generators, then $PP'=QQ'=\ldots$.

Now if $PQRS\ldots$ be on a sphere whose centre is O, since any curve on a sphere is a line of curvature; therefore the sphere and surface intersect at a constant angle.

Therefore, since OP, PP' are equal to OQ, QQ' respectively, and the angle OPP' is equal to the angle OQQ', OP' is equal to OQ'; and therefore $P'Q'R'S'\ldots$ is on a concentric sphere.

If O be on the surface, and if PP' be the generator through it, since OP is equal to OQ, O must be at the intersection of the generators through P and Q.

Hence, all the generators pass through the point O, and the surface must be a cone.

5. If $$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots + (-1)^n \frac{x^{2n}}{\lfloor 2n} X,$$

and $$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots + (-1)^n \frac{x^{2n-1}}{\lfloor 2n-1} X',$$

prove that $$\int_0^\infty \frac{X}{x}\,dx = \frac{\pi}{2} = \int_0^\infty \frac{X'}{x}\,dx.$$

By Maclaurin's theorem, if $f(x) = \sin x$,

$$X = (-1)^n f^{2n}(\theta x) = \sin\theta x,$$

and $$\int_0^\infty \frac{X}{x}\,dx = \int_0^\infty \frac{\sin\theta x}{x}\,dx = \frac{\pi}{2}.$$

Similarly, if $f(x) = \cos x$,

$$X' = (-1)^n f^{2n-1}(\theta x) = \sin\theta x, \text{ and } \int_0^\infty \frac{X'}{x} = \frac{\pi}{2}.$$

6. At each point of a closed curve are formed the rectangular hyperbola and the parabola of closest contact; shew that the arc of the curve described by the centre of the hyperbola will exceed the arc of the oval by twice the arc of the curve described by the focus of the parabola; provided that no parabola has five-pointic contact with the curve.

If $OPQRS$ be five consecutive points on the curve, and rectangular hyperbolas be drawn through $OPQR$, $PQRS$, they will have a common circle of curvature, the limit of the circle PQR.

If C, C' be the centres of the hyperbolas (fig. 71), and if CP, $C'P$ meet the common circle of curvature in c, c', then $CP = Pc$, $C'P = Pc'$, and therefore CC' will be parallel to cc', or the tangent at C to the locus of C will make with PC an angle equal to that which CP makes with the tangent at P to the oval.

The same proof applies to the locus of S, the focus of the parabola of closest contact, since $PS = \frac{1}{4}Pd$; (or to the pole of any curve, such as the equiangular spiral, and all the curves $r^n = a^n \cos n\theta$, in which the chord of curvature through the pole bears a constant ratio to the radius vector).

Now if the ellipse of five-pointic contact be drawn at P (since no parabola is supposed to have five-pointic contact, the conic of five-pointic contact must always be an ellipse, since it cannot for a closed oval always be a hyperbola), then in the reasoning concerning elementary arcs, this ellipse may be supposed to replace the oval.

Let C be the centre of this ellipse (fig. 72), H, S the centre of the rectangular hyperbola and focus of the parabola of closest contact at P.

Then PC will be a diameter of all three curves, and if P', H', S' be consecutive positions of P, H, S, $CP'H'$ will be a straight line, and, since HH', PP' make equal angles with PH, $lt\,\dfrac{HH'}{PP'} = \dfrac{CH}{CP}$.

Now $PH = \frac{1}{2}$ chord of curvature along $PC = \dfrac{CD^2}{CP}$; therefore $lt\,\dfrac{HH'}{PP'} = \dfrac{CH}{CP} = 1 + \dfrac{CD^2}{CP^2}$.

If PS, $P'S'$ meet in O, since PO, $P'O$ are equally inclined with PC, $P'C$ to the tangents at P, P'; therefore O, C will be foci of an ellipse touching the oval at P, P', or ultimately having four-pointic contact at P.

Therefore $$\frac{1}{CP} + \frac{1}{PO} = \frac{4}{PH} = 2\,\frac{CP}{CD^2},$$

$$PO = \frac{CD^2}{2CP - \dfrac{CD^2}{CP}} = \frac{SP.CP}{CP - SP}, \text{ for } SP = \frac{CD^2}{2CP}.$$

Therefore

$$lt\,\frac{SS'}{PP'} = \frac{OS}{OP} = 1 - \frac{SP}{OP} = 1 - \frac{CP - SP}{CP} = \frac{SP}{CP} = \frac{CD^2}{2CP^2},$$

and therefore $lt\,\dfrac{HH'}{PP'} - 2lt\,\dfrac{SS'}{PP'} = 1$; or the arc described by H = arc of the oval + 2 arc described by S.

7. A set of three conics pass through the three nodes of a trinodal quartic and touch the quartic; three points are determined by their remaining intersections; a second set of three points are similarly determined; prove that all six points lie on a second trinodal quartic having the same nodes as the first.

By triangular inversion the conics circumscribing the triangle formed by the nodes become straight lines touching the conic, which is the inverse of the quartic, and the proposition reduces to the well-known one, that if two triangles circumscribe the same conic their six vertices all lie on another conic.

8. SK, SK' are perpendiculars from a focus on the asymptotes of an hyperbola, and P is a point moving so that $KP.K'P$ is constant; prove that the tangent to the locus of P at a point where it meets the auxiliary circle of the hyperbola will be a tangent to the hyperbola also, and that the normal to the locus of P at this point will pass through S.

Forces F, F' along PK and PK' will have their resultant along the normal at P, if $F\frac{dr}{ds}+F'\frac{dr'}{ds}=0$, where r, r' stand for PK and PK'; but if $r'\frac{dr}{ds}+r\frac{dr'}{ds}=0$, then $\frac{F}{r'}=\frac{F'}{r}$, or the forces must be inversely as the distances.

If the forces were *as* the distances the resultant would act along PO (O being the middle point of KK'); hence, in the actual case, the resultant along the normal will make with PK an angle equal to OPK', or if a circle be drawn about KPK', the normal will pass through the intersection of the tangents at K, K'; that is, through S when this circle is the auxiliary circle of the hyperbola; and the tangent being at right angles to SP will touch the hyperbola.

9. A rough wire in the form of an equiangular spiral whose angle is $\cot^{-1}(2\mu)$ is placed with its plane vertical, and a heavy particle slides down it, coming to rest at its lowest point; prove that at the starting point the tangent makes with the horizon an angle $2\tan^{-1}\mu$, and that the velocity is greatest when the angle ϕ which the direction of motion makes with the horizon is given by the equation

$$(2\mu^2-1)\sin\phi+3\mu\cos\phi=2\mu.$$

Let the initial line SA (fig. 73) pass through the lowest point A, then the equation of the curve is $r = ae^{2\mu\theta}$ where $SA = a$.

The equations of motion are

$$mv\frac{dv}{ds} = -mg\sin\theta + \mu R,$$

$$mv^2\frac{d\theta}{ds} = -mg\cos\theta + R;$$

therefore $$v\frac{dv}{ds} - \mu v^2\frac{d\theta}{ds} = -g(\sin\theta - \mu\cos\theta),$$

or $$\frac{d}{ds}(v^2e^{-2\mu\theta}) = -2ge^{-2\mu\theta}(\sin\theta - \mu\cos\theta).$$

Now if $$\cot\alpha = 2\mu, \quad \frac{ds}{d\theta} = ae^{2\mu\theta}\operatorname{cosec}\alpha,$$

and $$\frac{d}{d\theta}(v^2e^{-2\mu\theta}) = -2ga\operatorname{cosec}\alpha(\sin\theta - \mu\cos\theta).$$

Integrating from $\theta = 0$,

$$v^2e^{-2\mu\theta} = 2ga\operatorname{cosec}\alpha(\cos\theta - 1 + \mu\sin\theta),$$

and therefore the particle started where

$$1 - \cos\theta = \mu\sin\theta, \text{ or } \tan\frac{\theta}{2} = \mu.$$

Also $v^2 = 2ga\operatorname{cosec}\alpha e^{2\mu\theta}(\cos\theta - 1 + \mu\sin\theta)$ is a maximum when

$$2\mu(\cos\theta - 1 + \mu\sin\theta) - \sin\theta + \mu\cos\theta = 0,$$

or $$(2\mu^2 - 1)\sin\theta + 3\mu\cos\theta = 2\mu.$$

10. If the horizontal distance of a projectile in a resisting medium from the point of projection be connected with the time by the equation $x = f(t)$, prove that the equation of the trajectory is

$$y = -gf(t)\int\frac{dt}{f'(t)} + g\int\frac{f(t)}{f'(t)}dt + Af(t) + B,$$

where A and B are constants.

In the case when $t = ax + bx^2$, shew that the equation of the trajectory is

$$y = x\tan\alpha - g\left(\tfrac{1}{2}a^2x^2 + \tfrac{2}{3}abx^3 + \tfrac{1}{3}b^2x^4\right).$$

The equations of motion are

$$\frac{d^2x}{dt^2} = -R\frac{dx}{ds}, \quad \frac{d^2y}{dt^2} = -g - R\frac{dy}{ds};$$

therefore $$\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} = -g\frac{dx}{dt}.$$

Dividing by $\left(\frac{dx}{dt}\right)^2$ and integrating,

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -g\int\frac{dt}{f'(t)} + A, \text{ since } \frac{dx}{dt} = f'(t);$$

therefore $$\frac{dy}{dt} = -gf'(t)\int\frac{dt}{f'(t)} + Af'(t),$$

and $$y = -gf(t)\int\frac{dt}{f'(t)} + g\int\frac{f(t)}{f'(t)} + Af(t) + B.$$

If $$t = ax + bx^2, \quad \frac{dt}{dx} = a + 2bx, \quad f'(t) = \frac{1}{a + 2bx},$$

$$\frac{dy}{dx} = -g\int(a + 2bx)^2\,dx + A$$

$$= \alpha - g\left(a^2x + 2abx^2 + \tfrac{4}{3}b^2x^3\right),$$

$$y = x\tan\alpha - g\left(\tfrac{1}{2}a^2x^2 + \tfrac{2}{3}abx^3 + \tfrac{1}{3}b^2x^4\right),$$

if the particle is projected from the origin at an angle α to the horizon.

11. A plane elliptic mirror swings on its major-axis, prove that the locus of a bright point, so placed that the mirror may throw a rectangular hyperbolic patch of light on a wall perpendicular to the axis of the mirror, is the spheroid generated by the motion of the mirror.

If P be the bright point and Q its image in the mirror, the quadric cone with vertex Q and passing through the elliptic boundary of the mirror will be cut by planes perpendicular to the major-axis of the ellipse in rectangular hyperbolas.

Therefore the plane through Q perpendicular to the major-axis will cut the cone in two straight lines at right angles; and if E, F be the points in which this plane cuts the ellipse, EQF being a right angle, Q lies on a circle described on EF as diameter, the plane of which is perpendicular to the plane of the mirror.

The locus of Q, and consequently also of P, will therefore be the spheroid generated by the motion of the mirror.

12. A sphere, radius a, rests between two parallel thin perfectly rough rods A and B in the same horizontal plane at a distance apart equal to $2b$; the sphere is turned about A till its centre is very nearly vertically over A; it is then allowed to fall back; prove that it will rock between A and B if $10b^2 < 7a^2$, and that θ_n the angle through which it will turn after the n^{th} impact is given by the equation

$$\cos\theta_n = \frac{\sqrt{(a^2-b^2)}}{a} + \frac{a-\sqrt{(a^2-b^2)}}{a}\left(1-\frac{10b^2}{7a^2}\right)^{2n}.$$

If C be the centre of the sphere at an instant of impact, if ω_{n-1}, ω_n be the angular velocities of the sphere just before and just after the n^{th} impact respectively, and if $\angle ACB = 2\alpha$, then

$$(a^2+k^2)\,\omega_n = k^2\omega_{n-1} + a^2\omega_{n-1}\cos 2\alpha,$$

and
$$\sin\alpha = \frac{b}{a},\ k^2 = \tfrac{2}{5}a^2;$$

therefore $$\frac{\omega_n}{\omega_{n-1}} = 1 - \frac{10b^2}{7a^2}.$$

Therefore $1 - \dfrac{10b^2}{7a^2}$ must be positive in order that the sphere may rock.

By the principle of energy,

$$\tfrac{1}{2}(a^2 + k^2)\omega_n^2 = ga(\cos\theta_n - \cos\alpha),$$

and if ω be the angular velocity of the sphere just before the first impact,

$$\tfrac{1}{2}(a^2 + k^2)\omega^2 = ga(1 - \cos\alpha);$$

therefore $$\frac{\omega_n^2}{\omega^2} = \frac{\cos\theta_n - \cos\alpha}{1 - \cos\alpha},$$

or $$\cos\theta_n = \cos\alpha + (1 - \cos\alpha)\frac{\omega_n^2}{\omega^2},$$

and $$\cos\alpha = \frac{\sqrt{(a^2 - b^2)}}{a}, \quad \frac{\omega_n}{\omega} = \left(1 - \frac{10b^2}{7a^2}\right)^n.$$

13. A cylindrical bullet of mass M is fired from a rifle, length $a + b$, of which a length b is originally occupied by the powder; the rifling is at a uniform pitch, making an angle α with the axis of the barrel, and $\tan\varepsilon$ is the coefficient of friction; supposing that the powder is all ignited before the shot stirs from its seat, shew that, neglecting the resistance of the air in the barrel, the velocity of the shot as it leaves the barrel being denoted by V,

$$V^2 = \frac{2\pi p_0 c^2}{M(\beta - 1)} \frac{b - b^\beta (a + b)^{1-\beta}}{1 + \frac{1}{2}\tan\alpha . \tan(\alpha + \varepsilon)},$$

where p_0 is the initial pressure behind the bullet, c the radius of the bore, and β the ratio of the specific heats of the gas at a constant pressure and at a constant volume, assuming that for all pressures and temperatures the law $pv^\beta = p_0 v_0^\beta$ holds.

Since the resultant action at every point of the rifling acting on the bullet makes an angle $\frac{1}{2}\pi + \alpha + \epsilon$ with the axis; therefore if R be the resolved part of the action parallel to the axis, then $Rc\cot(\alpha+\epsilon)$ will be the couple acting on the bullet due to the action of the rifling.

If P be the impressed force due to the powder at the time t, x the distance traversed by the bullet, and θ the angle turned through, then the equations of motion of the bullet are

$$M\frac{d^2x}{dt^2} = P - R,$$

$$M\frac{c^2}{2}\frac{d^2\theta}{dt^2} = Rc\cot(\alpha+\epsilon),$$

and since $x = c\cot\alpha.\theta$; therefore

$$\tfrac{1}{2}M\tan\alpha\frac{d^2x}{dt^2} = R\cot(\alpha+\epsilon),$$

and eliminating R,

$$M\{1 + \tfrac{1}{2}\tan\alpha\tan(\alpha+\epsilon)\}\frac{d^2x}{dt^2} = P.$$

Therefore $\qquad \tfrac{1}{2}MV^2\{1 + \tfrac{1}{2}\tan\alpha\tan(\alpha+\epsilon)\}$

$$= \int_0^a Pdx = \pi p_0 c^2 \int_0^a \left(1+\frac{x}{b}\right)^{-\beta} dx$$

$$= \frac{\pi p_0 c^2}{\beta - 1}\{b - b^\beta (a+b)^{1-\beta}\}.$$

14. If in an infinite mass of homogeneous incompressible fluid in equilibrium under finite fluid pressure only an infinitely long cylindrical column be suddenly annihilated, prove that no motion will take place.

If v be the velocity of the liquid at the distance r from the axis of the cavity, and if x be the radius of the cavity at the time t, then the equation of continuity is

$$rv = x\frac{dx}{dt} = \phi t,$$

and the equation of motion of the liquid is

$$\frac{1}{\rho} \cdot \frac{dp}{dr} = -\frac{Dv}{dt}$$

$$= -\frac{dv}{dt} - v\frac{dv}{dr}$$

$$= -\frac{\phi' t}{r} + \frac{(\phi t)^2}{r^3}.$$

If ϖ be the finite fluid pressure, ϖ will remain the pressure at infinity, and therefore

$$\frac{\varpi}{\rho} = \int_\infty^x \left\{-\frac{\phi' t}{r} + \frac{(\phi t)^2}{r^3}\right\} dr$$

$$= -\phi' t \int_\infty^x \frac{dr}{r} - \frac{(\phi t)^2}{2x^2}$$

$$= -\phi' t \int_\infty^x \frac{dr}{r} - \tfrac{1}{2}\left(\frac{dx}{dt}\right)^2.$$

Since $\int_\infty^x \frac{dr}{r}$ is infinite, $\phi' t$ must be zero in order that ϖ may be finite; and since ϕt is initially zero, therefore it is always zero, and no motion takes place.

Considering the liquid between two parallel planes perpendicular to the axis of the cavity, and at unit distance from each other; if motion were possible, the kinetic energy would be $\int_x^\infty 2\pi\rho r dr \cdot \tfrac{1}{2}v^2 = \pi\rho(\phi t)^2 \int_x^\infty \frac{dr}{r}$, which is infinite; and the work done would be the product of ϖ into the diminution of volume at infinity, that is, the product of ϖ into the diminution of area of the cavity, which is finite.

The kinetic energy and the work done cannot therefore be equal, and therefore no motion takes place.

15. Light falls normally through a very small hole on a plate of doubly refracting crystal of which the parallel faces are parallel to one of the circular sections of the surface of elasticity; shew that if t be the thickness of the plate, and the semi-axes of the surface of elasticity be proportional to λ, 1, λ', respectively, the area of the transverse section of the emergent cylinder of rays will be

$$\frac{\pi}{4}(\lambda^2-1)(1-\lambda'^2)\,t^2.$$

Let a, b, c be the optical constants of the crystal, that is, the ratio of the velocities of light along the axes of the crystal to the velocity of light in a vacuum.

Let O be the point of incidence of the light (fig. 74) $ACBB'$ the section of the wave surface made by the plane through the axes Ox and Oz of greatest and least elasticity of the crystal, DE the section of the equivalent sphere in air, supposed of radius unity.

The coefficients of restitution perpendicular to the axis Ox being as b^2 and c^2, therefore the ray-velocities along Ox are as b and c, and therefore $OB=b$, $OC=c$; similarly $OA=a$, $OB'=b$, and AC is an ellipse, BB' a circle.

If PQ be the common tangent of the ellipse AC and the circle BB', then OQ is perpendicular to a circular section of the ellipsoid of elasticity

$$a^2x^2+b^2y^2+c^2z^2=1.$$

The tangent plane to the wave surface through PQ will touch the surface in a circle, and if the crystal be cut perpendicularly to OQ, then the light incident normally at O will form inside the crystal a cone and will emerge in a cylindrical beam, the transverse section of which RS will be a circle.

If x, z be the coordinates of P, and if the angle POQ be denoted by α, then $\cos\alpha=\dfrac{b}{OP}$, and

$$\frac{x^2}{c^2}+\frac{z^2}{a^2}=1,\quad \frac{x^2}{c^4}+\frac{z^2}{a^4}=\frac{1}{b^2};$$

therefore
$$\frac{x^2}{c^2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=\frac{1}{b^2}-\frac{1}{a^2},$$

$$\frac{z^2}{a^2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=\frac{1}{c^2}-\frac{1}{b^2}.$$

$$OP^2=x^2+z^2=\frac{c^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+a^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)}{\frac{1}{c^2}-\frac{1}{a^2}},$$

$$\tan^2\alpha=\frac{OP^2}{b^2}-1$$

$$=\frac{c^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+a^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-b^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)}{b^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)}$$

$$=\frac{(a^2-b^2)(b^2-c^2)}{b^4}=(\lambda^2-1)(1-\lambda'^2),$$

and the area of the circle RS

$$=\tfrac{1}{4}\pi t^2\tan^2\alpha$$

$$=\tfrac{1}{4}\pi(\lambda^2-1)(1-\lambda'^2)t^2.$$

WEDNESDAY, *Jan.* 20, 1875. $1\frac{1}{2}$ to 4.

PROF. TAIT. Roman numbers.
MR. FREEMAN. Arabic numbers.

1. SHEW how to obtain a first integral of the differential equation

$$Rr + Ss + Tt + U(s^2 - rt) = V,$$

when it has a first integral of the form $F(u, v) = 0$, where R, S, T, U, V, u, v, are functions of x, y, z, p, q.

Obtain the complete integral of the equation

$$z(1+q^2)\,r - 2pqzs + z(1+p^2)\,t - z^2(s^2 - rt) + 1 + p^2 + q^2 = 0.$$

(Boole, *Differential Equations*, Supplementary Volume, pp. 125 to 141).

Comparing the proposed equation with the standard form, equation (21), p. 133 becomes

$$m^2 - 2mpqz + p^2q^2z^2 = 0,$$

the roots of which are each equal to pqz.

In this case it is possible to find three integrals of the system of differential equations of p. 139, which become

$$dz - pdx - qdy = 0 \text{ ....(1)},$$

$$dp + \frac{1+p^2}{z}\,dx + \frac{pq}{z}\,dy = 0,$$

$$dq + \frac{pq}{z}\,dx + \frac{1+q^2}{z}\,dy = 0;$$

and the last two, by reason of (1), reduce to

$$dx + zdp + pdz = 0 \text{ ....(2)},$$

$$dy + zdq + qdz = 0 \text{ ....(3)}.$$

The integrals of (2) and (3) are

$$x + pz = a, \quad y + qz = b,$$

and substituting for dx and dy from (2) and (3) in (1)

$$(1 + p^2 + q^2)\,dz + (pdp + qdq)\,dz = 0,$$

the integral of which is

$$z^2(1 + p^2 + q^2) = c^2.$$

Eliminating p and q from the three integrals, we have

$$(x-a)^2 + (y-b)^2 + z^2 = c^2,$$

which represents a sphere.

Now $\qquad a = \phi(c), \quad b = \psi(c), \quad F(a, b) = 0,$

are all first integrals of the given equation, and the complete integral is found by eliminating c between the equations

$$\{x - \phi(c)\}^2 + \{y - \psi(c)\}^2 + z^2 - c^2 = 0 \equiv f(x, y, z, c)$$

and $$\frac{df}{dc} = 0.$$

The complete integral is therefore the equation of a tubular surface, the central line of which is in the plane of xy.

A first integral of the equation is

$$F(x + pz, \; y + qz) = 0.$$

2. State the criterion for the selection of the combination weights of n independent measures of a magnitude. Determine the probable error of the result in terms of the probable errors of the n measures.

In the observation of the zenith distances of stars for the amplitude of a meridian divided into four sections by three stations intermediate between the extreme stations, a stars are observed at the first, second, third stations only; b stars at the second, third, fourth only; c stars at the third, fourth, fifth only; and the probable error of every observation of a star is e. Shew that there are only three independent modes of measuring the whole arc, and obtain equations for determining the combination weights of the three measures. In the case when $a=b=c$, prove that the square of the probable error of the result is $\dfrac{10e^2}{3a}$.

(Airy, *Errors of Observations*, § 64–70, and for the rider 80–82).

Consider the scheme

		Stations.				
		1	2	3	4	5
Number of stars observed	a	A_1	A_2	A_3		
	b		B_2	B_3	B_4	
	c			C_3	C_4	C_5
		Means of actual errors.				

The mean actual errors of possible measures of the whole arc are represented by

$$(A_1 - A_2) + (B_2 - B_3) + (C_3 - C_5) \ldots\ldots\ldots\ldots (1),$$

$$(A_1 - A_2) + (B_2 - B_4) + (C_4 - C_5) \ldots\ldots\ldots\ldots (2),$$

$$(A_1 - A_3) + (B_3 - B_4) + (C_4 - C_5) \ldots\ldots\ldots\ldots (3),$$

which represent three independent, though entangled measures of the arc.

Any other measure can be expressed in terms of (1), (2), and (3), for instance

$$(A_1 - A_3) + (C_3 - C_5) = (1) - (2) + (3).$$

Let x, y, z be the combination weights of (1), (2), and (3) respectively; then the actual error of the mean will be

$$\frac{x\{(A_1-A_2)+(B_2-B_3)+(C_3-C_5)\}+y\{(A_1-A_2)+(B_2-B_4)+(C_4-C_5)\}+z\{(A_1-A_3)+(B_3-B_4)+(C_4-C_5)\}}{x+y+z}$$

$$=\frac{(x+y+z)A_1-(x+y)A_2-zA_3+(x+y)B_2-(x-z)B_3-(y+z)B_4+xC_3+(y+z)C_4-(x+y+z)C_5}{x+y+z}.$$

The independent fallible quantities are now separated, and since (p. e. of $A_1)^2=\frac{e^2}{a}$, and so for the others; therefore $\left(\frac{\text{p. e. of result}}{e}\right)^2$

$$=\frac{\{(x+y+z)^2+(x+y)^2+z^2\}\frac{1}{a}+\{(x+y)^2+(x-z)^2+(y+z)^2\}\frac{1}{b}+\{x^2+(y+z)^2+(x+y+z)^2\}\frac{1}{c}}{(x+y+z)^2},$$

which is to be made a minimum; therefore, by § 69,

$$\{(x+y+z)+(x+y)\}\frac{1}{a}+\{(x+y)+(x-z)\}\frac{1}{b}+\{x+(x+y+z)\}\frac{1}{c}=C,$$

$$\{(x+y+z)+(x+y)\}\frac{1}{a}+\{(x+y)+(y+z)\}\frac{1}{b}+\{(y+z)+(x+y+z)\}\frac{1}{c}=C,$$

$$\{(x+y+z)+z\}\frac{1}{a}+\{(z-x)+(y+z)\}\frac{1}{b}+\{(y+z)+(x+y+z)\}\frac{1}{c}=C.$$

whence we can find $x:y:z$.

If $a=b=c$, then the equations become

$$6x+4y+\ z=Ca,$$
$$4x+6y+4z=Ca,$$
$$x+4y+6z=Ca;$$

and, therefore, $$\frac{x}{2}=-y=\frac{z}{2}.$$

Hence $$\left(\frac{\text{p.e. of result}}{e}\right)^2$$

$$=\frac{3^2+1+2^2+1+0+1+2^2+1+3^2}{3^2a}=\frac{10}{3a};$$

therefore $$(\text{p.e. of result})^2=\frac{10e^2}{3a}.$$

iii. Form the equations of motion of a frictionless liquid, including that of the bounding surface; and point out definitely what is to be understood by the velocity at any point.

Shew that no differentially irrotational motion (i.e. motion having a velocity-potential) can take place in an unmoved rigid simply-connected closed vessel completely filled with such a liquid.

If u, v, w the components of the velocity of the fluid at any given point (x, y, z) be given as functions of the time t, then x, y, z and t are independent variables; and the velocity in any direction will be measured by the volume of flow per unit of time and per unit of area across a small plane area placed at the point perpendicular to the given direction.

For if the velocity q be uniform and make an angle θ with the axis of x and we take a fixed plane area A (fig. 75) perpendicular to the axis of x and consider the fluid which crosses this area in the time t, this fluid will fill a cylinder of base A with generating lines parallel to the velocity and of length qt.

The volume of this cylinder is $Aqt\cos\theta$, and hence the flow across A per unit of time and per unit of area is $q\cos\theta$.

If the velocity is not uniform, the proposition is still differentially true at every point.

If $F(x, y, z, t) = 0$ be the equation of a surface which always contains the same particles of fluid, then in the infinitesimal time dt, x, y, z become respectively $x + udt$, $y + vdt$, $z + wdt$; and, therefore,

$$F(x + udt,\ y + vdt,\ z + wdt,\ t + dt) = 0;$$

therefore $$u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz} + \frac{dF}{dt} = 0,$$

the differential equation of a surface which always contains the same particles of fluid.

Two consecutive surfaces will by the principle of continuity remain consecutive throughout the motion and cannot penetrate each other except for discontinuous values of F.

Hence, in general, particles once in the bounding surface will always remain in the bounding surface, and if $F(x,y,z,t)=0$ be the equation of the bounding surface at the time t,

$$u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz} + \frac{dF}{dt} = 0.$$

If u, v, w be finite and continuous and the differential coefficients of a function ϕ, and the surfaces for which ϕ is a constant be drawn, then the vessel being acyclic, these surfaces are either closed surfaces or bounded entirely by the surface of the vessel.

A closed line within the region cutting any one of the surfaces must therefore cut the same surface in the opposite direction at some other point of its path; and therefore the integral of $udx + vdy + wdz$ round a closed line must be zero; hence ϕ must be a single valued function.

If dS denote an element of the surface of the vessel, dn an element of the inward drawn normal, then by Green's theorem

$$\iiint q^2 dx\,dy\,dz = -\iint \phi\frac{d\phi}{dn}dS + \iiint \phi\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right)dx\,dy\,dz,$$

and the first term on the right-hand side vanishes because $\frac{d\phi}{dn}$ is zero over the surface, and the second in consequence of the equation of continuity.

Therefore q must be zero, and no motion can take place in the interior.

If q became discontinuous or infinite we might have a vortex line in the interior of the vessel, or if the vessel were cyclic and ϕ many valued, we should have circulations of the liquid in the circuits of the vessel.

iv. Pressure is applied, according to an assigned law, to every point of a plane surface bounding an otherwise infinite isotropic solid. Find the resultant displacement at any point in the interior of the solid.

Work out the particular case in which the plane face is that of yz, and the pressure is perpendicular to it and proportional to $\sin \dfrac{2\pi y}{a}$.

(Thomson and Tait, *Natural Philosophy*, §§ 693 to 739).

Measuring the axis of x perpendicularly to the plane face into the interior of the solid and supposing the arbitrary pressure on the plane face a function of y only, then the stresses and strains are functions of x and y only, and the general equations of internal equilibrium reduce to

$$\left.\begin{aligned}\frac{dP}{dx}+\frac{dU}{dy}&=0\\ \frac{dU}{dx}+\frac{dQ}{dy}&=0\end{aligned}\right\}\text{......................(1).}$$

and if α, β be the displacements of a point originally at x, y

$$\left.\begin{aligned}P&=(m+n)\frac{d\alpha}{dx}+(m-n)\frac{d\beta}{dy}\\ Q&=(m-n)\frac{d\alpha}{dx}+(m+n)\frac{d\beta}{dy}\\ U&=n\left(\frac{d\beta}{dx}+\frac{d\alpha}{dy}\right)\end{aligned}\right\}\text{...........(2),}$$

where $m=k+\frac{1}{3}n$, k being the elasticity of volume, and n the elasticity of figure of the substance.

Equations (1) become

$$\left.\begin{aligned} m\frac{d\delta}{dx}+n\Delta^2\alpha=0 \\ m\frac{d\delta}{dy}+n\Delta^2\beta=0 \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(3),$$

where $$\delta=\frac{d\alpha}{dx}+\frac{d\beta}{dy}$$

is the superficial dilatation, and

$$\Delta^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}.$$

At the surface the stress being always normal, the stress-ellipse must be a circle; and, therefore, when $x=0$ we must have $P=Q$ and $U=0$.

If $P=pf(y)$ represent the distribution of traction and pressure on the plane face, then $\delta=\frac{P}{m}=\frac{p}{m}f(y)$ when $x=0$, and from equations (3) $\Delta^2\delta=0$; δ must also vanish when $x=\infty$.

Therefore by Fourier's theorem

$$\delta=\frac{p}{\pi m}\int_0^\infty\int_{-\infty}^\infty e^{-xu}f(u)\cos u(v-y)\,du\,dv,$$

and then since $\Delta^2\alpha=-\frac{m}{n}\frac{d\delta}{dx}$, $\Delta^2\beta=-\frac{m}{n}\frac{d\delta}{dy}$, α and β are the potentials due to densities $\frac{m}{4\pi n}\frac{d\delta}{dx}$ and $\frac{m}{4\pi n}\frac{d\delta}{dy}$ in the interior of the solid.

In the particular case where $f(y)=\sin\frac{2\pi y}{a}$, we must assume

$$P=\phi(x)\sin\frac{2\pi y}{a};$$

$$Q=\psi(x)\sin\frac{2\pi y}{a},\quad U=\chi(x)\cos\frac{2\pi y}{a},$$

and then from equations (1)

$$\phi'(x) - \frac{2\pi}{a}\chi(x) = 0,$$

$$\chi'(x) + \frac{2\pi}{a}\psi(x) = 0,$$

and, therefore, $$\chi(x) = \frac{a}{2\pi}\phi'(x),$$

$$\psi(x) = -\frac{a^2}{4\pi^2}\phi''(x),$$

and $$\delta = \frac{1}{2m}\left\{\phi(x) - \frac{a^2}{4\pi^2}\phi''(x)\right\}\sin\frac{2\pi y}{a}.$$

Putting $$\phi(x) - \frac{a^2}{4\pi^2}\phi''(x) = \varpi(x),$$

then since $\Delta^2\delta = 0$, therefore

$$\varpi''(x) - \frac{4\pi^2}{a^2}\varpi(x) = 0.$$

Therefore $$\varpi(x) = Ae^{-\frac{2\pi x}{a}},$$

or $$\phi''(x) - \frac{4\pi^2}{a^2}\phi(x) = \frac{4\pi^2}{a^2}Ae^{-\frac{2\pi x}{a}},$$

the solution of which is

$$\phi(x) = Be^{-\frac{2\pi x}{a}} - \frac{\pi}{a}Axe^{-\frac{2\pi x}{a}}.$$

Therefore $$\chi(x) = -Be^{-\frac{2\pi x}{a}} - \tfrac{1}{2}Ae^{-\frac{2\pi x}{a}} + \frac{\pi}{a}Axe^{-\frac{2\pi x}{a}},$$

and $$\psi(x) = -Be^{-\frac{2\pi x}{a}} - Ae^{-\frac{2\pi x}{a}} + \frac{\pi}{a}Axe^{-\frac{2\pi x}{a}}.$$

Now when $x = 0$, $P = Q = p\sin\frac{2\pi y}{a}$ and $U = 0$; therefore

$$\phi(0) = \psi(0) = p, \quad \chi(0) = 0;$$

and therefore $$A+2B=0,\quad B=p;$$

$$P=p\left(1+\frac{2\pi x}{a}\right)e^{-\frac{2\pi x}{a}}\sin\frac{2\pi y}{a},$$

$$U=-p\,\frac{2\pi x}{a}\,e^{-\frac{2\pi x}{a}}\cos\frac{2\pi y}{a},$$

$$Q=p\left(1-\frac{2\pi x}{a}\right)e^{-\frac{2\pi x}{a}}\sin\frac{2\pi y}{a},$$

and $$\delta=\frac{1}{2m}(P+Q)$$

$$=\frac{p}{m}e^{-\frac{2\pi x}{a}}\sin\frac{2\pi y}{a}.$$

Since $$\frac{d\alpha}{dx}=\frac{P+Q}{4m}+\frac{P-Q}{4n},$$

$$\frac{d\beta}{dy}=\frac{P+Q}{4m}-\frac{P-Q}{4n};$$

therefore $$\frac{d\alpha}{dx}=\tfrac{1}{2}p\left(\frac{1}{m}+\frac{2\pi x}{na}\right)e^{-\frac{2\pi x}{a}}\sin\frac{2\pi y}{a},$$

$$\frac{d\beta}{dy}=\tfrac{1}{2}p\left(\frac{1}{m}-\frac{2\pi x}{na}\right)e^{-\frac{2\pi x}{a}}\sin\frac{2\pi y}{a},$$

and therefore $$\alpha=-\frac{ap}{4\pi}\left\{\frac{1}{m}+\frac{1}{n}\left(\frac{2\pi x}{a}+1\right)\right\}e^{-\frac{2\pi x}{a}}\sin\frac{2\pi y}{a},$$

$$\beta=-\frac{ap}{4\pi}\left(\frac{1}{m}-\frac{2\pi x}{na}\right)e^{-\frac{2\pi x}{a}}\cos\frac{2\pi y}{a}.$$

5. Prove that in any substance the ratio of the specific heat at constant pressure to the specific heat at constant volume is equal to the ratio of the elasticity when no heat escapes to the elasticity at constant temperature.

Hence shew that

$$\left(\frac{dp}{dv}\right)_{(\phi\text{ const.})}\times\left(\frac{d\phi}{dt}\right)_{(v\text{ const})}=\left(\frac{dp}{dv}\right)_{(t\text{ const.})}\times\left(\frac{d\phi}{dt}\right)_{(p\text{ const.})},$$

where p, v, t are the pressure, volume and absolute temperature of the substance, and $\int t d\phi$ is the dynamical equivalent of the heat it has received from external sources during a series of operations.

It is proved in Maxwell's *Theory of Heat*, chap. IX., that

$$\frac{K_p}{K_v} = \frac{T.\dfrac{AK}{AL}}{T.\dfrac{AM}{AN}} = \frac{V.\dfrac{AN}{AL}}{V.\dfrac{AM}{AK}} = \frac{E_\phi}{E_\theta},$$

and referring to the diagram (fig. 76), we see that if

$$\phi_2 - \phi_1 = 1 \text{ and } T_2 - T_1 = 1,$$

$\dfrac{AK}{AL} = \left(\dfrac{d\phi}{dt}\right)_p$, K and L being on a line of equal pressure;

$\dfrac{AM}{AK} = \left(\dfrac{dp}{dv}\right)_t$, M and K being on an isothermal;

$\dfrac{AM}{AN} = \left(\dfrac{d\phi}{dt}\right)_v$, M and N being on a line of equal volume;

$\dfrac{AN}{AL} = \left(\dfrac{dp}{dv}\right)_\phi$, N and L being on an adiabatic;

whence the result.

6. Rays of plane polarised light are incident normally on a plate of uniaxal crystal; prove that the emergent rays are in general elliptically polarised, and find in what cases they would be circularly polarised.

If the elliptically polarised light be received normally on a second plate of uniaxal crystal (which could circularly polarise plane polarised light) in such a manner that the light emergent from the second plate is plane polarised, prove that

$$\tan\alpha = \tan\beta \sin 2\pi\frac{k}{\lambda} - \tan\gamma \cos 2\pi\frac{k}{\lambda},$$

where α is the angle from the principal plane of the first plate to the plane of polarisation of the incident light, β is the angle from the principal plane of the second plate to the plane of polarisation of the finally emergent light, γ is the angle from the principal plane of the first to that of the second plate, and k is the equivalent in air to the relative retardation of the ordinary and extraordinary rays caused by the first plate. The principal plane of each plate contains its optic axis and is normal to its parallel surfaces.

(*Œuvres de Verdet*, Tome VI., p. 60.)

The plane of the figure being perpendicular to the ray, let C be the point of incidence, Cx the principal plane of the plate, CP the plane of polarisation of the incident ray.

Inside the crystal the light is divided into the ordinary and extraordinary rays, the planes of polarisation of which are Cx and the plane Cy perpendicular to Cx.

The incidence being normal, the reflexions at the faces of the plate diminish in the same ratio the amplitudes of vibration of the ordinary and extraordinary rays; the amplitudes may, therefore, be represented by $a \sin \alpha$, $a \cos \alpha$.

If O, E be the equivalent lengths in air of the plate for the ordinary and extraordinary rays, then $O - E = k$, and the difference of phase of the rays at emergence is $\dfrac{k}{\lambda}$.

The displacement of the ether at emergence may therefore be represented by

$$x = a \sin \alpha \sin 2\pi \left(\frac{t}{T} + \frac{k}{\lambda}\right), \quad y = a \cos \alpha \sin 2\pi \frac{t}{T};$$

therefore $$\frac{x^2}{\sin^2 \alpha} - 2 \frac{xy}{\sin \alpha \cos \alpha} \cos 2\pi \frac{k}{\lambda} + \frac{y^2}{\cos^2 \alpha} = a^2 \sin^2 2\pi \frac{k}{\lambda},$$

which proves that the emergent ray is, in general, elliptically polarised.

If however $\tan \alpha = 1$ and $k = (2n+1)\frac{1}{4}\lambda$, then

$$x^2 + y^2 = \tfrac{1}{2}a^2,$$

and the emergent ray is circularly polarised.

If the ray be received on a second plate in which $k = \frac{1}{4}\lambda$, and of which the principal plane is Cx', then the component vibrations at emergence parallel and perpendicular to Cx' may be represented by

$$x' = a \sin\alpha \cos\gamma \sin 2\pi\left(\frac{t}{T} + \frac{k}{\lambda} + \tfrac{1}{4}\right)$$

$$+ a \cos\alpha \sin\gamma \sin 2\pi\left(\frac{t}{T} + \tfrac{1}{4}\right),$$

$$y' = - a \sin\alpha \sin\gamma \sin 2\pi\left(\frac{t}{T} + \frac{k}{\lambda}\right) + a \cos\alpha \cos\gamma \sin 2\pi\frac{t}{T}.$$

If the emergent ray be polarised in a plane making an angle β with Cx', then $\frac{x'}{y'} = \tan\beta$

$$= \frac{\left(\tan\alpha \cos 2\pi\frac{k}{\lambda} + \tan\gamma\right)\cos 2\pi\frac{t}{T} - \tan\alpha \sin 2\pi\frac{k}{\lambda} \sin 2\pi\frac{t}{T}}{\left(1 - \tan\alpha \tan\gamma \cos 2\pi\frac{k}{\lambda}\right)\sin 2\pi\frac{t}{T} - \tan\alpha \tan\gamma \sin 2\pi\frac{k}{\lambda} \cos 2\pi\frac{t}{T}},$$

a relation which must be independent of t, and therefore

$$\tan\beta\left(1 - \tan\alpha \tan\gamma \cos 2\pi\frac{k}{\lambda}\right) + \tan\alpha \sin 2\pi\frac{k}{\lambda} = 0,$$

$$\tan\alpha \tan\beta \tan\gamma \sin 2\pi\frac{k}{\lambda} + \tan\alpha \cos 2\pi\frac{k}{\lambda} + \tan\gamma = 0.$$

Multiply the first equation by $\sin 2\pi\frac{k}{\lambda}$, the second by $\cos 2\pi\frac{k}{\lambda}$, and add; $\tan\alpha \tan\beta \tan\gamma$ is eliminated, and the required condition is obtained.

vii. Explain briefly the nature of the analytic and synthetic processes by which Helmholtz shewed that the quality of a musical sound depends upon the number and intensity of the harmonics which accompany it, and which are, in general, objectively present.

Shew that two pure sounds, represented by increments of pressure in the external ear proportional to $\sin mt$ and $\sin(nt+\alpha)$ respectively, provided that they are of sufficient intensity to require us to take account of the square of the consequent disturbance of the membrane of the tympanum, give rise in the internal ear to the following new pure sounds

$$\sin\{(m+n)\,t+\alpha\} \text{ and } \sin\{(m-n)\,t-\alpha\},$$

in addition to their own first harmonics. Which of these new sounds is the louder, and what conditions are most favorable to its being heard distinctly?

Under what circumstances are such sounds produced objectively?

Musical sounds are distinguished (i) by their pitch, (ii) by their intensity or loudness, (iii) by their quality.

The pitch is determined by the number of vibrations in a second, the intensity by the amplitude of vibration, so that the quality can depend only on the form of the vibrations.

Fourier's theorem proves that any vibration can be resolved into a series of simple harmonic vibrations, having vibrational numbers which are once, twice, thrice, four times, &c. as great as the vibrational number of the given motion; and Helmholz has proved experimentally that these harmonic vibrations are objectively present in the air, and that the ear performs the resolution that Fourier's theorem shews is mathematically possible.

Analytically, the number and intensity of the harmonics can be found experimentally by the use of resonators applied to the ear, which reinforce the harmonic corresponding to the fundamental note of the resonator. The harmonics are proved to be objectively present by the fact that light membranes, &c. tuned to the harmonics are set in sympathetic vibration when the fundamental note is sounded.

Synthetically, the quality of any musical sound can be produced by a series of tuning forks, tuned to the fundamental note and its harmonics, and provided with resonance chambers, which are capable of producing variations of intensity and differences of phase in the harmonics.

The tuning forks being kept in vibration by electric currents, it is found that any musical sound can be reproduced by the combination of the prime tone with the upper harmonics in different intensities; and it is found also that the quality is independent of the phases of the harmonics.

The differential equation of the motion of the membrane of the tympanum may be written

$$\frac{d^2x}{dt^2} + p^2x = ax^2 + f\sin mt + g\sin(nt+\alpha).$$

For the first approximation, neglecting ax^2,

$$x = \frac{f\sin mt}{p^2 - m^2} + \frac{g\sin(nt+\alpha)}{p^2 - n^2},$$

the complementary function, which represents the proper tone of the membrane, being neglected because it rapidly dies away.

Substituting this value of x in ax^2, we have for a second approximation to the value of x the differential equation

$$\frac{d^2x}{dt^2} + p^2x = a\left\{\frac{f\sin mt}{p^2-m^2} + \frac{g\sin(nt+\alpha)}{p^2-n^2}\right\}^2$$

$$= \tfrac{1}{2}a\left\{\frac{f^2}{(p^2-m^2)^2} + \frac{g^2}{(p^2-n^2)^2}\right\} - \tfrac{1}{2}a\left\{\frac{f^2\cos 2mt}{(p^2-m^2)^2} + \frac{g^2\cos 2(nt+\alpha)}{(p^2-n^2)}\right\}$$

$$- \frac{afg}{(p^2-m^2)(p^2-n^2)}\left[\cos\{(m+n)t+\alpha\} - \cos\{(m-n)t-\alpha\}\right].$$

Hence, in addition to the first harmonics produced by the terms involving $\cos 2mt$ and $\cos 2(nt+\alpha)$, we shall have combinational tones represented by

$$\cos\{(m+n)t+\alpha\} \text{ and } \cos\{(m-n)t-\alpha\},$$

of amplitudes

$$\frac{afg}{(p^2-m^2)(p^2-n^2)\{p^2-(m+n)^2\}} \text{ and } \frac{afg}{(p^2-m^2)(p^2-n^2)\{p^2-(m-n)^2\}}$$

respectively.

The intensity of the differential tone is the greatest. It will be most easily heard if the generating sounds are less than an octave apart, because in that case the differential combinational tone is deeper than the generating sounds.

In general, a combinational tone is not reinforced by the proper resonator applied to the ear, shewing that the tone is produced in the ear itself. If, however, the generating sound is of sufficient intensity for the square of the displacement to be taken into account, the combinational tones will be produced objectively and will be reinforced by a resonator.

viii. Integrate the simultaneous equations:

$$\begin{cases} \dfrac{dw}{dt} + ax + by\cos nt + bz\sin nt = 0, \\[2mm] aw - \dfrac{dx}{dt} - by\sin nt + bz\cos nt = 0, \\[2mm] bw\cos nt + bx\sin nt - \dfrac{dy}{dt} - az = 0, \\[2mm] bw\sin nt - bx\cos nt + ay - \dfrac{dz}{dt} = 0. \end{cases}$$

Eliminating x, y, z

$$\begin{vmatrix} \dfrac{d}{dt}, & a, & b\cos nt, & b\sin nt \\[2mm] a, & -\dfrac{d}{dt}, & -b\sin nt, & b\cos nt \\[2mm] b\cos nt, & b\sin nt, & -\dfrac{d}{dt}, & -a \\[2mm] b\sin nt, & -b\cos nt, & a, & -\dfrac{d}{dt} \end{vmatrix} w = 0,$$

which expanded becomes

$$\frac{d^4w}{dt^4} + \{2(a^2+b^2) + n^2 - 2na\}\frac{d^2w}{dt^2}$$
$$+ \{(a^2+b^2)^2 + (a^2+b^2)(n^2-2na) - b^2n^2\} w = 0,$$

and there will result the same differential equation for x, y, and z.

The auxiliary equation is

$$m^4 + \{2(a^2+b^2)+n^2-2na\}\,m^2 + (a^2+b^2-na)^2 = 0,$$

or

$$(m^2+a^2+b^2-na)^2 + m^2n^2 = 0.$$

Hence m is of the form $\mu\sqrt{(-1)}$, where

$$\mu^2 \pm \mu n - (a^2+b^2-na) = 0.$$

If $\pm\mu_1$, $\pm\mu_2$ are the roots of this equation,

$$\mu_1 + \mu_2 = n,$$

and the solutions of the differential equations consist of the sum of the products of $\cos\mu_1 t$, $\sin\mu_1 t$, $\cos\mu_2 t$, $\sin\mu_2 t$ by arbitrary constants.

If we take $w = H\cos\mu_1 t$, we must take

$$x = H_1\sin\mu_1 t,\quad y = H_2\sin\mu_2 t,\quad z = H_3\cos\mu_2 t,$$

and substituting in the differential equations we must have

$H_1 = H$, $H_3 = -H_2$ and $-(\mu_1-a)H - bH_2 = 0$, $bH - (\mu_2-a)H_2 = 0$.

Therefore $$\frac{H_2}{H} = \frac{a-\mu_1}{b} = \frac{-b}{a-\mu_2},$$

which satisfies the auxiliary equation, since

$$\mu_1+\mu_2 = n,\quad \mu_1\mu_2 = na - a^2 - b^2.$$

Similarly, if we take $w = I\sin\mu_1 t$, we must take

$$x = -I\cos\mu_1 t,\quad y = I_2\cos\mu_2 t,\quad z = I_2\sin\mu_2 t,$$

and $$(\mu_1-a)I + bI_2 = 0,\quad -bI + (\mu_2-a)I_2 = 0.$$

If we take $w = J\cos\mu_2 t$, we must take

$$x = J\sin\mu_2 t,\quad y = J_2\sin\mu_1 t,\quad z = -J_2\cos\mu_1 t,$$

and $$-(\mu_2-a)J - bJ_2 = 0,\quad bJ - (\mu_1-a)J_2 = 0.$$

If we take $w = K\sin\mu_2 t$, we must take

$$x = -K\cos\mu_2 t,\quad y = K_2\cos\mu_1 t,\quad z = K_2\sin\mu_1 t,$$

and $$(\mu_2-a)K + bK_2 = 0;\quad -bK + (\mu_2-a)K_2 = 0.$$

Therefore the solution of the equations is

$$w = H\cos\mu_1 t + I\sin\mu_1 t + J\cos\mu_2 t + K\sin\mu_2 t,$$

$$x = H\sin\mu_1 t - I\cos\mu_1 t + J\sin\mu_2 t - K\cos\mu_2 t,$$

$$y = \frac{a-\mu_1}{b} H\sin\mu_2 t + \frac{a-\mu_1}{b} I\cos\mu_2 t + \frac{a-\mu_2}{b} J\sin\mu_1 t + \frac{a-\mu_2}{b} K\cos\mu_1 t,$$

$$z = -\frac{a-\mu_1}{b} H\cos\mu_2 t + \frac{a-\mu_1}{b} I\sin\mu_2 t - \frac{a-\mu_2}{b} J\cos\mu_1 t + \frac{a-\mu_2}{b} K\sin\mu_1 t,$$

involving the four arbitrary constants H, I, J, K.

ix. What is the nature of the analogy between the bending of a flexible rod and the motion of a rigid body?

Work it out in full for the case of a pendulum which *just* makes a complete revolution in a vertical plane.

(Thomson and Tait, *Natural Philosophy*, §§ 593—611).

Draw two planes of reference at right angles to one another through the elastic central line of the rod when straight, cutting the normal section at P in PK and PL.

Let the rod be bent into any curve, of which PT is the tangent to the central line at the point P; and let τ be the twist at P, and κ, λ the component curvatures in the planes perpendicular to PK and PL.

If a rigid body be taken which moves about a fixed point O, and which has component angular velocities τ, κ, λ about axes OT', OK', OL' fixed in the body; then if P move along the central line with unit velocity, the lines OT', OK', OL', if initially parallel, will always remain parallel to PT, PK, PL.

The twist and flexures τ, κ, λ being proportional to the impressed couples, the energy w of the stress per unit length of the rod is a quadratic function of τ, κ, λ, and therefore

$$w = \tfrac{1}{2}(A\tau^2 + B\kappa^2 + C\lambda^2 + 2D\kappa\lambda + 2E\lambda\tau + 2F\tau\kappa),$$

where A, B, C, F, G, H are constant, since the rod is supposed uniform.

If A, B, C, F, G, H be the moments and products of inertia of the rigid body about the axes OT', OK', OL', then w is the kinetic energy of the body.

The component couples acting on the rod at P perpendicular to PT, PK, PL, and the component moments of momentum of the rigid body about OT', OK', OL' are therefore $\frac{dw}{d\tau}$, $\frac{dw}{d\kappa}$, $\frac{dw}{d\lambda}$ respectively; and the resultant stress couple at the point P and the corresponding resultant moment of momentum of the body will be equal.

If the rod be bent by a pair of balancing wrenches at each end, the stress couple at any point P will be the resultant of the couple G of either wrench and of the couple formed by bringing the force R of the wrench to the point P.

The impressed couple that must act on the rigid body in order to make the body move in the prescribed manner about O will be the rate of variation per unit of length at P of the stress couple, which is $R \sin\alpha$, where α is the inclination of PT to the axis of the wrench.

Hence a force R, parallel to the axis of the wrench, acting through a point in OT' at unit distance from O, will make the rigid body move in the required manner.

In the particular case $\tau = 0$, $\lambda = 0$, $G = 0$; and the rod takes the shape of (fig. 77), where the curvature is proportional to the distance from the line of force.

If a be the length of the simple equivalent pendulum of the rigid body, and also the mean proportional between the radius of curvature at P and distance of P from the line of force; then if P move with velocity $\sqrt{(ga)}$, instead of unit velocity, the tangent at P if initially will always be perpendicular to OT' (fig. 78).

At the time t, measured from the instant when P is at A and OT' in the vertical position OA', let the angle $A'OT' = \theta$, and the arc $AP = s$; then $s = \sqrt{(ga)}\, t$.

The equation of motion of the pendulum is

$$\tfrac{1}{2}a^2\left(\frac{d\theta}{dt}\right)^2 = ga\,(1 + \cos\theta),$$

or $$\frac{dt}{d\theta} = \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\sec\frac{\theta}{2};$$

therefore $$t = \sqrt{\left(\frac{a}{g}\right)}\log\tan\frac{\pi+\theta}{4},$$

or $$\tan\frac{\pi+\theta}{4} = e^{\sqrt{\left(\frac{g}{a}\right)}t},$$

and the intrinsic equation of the curve AP is

$$\tan\frac{\pi+\theta}{4} = e^{\frac{s}{a}}.$$

10. Define an electric image, and find the surface density on an uninsulated spherical conductor (radius a) under the influence of a quantity e of electricity at an external point at a distance f from the centre of the sphere.

When the sphere is insulated and the whole charge on the sphere is $-e\frac{a^3}{f^3}$, find the position of the line of no electrification on the surface of the sphere, and the quantities of electricity on each side of this line.

(Maxwell, *Electricity*, § 157).

When uninsulated the charge induced on the sphere is $-e\frac{a}{f}$, and the surface density at any point is $-e\frac{f^2-a^2}{4\pi ar^3}$, where r is the distance from the influencing point.

Hence if the sphere be insulated and have a charge $-e\frac{a^3}{f^3}$, we must superpose on the preceding system a charge $e\left(\frac{a}{f}-\frac{a^3}{f^3}\right)$, uniformly distributed with surface density $e\frac{f^2-a^2}{4\pi af^3}$; and therefore the density at any point will be

$$e\frac{f^2-a^2}{4\pi a}\left(\frac{1}{f^3}-\frac{1}{r^3}\right).$$

At the line of no electrification on the surface of the sphere $r=f$; hence if A be the influencing point (fig. 79), B the image, C the centre of the sphere, and EE' the line of no electrification; then $AE=AC$, and therefore $BE=EC$.

The quantity of electricity on EDE'

$$=e\frac{f^2-a^2}{4\pi a}\int\left(\frac{1}{f^3}-\frac{1}{r^3}\right)2\pi a^2\sin\theta\,d\theta;$$

and $\qquad r^2=a^2-2af\cos\theta+f^2,\quad af\sin\theta\,d\theta=r\,dr.$

Therefore the quantity of electricity on EDE'

$$=\tfrac{1}{2}e\left(1-\frac{a^2}{f^2}\right)\int_f^{f+a}\left(\frac{r}{f^2}-\frac{f}{r^2}\right)dr$$

$$=\tfrac{1}{2}e\left(1+\frac{a^2}{f^2}\right)\left(\frac{r^2}{2f^2}+\frac{f}{r}\right)_f^{f+a}$$

$$=\tfrac{1}{4}e\frac{a^2}{f^2}\left(1-\frac{a}{f}\right)\left(3+\frac{a}{f}\right),$$

and therefore the quantity on EdE'

$$=-e\frac{a^3}{f^3}-\tfrac{1}{4}e\frac{a^2}{f^2}\left(1-\frac{a}{f}\right)\left(3+\frac{a}{f}\right)$$

$$=-\tfrac{1}{4}e\frac{a^2}{f^2}\left(1+\frac{a}{f}\right)\left(3-\frac{a}{f}\right).$$

xi. What is meant by the specific inductive capacity of a dielectric?

Investigate the electrostatic capacity, per unit of length, of a submarine cable, the diameter of the core being d, and the external diameter of the insulating sheath D.

Assuming the leakage to bear, at all points, the same ratio to the charge, form the equation for the transmission of electric potential along the cable: and shew from it that, *ceteris paribus*, the time necessary for a definite electrical operation is as the square of the cable's length. If the leakage be considerable, how must the battery-power depend on the length of the cable in order that slow signals may be of a given intensity?

(Maxwell, *Electricity*, §§ 52, 126; Stokes and Thomson, *Proceedings of the Royal Society*, VII.)

Let c be the electrostatic capacity per unit of length, so that cvl is the quantity of electricity required to charge a length l of the cable up to potential v.

Then $c = \frac{1}{2} \frac{K}{\log \frac{D}{d}}$, where K is the specific inductive capacity of the dielectric.

Let k denote the galvanic resistance of the cable, and let γ denote the strength at the time t of the current at a point P of the cable at a distance x from one end.

Let h denote the ratio of the leakage to the charge per unit of time.

The potential at the outside of the cable may be taken at each instant as zero; hence at the time t the quantity of electricity on a length dx of the cable at P will be $cvdx$.

The quantity that leaves the element dx in the time dt for the adjacent parts of the cable will be $dt \frac{d\gamma}{dx} dx$, and the leakage in the same time will be $hcvdxdt$.

Therefore
$$-cdx \frac{dv}{dt} dt = dt \frac{d\gamma}{dx} dx + hcvdxdt,$$

or
$$-\frac{d\gamma}{dx} = c\left(\frac{dv}{dt} + hv\right).$$

But the electromotive force at P is $-\frac{dv}{dx}$, and therefore $k\gamma = -\frac{dv}{dx}$; therefore

$$\frac{d^2v}{dx^2} = ck\left(\frac{dv}{dt} + hv\right),$$

the differential equation for the transmission of electric potential along the cable.

This is the differential equation for the propagation of the temperature v in a bar, of which c is the specific heat per unit of volume, and $\frac{1}{k}$, h the coefficients of interior and exterior conductibility of the bar per unit of length.

If we assume $v = e^{-ht}\phi$, the differential equation becomes

$$\frac{d^2\phi}{dx^2} = ck\frac{d\phi}{dt}.$$

The consideration of the dimensions of this equation shews that two cables will be similar, provided the squares of the lengths x, measured to similarly situated points, and therefore the squares of the whole lengths l, vary as the times divided by ck; or the time of an electrical operation is proportional to ckl^2.

Taking into consideration the leakage, the potential diminishes as e^{-ht}, and the time varies as the square of the length of the cable; hence the battery power must vary as $e^{\lambda l^2}$.

THURSDAY, *Jan.* 21, 1875. 9 to 12.

MR. COCKSHOTT. Roman numbers.
MR. GREENHILL. Arabic numbers.

1. EXPLAIN the general principle of reciprocal polars. Shew that the reciprocal of a circle with respect to a point is a conic section, and determine the nature and magnitude of this conic.

The diagonals of a quadrilateral inscribed in a circle intersect at right angles in a fixed point. Prove that the sides of the quadrilateral touch a fixed conic.

The angular points of a rectangle circumscribing a conic lie on the director circle.

Reciprocating with respect to a focus proves that if the diagonals of a quadrilateral inscribed in a circle intersect at right angles in a fixed point, the sides of the quadrilateral will touch a conic, of which the fixed point and the centre of the circle are foci.

ii. Prove by changing the order of integration, or otherwise, that

$$\int_0^x \frac{dy}{\sqrt{(x-y)}} \int_0^y \frac{f'(\xi)\,d\xi}{\sqrt{(y-\xi)}} = \pi \{f(x) - f(0)\}.$$

Prove also that

$$\int_0^{x_1} \frac{dx_2}{(x_1-x_2)^{\frac{n-1}{n}}} \int_0^{x_2} \frac{dx_3}{(x_2-x_3)^{\frac{n-1}{n}}} \cdots \int_0^{x_n} \frac{f'(\xi)\,d\xi}{(x_n-\xi)^{\frac{n-1}{n}}}$$

$$= \left\{\Gamma\left(\frac{1}{n}\right)\right\}^n \{f(x_1) - f(0)\}.$$

Changing the order of integration,

$$\int_0^x \frac{dy}{\sqrt{(x-y)}} \int_0^y \frac{f'(\xi)\,d\xi}{\sqrt{(y-\xi)}} = \int_0^x f'(\xi)\,d\xi \int_\xi^x \frac{dy}{\sqrt{(x-y)}\sqrt{(y-\xi)}}$$

$$= \pi \int_0^x f'(\xi)\,d\xi = \pi\,\{f(x)-f(0)\};$$

for

$$\int_\xi^x \frac{dy}{\sqrt{\{(x-y)(y-\xi)\}}} = \pi.$$

Similarly,

$$\int_0^x \frac{dy}{(x-y)^{\frac{n-1}{n}}} \int_0^y \frac{f'(\xi)\,d\xi}{(y-\xi)^{\frac{n-r}{n}}}$$

$$= \int_0^y f'(\xi)\,d\xi \int_\xi^x \frac{dy}{(x-y)^{\frac{n-1}{n}}(y-\xi)^{\frac{n-r}{n}}}$$

$$= \int_0^x \frac{f'(\xi)\,d\xi}{(x-\xi)^{\frac{n-r-1}{n}}} \int_0^1 \frac{dz}{(1-z)^{\frac{n-1}{n}} z^{\frac{n-r}{n}}}$$

$$= B\left(\frac{r}{n}, \frac{1}{n}\right) \int_0^x \frac{f'(\xi)\,d\xi}{(x-\xi)^{\frac{n-r-1}{n}}};$$

and therefore, by successive changes in the order of integration, the multiple integral becomes

$$B\left(\frac{1}{n}, \frac{1}{n}\right) B\left(\frac{2}{n}, \frac{1}{n}\right) \ldots B\left(\frac{n-1}{n}, \frac{1}{n}\right) \int_0^{x_1} f'(\xi)\,d\xi$$

$$= \left\{\Gamma\left(\frac{1}{n}\right)\right\}^n \{f(x_1)-f(0)\}.$$

iii. If u and $\phi(x, y)$ are two functions of x and y, which become U and $\Phi(X, Y)$ by a linear transformation in which the modulus is unity, prove that

$$\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right) u = \Phi\left(\frac{d}{dY}, -\frac{d}{dX}\right) U.$$

If $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 1$ represents a hyperbola and its conjugate referred to rectangular axes, form the equation which determines the lengths of the semi-axes.

Transformed to the axes, the equation will become

$$\left(\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}\right)^2=1,$$

where α and β are the semi-axes.

Therefore $$\left(\frac{1}{\alpha^2}\frac{d^2}{dy^2}-\frac{1}{\beta^2}\frac{d^2}{dx^2}\right)^2\left(\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}\right)^2$$

$$=\left(a\frac{d^4}{dy^4}-b\frac{d^4}{dx\,dy^3}+\ldots\right)(ax^4+bx^3y+\ldots),$$

or $$\frac{\lfloor\!\underline{4}}{\alpha^4\beta^4}+\frac{2^4}{\alpha^4\beta^4}+\frac{\lfloor\!\underline{4}}{\alpha^4\beta^4}=2\lfloor\!\underline{4}\,ae-2\lfloor\!\underline{3}\,bd+2^2e^2,$$

or $$\frac{1}{\alpha^4\beta^4}=\frac{12ae-3bd+c^2}{16}.$$

Again performing the equation $\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)^2$ on each function

$$\frac{\lfloor\!\underline{4}}{\alpha^4}-\frac{2^4}{\alpha^2\beta^2}+\frac{\lfloor\!\underline{4}}{\beta^4}=\lfloor\!\underline{4}\,a+2^3c+\lfloor\!\underline{4}\,e,$$

or $$\frac{1}{\alpha^4}+\frac{1}{\beta^4}=\frac{2}{3}\frac{1}{\alpha^2\beta^2}+a+\tfrac{1}{3}c+e$$

$$=\frac{\sqrt{(12ae-3bd+c^2)}}{6}+a+\tfrac{1}{3}c+e.$$

Hence the required equation is

$$1-R^4\left\{\frac{\sqrt{(12ae-3bd+e^2)}}{6}+a+\tfrac{1}{3}c+e\right\}+R^8\frac{12ae-3bd+c^2}{16}=0.$$

iv. State the conditions that the integral of a given function of x, y and the differential coefficients of y with respect to x shall be a maximum or minimum by the variation of the form of the function connecting x and y, and explain how the limits of integration and the constants introduced by the integration of the differential equation are determined when no conditions are imposed by the problem on the values of x and y or the differential coefficients at the limits of integration.

A lamina of given mass is symmetrical with respect to an axis and its density at any point varies as the square of the abscissa measured from one end of its axis; if the attraction upon a particle on the axis be a maximum, prove that the lamina is bounded by the oval $r^2 = \left(\frac{32m}{3\pi\sigma}\right) \cos\theta$, where m is the given mass and σ the density at unit distance, assuming the law of attraction to be that of the inverse square of the distance.

The attraction of the element ydx of the lamina is

$$\int \frac{\sigma x^3 dx\, dy}{(x^2+y^2)^{\frac{3}{2}}} = 2\sigma \frac{xy\, dx}{\sqrt{(x^2+y^2)}},$$

and therefore the attraction of the lamina is

$$2\sigma \int_{x_0}^{x_1} \frac{xy\, dx}{\sqrt{(x^2+y^2)}}.$$

The mass of the lamina is given by

$$m = 2\sigma \int_{x_0}^{x_1} x^2 y\, dx.$$

Let $V = \frac{xy}{\sqrt{(x^2+y^2)}} - \frac{x^2 y}{a^2}$; then we have to investigate the maximum value of $\int_{x_0}^{x_1} V dx$.

The condition $N - \frac{dP}{dx} + \ldots = 0$ reduces to $N = 0$, that is,

$$\frac{x}{\sqrt{(x^2+y^2)}} - \frac{xy^2}{(x^2+y^2)^{\frac{3}{2}}} - \frac{x^2}{a^2} = 0,$$

or $$(x^2+y^2)^{\frac{3}{2}} - a^2 x = 0;$$

therefore $$r^2 = a^2 \cos\theta,$$

the equation of the curve bounding the lamina.

At the limits $V_1 dx_1 - V_0 dx_0$ must vanish, and therefore $V_1 = 0$, $V_0 = 0$, which bends to $y_1 = 0$ and $y_0 = 0$. Thus the lamina must be bounded by the whole closed curve.

To determine a, since the mass of the lamina is m, therefore

$$m = \iint \sigma r^3 \cos^2\theta\, dr\, d\theta$$

$$= \tfrac{1}{4}\sigma a^4 \int_0^{\frac{1}{2}\pi} \cos^4\theta\, d\theta$$

$$= \tfrac{1}{4}\sigma a^4 \frac{3.1}{4.2}\frac{\pi}{2};$$

therefore $$a^4 = \frac{32m}{3\pi\sigma}.$$

5. Prove that

$$\iint \ldots x^{l-1} y^{m-1} z^{n-1} \ldots f(x+y+z+\ldots)\, dx\, dy\, dz \ldots,$$

the integral being so taken as to give the variables all positive values consistent with the condition that $x+y+z+\ldots$ is not greater than c, is equal to

$$\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\ldots}{\Gamma(l+m+n+\ldots)} \int_0^c f(h)\, h^{l+m+n+\ldots-1} dh.$$

If a rod be divided into p pieces at random prove that the chance that none of the pieces shall be less than $\frac{1}{m}$th of the whole, where m is greater than p, is $\left(1 - \frac{p}{m}\right)^{p-1}$.

If x be the distance of the n^{th} point of division from one end, then if each piece be greater than $\frac{1}{m}$th of the whole, we must have x greater than $\frac{n}{m}$ and $1 - x$ greater than $\frac{p-n}{m}$; and therefore

$$1 - \frac{p-n}{m} > x > \frac{n}{m}.$$

Hence, each point of division has a favourable range $1-\frac{p}{m}$ of the length of the rod when each part is not less than $\frac{1}{m}$th of the whole, and since there are $p-1$ points of division, the required chance is $\left(1-\frac{p}{m}\right)^{p-1}$.

6. State what general class of integrals can be reduced to elliptic integrals.

Distinguish the three species of elliptic integrals, and explain the terms sinam, cosam, Δam.

Prove that

$$\sin\left[\text{am}\,(u+v)+\text{am}\,(u+v)\right]=\frac{2\,\text{sinam}\,u\,\text{cosam}\,u\,\Delta\text{am}\,v}{1-k^2\,\text{sinam}^2 u\,\text{sinam}^2 v}.$$

Writing for shortness $x=\text{sinam}\,u$, $y=\text{sinam}\,v$, then

$$\text{sinam}\,(u+v)=\frac{x\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}+y\sqrt{(1-x)^2}\sqrt{(1-k^2x^2)}}{1-k^2x^2y^2}$$

$$=\frac{A+A'}{D}\text{ suppose,}$$

$$\text{cosam}\,(u+v)=\frac{\sqrt{(1-x^2)(1-y^2)}-xy\sqrt{(1-k^2x^2)}\sqrt{(1-k^2y^2)}}{1-k^2x^2y^2}$$

$$=\frac{B-B'}{D}\text{ suppose;}$$

and therefore

$$\text{sinam}\,(u-v)=\frac{A-A'}{D},\quad \text{cosam}\,(u-v)=\frac{B+B'}{D}.$$

Therefore $\sin\left[\text{am}\,(u+v)+\text{am}\,(u-v)\right]$

$$=\text{sinam}\,(u+v)\,\text{cosam}\,(u-v)+\text{sinam}\,(u-v)\,\text{cosam}\,(u+v)$$

$$=\frac{(A+A')(B+B')+(A-A')(B-B')}{D^2}$$

$$=2\,\frac{AB+A'B'}{D^2},$$

and $$AB+A'B'$$

$$=x\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}\sqrt{(1-x^2)}\sqrt{(1-y^2)}$$
$$+y\sqrt{(1-x^2)}\sqrt{(1-k^2x^2)}\,xy\sqrt{(1-k^2x^2)}\sqrt{(1-k^2y^2)}$$
$$=x\sqrt{(1-x^2)}\sqrt{(1-k^2y^2)}\{1-y^2+y^2(1-k^2x^2)\}$$
$$=x\sqrt{(1-x^2)}\sqrt{(1-k^2y^2)}\,D.$$

Hence $$2\frac{AB+A'B'}{D^2}=\frac{2x\sqrt{(1-x^2)}\sqrt{(1-k^2y^2)}}{1-k^2x^2y^2},$$

the required result.

vii. Prove by Newton's method that an orbit similar and equal to the apparent orbit of P round S in motion may be described round S fixed by the action of the same central force, and that the periodic time will be increased in the ratio of

$$\sqrt{(S+P)}:\sqrt{(S)}.$$

Prove that if two similar orbits be similarly described, $\alpha=\frac{\beta\gamma^2}{\varepsilon}$, $\lambda=\frac{\alpha}{\gamma}$; where α is the ratio of homologous linear dimensions of the orbits, β is the ratio of homologous forces, γ the ratio of the periodic times, ε the ratio of the masses, and λ the ratio of homologous velocities.

If PQ, pq be similar elementary arcs of the two orbits; PR, pr the tangents at P, p and RQ, rq the subtenses in the direction of the resultant impressed forces, then

$$\alpha=\frac{PR}{pr}=\frac{RQ}{rq},$$

or $$\alpha=\frac{vt}{v't'}=\frac{\frac{1}{2}\cdot\frac{P}{M}t^2}{\frac{1}{2}\cdot\frac{P'}{M'}t'^2},$$

where v, v' are the velocities at P, p respectively,
P, P'.........impressed forces................,
M, M' ....masses,
t, t' ....times of describing PQ, pq.

Therefore $\alpha = \lambda\gamma = \dfrac{\beta\gamma^2}{\varepsilon}$, the required relations.

8. Prove the differential equations for the motion of the moon supposing its orbit in the ecliptic,

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{H^2u^2} - \frac{T}{H^2u^3} \cdot \frac{du}{d\theta}, \text{ and } \frac{dH^2}{d\theta} = \frac{2T}{u^3}.$$

If
$$P = \mu u^2 - \frac{m^2n^2}{u} \{\tfrac{1}{2} + \tfrac{3}{2}\cos 2(1-m)\theta\},$$

$$T = -\frac{m^2n^2}{u} \cdot \tfrac{3}{2} \sin 2(1-m)\theta,$$

and we assume as integrals of the equations of motion

$$au = 1 + a_1 \cos 2(1-m)\theta, \quad H = na^2\{1 + h_1 \cos 2(1-m)\theta\},$$

and neglect squares and products of m^2, h_1 and a_1, prove that

$$h_1 = \frac{3m^2}{4(1-m)}, \quad a_1 = \frac{\tfrac{3}{2}m^2 \dfrac{2-m}{1-m}}{4(1-m)^2 - 1}, \quad \mu = n^2a^3(1 + \tfrac{1}{2}m^2),$$

and
$$\theta = nt + \varepsilon + \frac{2a_1 + h_1}{2(1-m)} \sin 2(1-m)(nt + \varepsilon).$$

To the required order of approximation

$$H^2 = n^2a^4\{1 + 2h_1 \cos 2(1-m)\theta\},$$

$$\frac{dH^2}{d\theta} = -4n^2a^4h_1(1-m) \sin 2(1-m)\theta,$$

$$\frac{2T}{u^3} = -3m^2n^2a^4 \sin 2(1-m)\theta;$$

therefore
$$h_1 = \frac{3m^2}{4(1-m)}.$$

Also $$\frac{d^2u}{d\theta^2}+u=\frac{1}{a}-\frac{a_1}{a}\{4(1-m)^2-1\}\cos 2(1-m)\theta,$$

$$\frac{P}{H^2u^2}=\frac{\mu}{H^2}-\frac{m^2n^2}{H^2u^3}\{\tfrac{1}{2}+\tfrac{3}{2}\cos 2(1-m)\theta\}$$

$$=\frac{\mu}{n^2a^4}\{1-2h_1\cos 2(1-m)\theta\}-\frac{m^2}{a}\{\tfrac{1}{2}+\tfrac{3}{2}\cos 2(1-m)\theta\},$$

and $$\frac{T}{h^2u^3}\frac{du}{d\theta}=0$$

to the required order.

Therefore, equating coefficients of like terms,

$$\frac{1}{a}=\frac{\mu}{n^2a^4}-\frac{1}{2}\frac{m^2}{a},$$

and $$\frac{a_1}{a}\{4(1-m)^2-1\}=\frac{2\mu h_1}{n^2a^4}+\frac{3}{2}\frac{m^2}{a};$$

or $$\mu=n^2a^3(1+\tfrac{1}{2}m^2),$$

and $$a_1\{4(1-m)^2-1\}=2h_1+\tfrac{3}{2}m^2$$

$$=\frac{3m^2}{2(1-m)}+\tfrac{3}{2}m^2$$

$$=\tfrac{3}{2}m^2\frac{2-m}{1-m}.$$

Also $$\frac{d\theta}{dt}=Hu^2;$$

therefore $$n\frac{dt}{d\theta}=\frac{n}{Hu^2}=1-(2a_1+h_1)\cos 2(1-m)\theta,$$

$$nt+\varepsilon=\theta-\frac{2a_1+h_1}{2(1-m)}\sin 2(1-m)\theta,$$

and, inverting the series,

$$\theta=nt+\varepsilon+\frac{2a_1+h_1}{2(1-m)}\sin 2(1-m)(nt+\varepsilon)$$

to the required order of approximation.

9. If a gravitating particle of mass m' be placed at the point $(x'y'z')$, prove that the work required to move a particle of unit mass from the point (xyz) to an infinite distance is

$$m'\{(x-x')^2+(y-y')^2+(z-z')^2\}^{-\frac{1}{2}}.$$

Prove also that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\left[m'\{(x-x')^2+(y-y')^2+(z-z')^2\}^{-\frac{1}{2}}\right]=0,$$

except when $x=x'$, $y=y'$, $z=z'$.

Extend these theorems by replacing the particle m' by gravitating bodies of finite extent and find the potential at any point due to a hollow sphere of uniform density.

Prove that the pressure per unit of length on any normal section of a spherical shell of mass m and radius a due to the mutual gravitation of the particles tends to the limit $\frac{m^2}{16\pi a^3}$ as the thickness of the shell is indefinitely diminished.

If $$r^2=(x-x')^2+(y-y')^2+(z-z')^2,$$

the work required to move a particle of unit mass from a distance r from the particle m' to an infinite distance is

$$\int_r^\infty m'\frac{dr}{r^2}=\frac{m'}{r}.$$

Also $\frac{d}{dx}\left(\frac{1}{r}\right)=-\frac{x-x'}{r^3}$, $\frac{d^2}{dx^2}\left(\frac{1}{r}\right)=\frac{3(x-x')^2-r^2}{r^5}$;

and therefore, provided r does not vanish,

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\frac{1}{r}=0,$$

or $$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\left[m'\{(x-x')+(y-y')^2+(z-z')^2\}^{-\frac{1}{2}}\right]=0.$$

If the particle m' be replaced by bodies of finite extent, then the preceding propositions, being true for every particle of the bodies, are true by addition for the whole bodies; and, therefore, if V be the work required to move a particle of unit mass from a point (xyz) to an infinite distance, and if ρ' be the density at the point $(x'y'z')$, then

$$V = \iiint \rho' dx' dy' dz' \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{-\frac{1}{2}},$$

and if (xyz) be a point in free space

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0.$$

If, however, ρ be the density at the point (xyz),

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0.$$

For a hollow sphere of uniform density ρ, V depends only on the distance from the centre, and taking the centre of the sphere as origin, and $r^2 = x^2 + y^2 + z^2$,

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} + 4\pi\rho = 0;$$

therefore $$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) + 4\pi\rho r^2 = 0,$$

$$r^2 \frac{dV}{dr} = -\int_0^r 4\pi\rho r^2 dr = -m_1,$$

where m_1 is the whole amount of matter within the spherical surface of radius r.

If a, b be the external and internal radii of the shell, then

(i) When $r > a$, $m_1 = \frac{4}{3}\pi\rho(a^3 - b^3) = m$, $\dfrac{dV}{dr} = -\dfrac{m}{r^2}$,

and $$V = \frac{m}{r},$$

where m is the mass of the shell.

(ii) $a > r > b$, $m_1 = \frac{4}{3}\pi\rho(r^3 - b^3)$, $\dfrac{dV}{dr} = -\frac{4}{3}\pi\rho\left(r - \dfrac{b^3}{r^2}\right)$,

$$V = C - \tfrac{2}{3}\pi\rho r^2 - \tfrac{4}{3}\pi\rho\frac{b^3}{r},$$

and when $$r=a, \quad V=\tfrac{4}{3}\pi\rho\left(a^2-\frac{b^3}{a}\right);$$

therefore $$V=\tfrac{2}{3}\pi\rho\,(a^2-r^2)+\tfrac{4}{3}\pi\rho\left(a^2-\frac{b^3}{r}\right).$$

(iii) $$b>r, \quad m_1=0, \quad \frac{dV}{dr}=0,$$

and V is constant and equal to the value it has when $r=b$; therefore $V=2\pi\rho\,(a^2-b^2)$.

The resultant force of attraction on one half of the shell

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_b^a -\frac{dV}{dr}\cos\theta\,\rho\,dr\,r\,d\theta\,r\sin\theta\,d\phi$$

$$= -\pi\rho\int_b^a r^2\frac{dV}{dr}dr$$

$$= \tfrac{4}{3}\pi^2\rho^2\int_b^a (r^3-b^3)\,dr$$

$$= \tfrac{4}{3}\pi^2\rho^2\left\{\tfrac{1}{4}(a^4-b^4)-b^3(a-b)\right\}$$

$$= \tfrac{1}{3}\pi^2\rho^2(a-b)^2(a^2+2ab+3b^2)$$

$$= \tfrac{3}{16}m^2\frac{(a-b)^2(a^2+2ab+3b^2)}{(a^3-b^3)^2}$$

$$= \tfrac{3}{16}m^2\frac{a^2+2ab+3b^2}{(a^2+ab+b^2)^2},$$

which becomes $\dfrac{m^2}{8a^2}$ when b is made equal to a.

Therefore the pressure per unit of length on any normal section will be $\dfrac{m^2}{16\pi a^3}$.

10. Establish the equation of continuity of a fluid, (i) by considering the fluid which enters and leaves a fixed element of space, (ii) by following the motion of an element of the fluid.

Find the rate at which momentum in the direction of the axis of x of the fluid which is instantaneously within the element of space $dx\,dy\,dz$ is increased on account of the fluid which enters and leaves that element.

Hence prove that

$$\frac{d(u\rho)}{dt}+\frac{d(u^2\rho)}{dx}+\frac{d(uv\rho)}{dy}+\frac{d(uw\rho)}{dz}+\frac{dp}{dx}-\rho X=0$$

with the usual notation, and deduce the ordinary equations of fluid motion.

Considering only what takes place in the interior and on the surface of the element of space $dx\,dy\,dz$ at the point xyz, and considering only the components of the force and momentum parallel to the axis of x, then the momentum of the fluid which is in $dx\,dy\,dz$ at the time t is $u\rho\,dx\,dy\,dz$, and the momentum of the fluid which is in $dx\,dy\,dz$ at the time $t+dt$ is $u\rho\,dx\,dy\,dz+\frac{d(u\rho)}{dt}dt\,dx\,dy\,dz$.

In the time $t+dt$

(1) momentum $u^2\rho\,dt\,dy\,dz$ enters the element of space $dx\,dy\,dz$ by the face x, and momentum

$$u^2\rho\,dt\,dy\,dz+\frac{d(u^2\rho)}{dx}dt\,dx\,dy\,dz$$

leaves by the face $x+dx$;

(2) momentum $uv\rho\,dt\,dz\,dx$ enters by the face y, and momentum $uv\rho\,dt\,dz\,dx+\frac{d(uv\rho)}{dy}dt\,dx\,dy\,dz$ leaves by the face $y+dy$;

(3) momentum $uw\rho\,dt\,dx\,dy$ enters by the face z, and momentum $uw\rho\,dt\,dx\,dy+\frac{d(uw\rho)}{dz}dt\,dx\,dy\,dz$ leaves by the face $z+dz$.

Hence the rate at which momentum in the direction of the axis of x is being generated within the element of space $dx\,dy\,dz$ per unit of volume is

$$\frac{d(u\rho)}{dt}+\frac{d(u^2\rho)}{dx}+\frac{d(uv\rho)}{dy}+\frac{d(uw\rho)}{dz}.$$

Considering the pressures which act on the element $dxdydz$, the pressure p on the face x generates in the time dt the momentum $pdtdydz$, and the pressure $p+\frac{dp}{dx}dx$ on the face $x+dx$ generates in the time dt the momentum

$$-pdtdydz-\frac{dp}{dx}dtdxdydz.$$

The pressures on the faces parallel to the axis of x will generate no momentum parallel to the axis of x.

The impressed force X generates in the time dt the momentum $\rho Xdtdxdydz$.

Hence the rate at which momentum in the direction of the axis of x is generated within the element of space $dxdydz$ by the pressure of the surrounding fluid on the element and by the impressed force per unit of volume is

$$\rho X-\frac{dp}{dx}.$$

Therefore

$$\rho X-\frac{dp}{dx}=\frac{d(u\rho)}{dt}+\frac{d(u^2\rho)}{dx}+\frac{d(uv\rho)}{dy}+\frac{d(uw\rho)}{dz},$$

the required equation.

Combined with the equation of continuity

$$\frac{d\rho}{dt}+\frac{d(u\rho)}{dx}+\frac{d(v\rho)}{dy}+\frac{d(w\rho)}{dz}=0,$$

this equation reduces to the ordinary form

$$X-\frac{1}{\rho}\frac{dp}{dx}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}.$$

xi. If the velocity of normal propagation of a plane wave of light-vibrations in a crystal in a direction (l, m, n) referred to a system of rectangular axes be given by the equation

$$\frac{l^2}{a^2-v^2}+\frac{m^2}{b^2-v^2}+\frac{n^2}{c^2-v^2}=0,$$

prove that when a disturbance spreads from the origin of coordinates, the locus of all particles in the same state of vibration is a surface of the form

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0.$$

If a, b, c are unequal and in descending order of magnitude, prove that the surface has four real nodes, and that the equation of the normal cone at one of them referred to axes parallel to axes of the surface is

$$x^2(b^2-c^2)+y^2(a^2-c^2)+z^2(a^2-b^2)=\frac{xz}{ac}(a^2+c^2)\sqrt{(a^2-b^2)}\sqrt{(b^2-c^2)}.$$

In finding in the usual way the envelope of the plane

$$lx+my+nz=v,$$

subject to the conditions

$$\frac{l^2}{a^2-v^2}+\frac{m^2}{b^2-v^2}+\frac{n^2}{c^2-v^2}=0,$$

$$l^2+m^2+n^2=1,$$

we obtain

$$\frac{x}{r^2-a^2}=\frac{lv}{v^2-a^2},$$

$$\frac{y}{r^2-b^2}=\frac{mv}{v^2-b^2},$$

$$\frac{z}{r^2-c^2}=\frac{nv}{v^2-c^2},$$

from which we find the required equation of the wave surface.

At the four real nodes $y=0$, $r=b$, and

$$\frac{a^2x^2}{a^2-b^2}=\frac{c^2z^2}{b^2-c^2}=\frac{a^2c^2}{a^2-c^2}.$$

Therefore
$$\frac{x}{a^2-b^2}=\frac{lv}{a^2-v^2},$$

$$\frac{z}{b^2-c^2}=\frac{nv}{v^2-c^2};$$

or
$$\frac{a^2}{v} - v = (a^2 - b^2)\frac{l}{x},$$
$$v - \frac{c^2}{v} = (b^2 - c^2)\frac{n}{z}.$$

Therefore
$$\frac{a^2 - c^2}{v} = (a^2 - b^2)\frac{l}{x} + (b^2 - c^2)\frac{n}{z},$$
$$(a^2 - c^2)\, v = c^2 (a^2 - b^2)\frac{l}{x} + a^2 (b^2 - c^2)\frac{n}{z}.$$

Eliminating v,
$$(a^2-c^2)^2 = c^2(a^2-b^2)^2\frac{l^2}{x^2} + (a^2-b^2)(b^2-c^2)(a^2+c^2)\frac{ln}{xz} + a^2(b^2-c^2)\frac{n^2}{z^2},$$
or $(a^2 - c^2)(l^2 + m^2 + n^2)$
$$= (a^2 - b^2)\, l^2 + \frac{a^2 + c^2}{ac}\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}\, lz + (b^2 - c^2)n^2,$$
or $l^2 (b^2 - c^2) + m^2 (a^2 - c^2) + n^2 (a^2 - b^2)$
$$= \frac{ln}{ac}(a^2 + c^2)\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}.$$

xii. Find the intensities of the reflected and refracted rays, when light polarized perpendicularly to the plane of incidence falls on a refracting surface of glass, stating clearly the assumptions that are made.

If common light fall on a series of parallel plates of glass at an angle $\tan^{-1}\mu$, where μ is the coefficient of refraction for glass, prove that the light reflected at any of the surfaces of the plates will be completely polarized in the plane of incidence.

The assumptions rest on four principles (*Œuvres de Verdet*, tome v., p. 397).

1. The principle of energy, by which the energy of the incident ray is equal to the energy of the reflected and refracted rays.

2. The principle of continuity, by which the differences of velocity and displacement of points indefinitely near to one another on opposite sides of the surface of separation of the media are infinitesimally small.

3. The principle of sudden change at the surface of separation, by which it is assumed that the change from the incident to the reflected and refracted ray takes place immediately.

4. The principle of the constitution of the ether, in which it is assumed that the pressure of the ether is the same in all media, and that the difference of the velocity of light is due to the difference of the densities of the ether, so that the velocity of light in a medium is inversely proportional to the square root of the density of the ether.

If a denote the amplitude of the incident, b of the reflected and c of the refracted ray, where b and c are estimated positively when the direction of the vibration of the reflected or refracted ray coincides with the direction of vibration of the incident ray when the reflected or refracted ray is turned so as to be in the prolongation of the incident ray (fig. 80), and if i, i' be the angles of incidence and refraction; then, by principles 1 and 4,

$$\frac{a^2-b^2}{c^2}=\frac{\sin i\cos i'}{\sin i'\cos i},$$

and, by principles 2 and 3,

$$(a-b)\cos i=c\cos i'.$$

Therefore $$\frac{a+b}{c}=\frac{\sin i}{\sin i'},$$

and $$\frac{b}{a}=\frac{\tan(i-i')}{\tan(i+i')},\quad \frac{c}{a}=\frac{2\sin i'\cos i}{\sin(i+i')\cos(i-i')},$$

If $i+i'=90°$, then $\tan i=\mu$, and the light which is reflected at the surface will be completely polarized in the plane of incidence.

If the light fall on a series of parallel plates, then, since $i + i' = 90°$, the light after any number of refractions, internal or external, will always be incident at the polarizing angle, and the reflected light will be completely polarized in the plane of incidence.

THURSDAY, *Jan.* 21, 1875. $1\frac{1}{2}$ to 4.

PROF. TAIT. Roman numbers.
MR. GREENHILL. Arabic numbers.

1. IF a binary quantic contain a linear factor α times and not more, prove that the Hessian will contain the same linear factor $2\alpha-2$ times and not more.

Find the conditions that a binary quartic may be a perfect square, and considering the coefficients as being each of the order unity, shew that the order of the system is equal to 4.

The repeated factor may without loss of generality be taken to be x, the quantic is then $x^\alpha\phi$, and it is to be shown that the Hessian contains the factor $x^{2\alpha-2}$, and not any higher power of x. The first differential coefficients of $x^\alpha\phi$ are

$$\alpha x^{\alpha-1}\phi + x^\alpha \frac{d\phi}{dx},\ x^\alpha\frac{d\phi}{dy};$$

and hence the Hessian is

$$\left\{\alpha(\alpha-1)x^{\alpha-2}\phi + 2\alpha x^{\alpha-1}\frac{d\phi}{dx} + x^\alpha\frac{d^2\phi}{dx^2}\right\} x^\alpha\frac{d^2\phi}{dy^2}$$

$$-\left\{\alpha x^{\alpha-1}\frac{d\phi}{dy} + x^\alpha\frac{d^2\phi}{dx\,dy}\right\}^2$$

$$=\alpha x^{2\alpha-2}\left\{(\alpha-1)\phi\frac{d^2\phi}{dy^2} - \alpha\left(\frac{d\phi}{dy}\right)^2\right\} + \text{terms in } x^{2\alpha-1} \text{ and } x^{2\alpha}.$$

Hence the Hessian contains the factor $x^{2\alpha-2}$, and it only remains to show that it does not contain any higher power of x; this is so if the coefficient of $x^{2\alpha-2}$ does not vanish for $x=0$.

Suppose ϕ is of the order n, and write A, B, C for its second differential coefficients, we have

$$\phi = \frac{1}{n(n-1)}(Ax^2 + 2Bxy + Cy^2),$$

$$\frac{d\phi}{dy} = \frac{1}{n-1}(Bx + Cy)$$

$$\frac{d^2\phi}{dy^2} = C;$$

and thence the coefficient of $x^{2\alpha-2}$ is

$$= \frac{\alpha - 1}{n\,(n-1)}\{Ax^2 + 2Bxy + Cy^2\}\,C - \frac{\alpha}{(n-1)^2}(Bx + Cy)^2,$$

which for $x = 0$ becomes

$$\frac{1}{n\,(n-1)^2}\{(n-1)(\alpha-1) - n\alpha\}\,C^2y^2, \quad = \frac{\alpha + n - 1}{n\,(n-1)^2}\,C^2y^2.$$

But the quantic $x^\alpha\phi$ contains the linear factor α (and not more) times, hence ϕ does not contain the linear factor x, and consequently its second differential coefficient C does not vanish with x; and the remaining factor $\dfrac{\alpha + n - 1}{n\,(n-1)^2}$ does not vanish for any positive integral values of α or n; hence the Hessian contains a non-vanishing term in $x^{2\alpha-2}$, or it contains the linear factor $2\alpha - 2$ (and not more) times.

If a binary quartic contains a linear factor twice, then the Hessian contains the same factor twice; and hence if the quartic is a perfect square (that is, if it contains two linear factors each twice), the Hessian will contain the same factors each twice, or it will be a mere constant multiple of the quartic. Taking the quartic to be

$$(a, b, c, d, e \between x, y)^4,$$

the conditions are

$$\frac{ac - b^2}{a} = \frac{2\,(ad - bc)}{4b} = \frac{ae + 2bd - 3c^2}{6c} = \frac{2\,(be - cd)}{4d} = \frac{ce - d^2}{e},$$

these being, of course, equivalent to a two-fold relation between the coefficients (a, b, c, d, e).

The order of the system is equal to the number of solutions obtained by combining with the foregoing a number of arbitrary linear relations sufficient to render the system determinate; that is, 2 arbitrary linear relations. The conditions express that there exist quantities (α, β, γ), such that

$$(a, b, c, d, e \between x, y)^4 = (\alpha x^2 + 2\beta xy + \gamma y^2)^2$$

identically, viz. that we have

$$a, 4b, 6c, 4d, e = (\alpha^2, 4\alpha\beta, 2\alpha\gamma + 4\beta^2, 4\beta\gamma, \gamma^2).$$

Imagining these values substituted in the arbitrary linear relations, we have for the determination of the ratios $\alpha : \beta : \gamma$ two equations of the form $(\alpha, \beta, \gamma)^2 = 0$ giving 4 systems of values of α, β, γ, and therefore also 4 systems of values of (a, b, c, d, e); or the order of the system in (a, b, c, d, e) is 4.

2. State and prove Sturm's theorem for determining the number and position of the real roots of an equation.

If $$X_n = \frac{1}{\lfloor n} \cdot \frac{d^n}{dx^n} \left(\frac{x^2-1}{2}\right)^n,$$

prove that $$nX_n = (2n-1)\,xX_{n-1} - (n-1)\,X_{n-2},$$

and hence shew that the roots of the equations $X_n = 0$, $X_{n-1} = 0, \ldots$ are all real and between -1 and 1.

Putting $\dfrac{x^2-1}{2} = z$, then

$$\begin{aligned} & nX_n - (2n-1)\,xX_{n-1} \\ = \ & \frac{1}{\lfloor n-1} \left\{ \frac{d^n z^n}{dx^n} - (2n-1)\,x\,\frac{d^{n-1}z^{n-1}}{dx^{n-1}} \right\} \\ = \ & \frac{1}{\lfloor n-1} \left\{ n \cdot \frac{d^{n-1}xz^{n-1}}{dx^{n-1}} - (2n-1)\,\frac{d^{n-1}xz^{n-1}}{dx^{n-1}} \right. \\ & \qquad \left. + (2n-1)(n-1)\,\frac{d^{n-2}z^{n-1}}{dx^{n-2}} \right\} \end{aligned}$$

$$= -\frac{1}{\lfloor n-1}\left\{(n-1)\frac{d^{n-1}xz^{n-1}}{dx^{n-1}} - (2n-1)(n-1)\frac{d^{n-2}z^{n-1}}{dx^{n-2}}\right\}$$

$$= -\frac{1}{\lfloor n-2}\frac{d^{n-2}}{dx^{n-2}}\left\{\frac{dxz^{n-1}}{dx} - (2n-1)z^{n-1}\right\}$$

$$= -\frac{1}{\lfloor n-2}\frac{d^{n-2}}{dx^{n-2}}\left\{(n-1)x^2z^{n-2} + z^{n-1} - (2n-1)z^{n-1}\right\}$$

$$= -\frac{n-1}{\lfloor n-2}\frac{d^{n-2}z^{n-2}}{dx^{n-2}} = -(n-1)X_{n-2}.$$

Considering the series of functions X_n, X_{n-1}, ..., X_0, we see that $X_0 = 1$, and that two consecutive functions cannot therefore vanish for the same value of x; also that when a function vanishes, the adjacent functions are of opposite sign; hence the functions may be considered as a series of Sturm's functions.

When $x=-1$, $X_m = (-1)^m$, and when $x=1$, $X_m = 1$; therefore the roots of the functions are all real and comprised between -1 and 1, and the roots of the equation $X_{n-1}=0$ separate the roots of the equation $X_n = 0$.

3. If $\psi t = \dfrac{a+bt}{c+et}$, prove that

$$\psi^y t = \frac{b+c}{2e\cos\alpha}\cdot\frac{b\sin y\alpha - c\sin(y+2)\alpha - 2et\cos\alpha\sin(y+1)\alpha}{b\sin(y-1)\alpha - c\sin(y+1)\alpha - 2et\cos\alpha\sin y\alpha} - \frac{c}{e},$$

where $(b+c)^2 = 4(bc-ae)\cos^2\alpha$, and hence prove that the condition that ψ is a periodic function of the xth order is

$$\alpha = \frac{i\pi}{x} \text{ or } e = -\frac{b^2 - 2bc\cos\frac{2i\pi}{x} + c^2}{4a\cos^2\frac{i\pi}{x}},$$

i being an integer not a multiple of x.

Find $\psi^y t$ when $(b+c)^2 > 4(bc-ae)$, and discuss the case when $(b+c)^2 = 4(bc-ae)$.

Let $\psi^y t = u_y$; then

$$u_{y+1} = \frac{a + bu_y}{c + eu_y}, \text{ or } u_{y+1}(c + eu_y) = a + bu_y.$$

Let $u_y = \frac{v_{y+1}}{v_y} - \frac{c}{e}$; then

$$\left(\frac{v_{y+2}}{v_{y+1}} - \frac{c}{e}\right) e \frac{v_{y+1}}{v_y} = a + b\left(\frac{v_{y+1}}{v_y} - \frac{c}{e}\right),$$

or $$e^2 v_{y+2} - e(b + c) v_{y+1} + bc - ae = 0.$$

Hence $v_y = A\beta^y + B\gamma^y$, where β, γ are the roots of the equation

$$e^2 z^2 - e(b + c) z + bc - ae = 0.$$

If $(b + c)^2 < 4(bc - ae)$, the roots of this equation are impossible; and, putting $(b + c)^2 = 4(bc - ae)\cos^2\alpha$, the roots are

$$\frac{b + c}{2e \cos\alpha}\{\cos\alpha \pm \sqrt{(-1)}\sin\alpha\}.$$

Therefore

$$u_y = \frac{b + c}{2e\cos\alpha}\,\frac{C\sin(y + 1)\alpha + D\cos(y + 1)\alpha}{C\sin y\alpha + D\cos y\alpha} - \frac{c}{e};$$

and putting $y = 0$, $u_0 = t$, and

$$\frac{b + c}{2e\cos\alpha}\left(\frac{C}{D}\sin\alpha + \cos\alpha\right) - \frac{c}{e} = t;$$

therefore $$\frac{C}{D} = \frac{2et\cos\alpha - (b - c)\cos\alpha}{(b + c)\sin\alpha},$$

and $$\frac{C\sin(y + 1)\alpha + D\cos(y + 1)\alpha}{C\sin y\alpha + D\cos y\alpha}$$

$$= \frac{2et\cos\alpha\sin(y+1)\alpha - (b-c)\cos\alpha\sin(y+1)\alpha + (b+c)\sin\alpha\cos(y+1)\alpha}{2et\cos\alpha\sin y\alpha - (b - c)\cos\alpha\sin y\alpha + (b + c)\sin\alpha\cos y\alpha}$$

$$= \frac{b\sin y\alpha - c\sin(y + 2)\alpha - 2et\cos\alpha\sin(y + 1)\alpha}{b\sin(y - 1)\alpha - c\sin(y + 1)\alpha - 2et\cos\alpha\sin y\alpha}.$$

Hence $\psi^x t = t$ if $\alpha = \frac{i\pi}{x}$, and then, since

$$(b+c)^2 = 4(bc - ae)\cos^2\alpha,$$

$$4ae\cos^2\frac{i\pi}{x} + b^2 - 2bc\cos\frac{2i\pi}{x} + c^2 = 0.$$

If $(b+c)^2 > 4(bc - ae)$, β and γ are real, and

$$u_y = \frac{A\beta^{y+1} + B\gamma^{y+1}}{A\beta^y + B\gamma^y} - \frac{c}{e};$$

and putting $y = 0$, $u_0 = t$, and

$$\frac{A\beta + B\gamma}{A+B} = t + \frac{c}{e};$$

therefore $$\frac{A}{B} = -\frac{\gamma - t - \frac{c}{e}}{\beta - t - \frac{c}{e}},$$

and $$u_y = \frac{\left(\gamma - t - \frac{c}{e}\right)\beta^{y+1} - \left(\beta - t - \frac{c}{e}\right)\gamma^{y+1}}{\left(\gamma - t - \frac{c}{e}\right)\beta^y - \left(\beta - t - \frac{c}{e}\right)\gamma^y} - \frac{c}{e},$$

and the function cannot be periodic.

If $(b+c)^2 = 4(bc - ae)$, then $\beta = \gamma = \frac{b+c}{2}$, and

$$v_y = (A + By)\beta^y,$$

$$u_y = \frac{\beta - \left(\beta - t - \frac{c}{e}\right)(y+1)}{\beta - \left(\beta - t - \frac{c}{e}\right)y}\beta - \frac{c}{e},$$

4. Find the measures of curvature and tortuosity at any point of a given curve.

If at any point of a curve the osculating helix be drawn having the same curvature and tortuosity as the curve, prove that the axis lies along the shortest distance between consecutive principal normals, and that if along the curve and the helix equal arcs δs be measured from the point of contact and on the same side of it, the distance between the ends of these arcs will be ultimately $\frac{\delta s^3}{6\rho^2} \cdot \frac{d\rho}{ds}$, where ρ is the radius of curvature.

The osculating helix touches the curve and has the same osculating plane as the curve at two consecutive points; hence the curve and the helix will have the same principal normals at two consecutive points, and therefore the axis of the helix lies along the shortest distance between consecutive principal normals.

If l, m, n be the direction-cosines of the principal normal at the point xyz, and if δx, $\delta x'$, be the relative abscissa of the ends of equal arcs δs measured on the curve and on the osculating helix, then since $\frac{dx}{ds}$ and $\frac{d^2x}{ds^2}$ are the same for the curve and for the helix, therefore

$$\delta x' - \delta x = \frac{\delta s^3}{6}\left(\frac{d^3x'}{ds^3} - \frac{d^3x}{ds^3}\right) \text{ to the 3rd order.}$$

Now $\frac{d^2x}{ds^2} = \frac{l}{\rho}$, and $\frac{dl}{ds}$ is the same for the curve and the helix, but in the helix $\frac{d\rho}{ds} = 0$.

Therefore $$\frac{d^3x}{ds^3} = \frac{1}{\rho}\frac{dl}{ds} - \frac{l}{\rho^2}\frac{d\rho}{ds},$$

and $$\frac{d^3x'}{ds^3} = \frac{1}{\rho}\frac{dl}{ds};$$

therefore $$\frac{d^3x'}{ds^3} - \frac{d^3x}{ds^3} = \frac{l}{\rho^2}\frac{d\rho}{ds},$$

$$\delta x' - \delta x = l\frac{\delta s^3}{6\rho^2}\frac{d\rho}{ds};$$

and therefore the required distance is $\frac{\delta s^3}{6\rho^2}\frac{d\rho}{ds}$ to the 3rd order.

v. State, briefly, to what classes of enquiries Laplace's coefficients are most directly applicable, and mention the properties which render them so useful.

Show that the coefficient of h^i in the expansion of $(1-2\mu h+h^2)^{-\frac{1}{2}}$ is

$$Q_i=\frac{1}{\underline{|i}}\left(\frac{d}{d\mu}\right)^i\left(\frac{\mu^2-1}{2}\right)^i.$$

Also prove that

$$\int_{-1}^{+1} Q_i Q_j d\mu=0, \text{ or }=\frac{2}{2i+1},$$

according as i and j are different, or equal, positive integers.

Spherical Harmonics or Laplace's coefficients are used to express in converging series the potential and attraction of bodies, and the velocity function of liquids.

The usefulness of these functions depends on the fundamental property that a harmonic distribution of density on a spherical surface produces potential, which is the allied solid harmonic of positive degree in the interior and of negative degree in the exterior of the sphere; and, similarly, for the velocity function of a liquid.

Let $$(1-2\mu h+h^2)^{\frac{1}{2}}=1-xh,$$

then as μ increases from -1 to $+1$, x also increases from -1 to $+1$; also

$$1-2\mu h+h^2=1-2xh+x^2h^2;$$

therefore $$\mu=x+\tfrac{1}{2}h(1-x^2),$$

or $$x=\mu+h\frac{x^2-1}{2},$$

and therefore by Lagrange's theorem

$$x=\mu+h\frac{\mu^2-1}{2}+\ldots+\frac{h^n}{\underline{|n}}\left(\frac{d}{d\mu}\right)^{n-1}\left(\frac{\mu^2-1}{2}\right)^n+\ldots.$$

Also
$$\frac{d\mu}{dx} = 1 - xh$$
$$= (1 - 2\mu h + h^2)^{\frac{1}{2}};$$

therefore
$$\frac{dx}{d\mu} = (1 - 2\mu h + h^2)^{-\frac{1}{2}},$$

and therefore
$$Q_i = \frac{1}{\lfloor\underline{i}} \left(\frac{d}{d\mu}\right)^i \left(\frac{\mu^2 - 1}{2}\right)^i.$$

Integrating by parts, $\int_{-1}^{1} Q_i Q_j d\mu$

$$= \frac{1}{\lfloor\underline{i}} \frac{1}{\lfloor\underline{j}} \left\{\left(\frac{d}{d\mu}\right)^{i-1} \left(\frac{\mu^2 - 1}{2}\right)^{i-1} \left(\frac{d}{d\mu}\right)^j \left(\frac{\mu^2 - 1}{2}\right)^j\right\}_{-1}^{1}$$
$$- \frac{1}{\lfloor\underline{i}} \frac{1}{\lfloor\underline{j}} \int_{-1}^{1} \left(\frac{d}{d\mu}\right)^{i-1} \left(\frac{\mu^2 - 1}{2}\right)^i \left(\frac{d}{d\mu}\right)^{j+1} \left(\frac{\mu^2 - 1}{2}\right)^j d\mu;$$

the first part vanishes at both limits, because $\left(\frac{d}{d\mu}\right)^{i-1} \left(\frac{\mu^2 - 1}{2}\right)^i$ contains $\mu^2 - 1$ as a factor.

Hence, continuing the process of integration by parts,

$$\int_{-1}^{1} Q_i Q_j d\mu = \frac{(-1)^j}{\lfloor\underline{i}\lfloor\underline{j}} \int_{-1}^{1} \left(\frac{d}{d\mu}\right)^{i-j} \left(\frac{\mu^2 - 1}{2}\right)^i \left(\frac{d}{d\mu}\right)^{2j} \left(\frac{\mu^2 - 1}{2}\right)^j d\mu$$
$$= \frac{(-1)^i \lfloor\underline{2j}}{2^j \lfloor\underline{i}\lfloor\underline{j}} \int_{-1}^{1} \left(\frac{d}{d\mu}\right)^{i-j} \left(\frac{\mu^2 - 1}{2}\right)^i d\mu.$$

If i and j are different, this is

$$\frac{(-1)^i \lfloor\underline{2j}}{2^j \lfloor\underline{i}\lfloor\underline{j}} \left(\frac{d}{d\mu}\right)^{i-j-1} \left(\frac{\mu^2 - 1}{2}\right)^i,$$

which vanishes at both limits.

But if i and j are equal, this is

$$\frac{(-1)^i \lfloor\underline{2i}}{2^i (\lfloor\underline{i})^2} \int_{-1}^{1} \left(\frac{\mu^2 - 1}{2}\right)^i d\mu;$$

$$\text{and} \int_{-1}^{1} (\mu^2-1)^i d\mu = \left\{\mu(\mu^2-1)\right\}_{-1}^{1} - 2i\int_{-1}^{1} \mu^2(\mu^2-1)^{i-1} d\mu$$

$$= -\frac{2i}{2i+1}\int_{-1}^{1} (\mu^2-1)^{i-1} d\mu$$

$$= (-1)^i \frac{2i(2i-2)\ldots 2}{(2i+1)\ldots 3}\int_{-1}^{1} d\mu$$

$$= 2(-1)^i \frac{2i(2i-2)\ldots 2}{(2i+1)\ldots 3}.$$

Therefore

$$\int_{-1}^{1} Q_i^2 d\mu$$

$$= 2\,\frac{\lfloor 2i \; 2i(2i-2)\ldots 2}{2^{2i}(\lfloor i)^2 (2i+1)(2i-1)\ldots 3}$$

$$= \frac{2}{2i+1}.$$

6. Assuming that if V be the potential at a point outside the spheroid at a distance r from the centre of a homogeneous spheroid of small ellipticity, the equation to whose bounding surface is $r = k(1 - \frac{2}{3}\varepsilon Q_2)$, where k is the mean radius of the spheroid, ρ the density, ε the ellipticity, and Q_2 the zonal harmonic of the second order

$$V = \frac{4\pi\rho k^3}{3r} - \frac{8\pi\rho k^5}{15r^3}.\varepsilon Q_2,$$

deduce the corresponding expression for V when the spheroid, instead of being homogeneous, is composed of strata of equal density, the general equation to which is $r = k(1 - \frac{2}{3}\varepsilon Q_2)$, where ε the ellipticity and ρ the density are functions of k.

If the spheroid be revolving with angular velocity ω about its axis, and the external surface be a level surface for gravitation and centrifugal force, prove that at an external point

$$V = \frac{M}{r} - \frac{2Mk_0^2}{3r^3}(\varepsilon_0 - \tfrac{1}{2}m)\, Q_2,$$

where ε_0, k_0 refer to the external surface, M is the mass of the spheroid, $m = \dfrac{\omega^2 k_0^{\,3}}{M}$, and m is considered of the same order as ε_0.

Hence prove that at a point on the surface of the spheroid the resultant of gravitation and centrifugal force is

$$\frac{M}{k_0^{\,2}}\left(1 - \tfrac{2}{3}m\right)\left\{1 + \tfrac{2}{3}\left(\tfrac{5}{2}m - \varepsilon_0\right) Q_2\right\}.$$

The homogeneous spheroid may be considered as made up of a sphere of radius k and of a distribution of surface density $-\frac{2}{3}\rho k\varepsilon Q_2$ on a spherical surface of radius k.

By § 536 of Thomson and Tait's *Natural Philosophy*, the potential of this harmonic distribution of density at an external point is $-\dfrac{8\pi\rho k^5 \varepsilon Q_2}{15r^3}$; and, therefore, for the homogeneous spheroid

$$V = \frac{4\pi\rho k^3}{3r} - \frac{8\pi\rho k^5 \varepsilon Q_2}{15r^3}.$$

Hence, if dV be the potential at any external point of the stratum of density ρ, contained by the spheroids of mean radii k and $k + dk$, and ellipticities ε and $\varepsilon + d\varepsilon$,

$$dV = \frac{4\pi\rho}{3r}\frac{d(k^3)}{dk}dk - \frac{8\pi\rho}{15r^3}\frac{d(k^5\varepsilon)}{dk}Q_2 dk;$$

and, therefore, for the heterogeneous spheroid

$$V = \frac{4\pi}{r}\int_0^{k_0}\rho k^2 dk - \frac{8\pi Q_2}{15r^3}\int_0^{k_0}\rho\frac{d}{dk}(k^5\varepsilon)\,dk$$

$$= \frac{M}{r} - \frac{8\pi E}{15r^3}Q_2,$$

if M denote the mass of the spheroid, and E denote the integral $\int_0^{k_0}\rho\dfrac{d}{dk}(k^5\varepsilon)\,dk$.

The external surface being a level surface we must have

$$V + \tfrac{1}{2}\omega^2(x^2 + y^2) = C,$$

or

$$\frac{M}{r} - \frac{8\pi E}{15r^3}Q_2 + \tfrac{1}{3}\omega^2 r^2(1 - Q_2) = C,$$

a constant, over the surface.

Putting $r = k_0(1 - \frac{2}{3}\epsilon_0 Q_2)$, and retaining only terms of the first order

$$\frac{M}{k_0}(1 + \tfrac{2}{3}\epsilon_0 Q_2) - \frac{8\pi E}{15k_0^3} Q_2 + \tfrac{1}{3}\omega^2 k_0^2 (1 - Q_2) = C,$$

or
$$\frac{M}{k_0}(1 + \tfrac{1}{3}m) + \left\{\frac{2}{3}\frac{M}{k_0}(\epsilon_0 - \tfrac{1}{2}m) - \frac{8\pi E}{15k_0^3}\right\} Q_2 = C,$$

for all values of Q_2.

Therefore
$$\frac{2}{3}\frac{M}{k_0}(\epsilon_0 - \tfrac{1}{2}m) - \frac{8\pi E}{15k_0^3} = 0,$$

or
$$\frac{8\pi E}{15} = \tfrac{2}{3}Mk_0^2(\epsilon_0 - \tfrac{1}{2}m),$$

and
$$V = \frac{M}{r} - \frac{2Mk_0^2}{3r^3}(\epsilon_0 - \tfrac{1}{2}m) Q_1.$$

On the surface the resultant of gravitation and centrifugal force to the order considered is

$$-\frac{dV}{dr} - \tfrac{2}{3}\omega^2 r(1 - Q_2)$$

$$= \frac{M}{k_0^2}(1 + \tfrac{4}{3}\epsilon_0 Q_2) - 2\frac{M}{k_0^2}(\epsilon_0 - \tfrac{1}{2}m) Q_1 - \tfrac{2}{3}\omega^2 k_0(1 - Q_2)$$

$$= \frac{M}{k_0^2}(1 - \tfrac{2}{3}m) + \tfrac{2}{3}\frac{M}{k_0^2}(\tfrac{5}{2}m - \epsilon_0) Q_2$$

$$= \frac{M}{k_0^2}(1 - \tfrac{2}{3}m)\{1 + \tfrac{2}{3}(\tfrac{5}{2}m - \epsilon_0) Q_2\}$$

to the order considered.

vii. Precession and Nutation are sometimes said to be due to the attraction of the sun and moon on the protuberant portions of the earth towards the equator. Why is this not necessarily true? Shew how to put the statement in a correct form.

What is the terrestrial constant upon which Precession and Nutation depend; and how does the determination of it from observation give us information, as to the interior distribution of the earth's mass, which we cannot obtain so accurately from pendulum observations.

(Thomson and Tait, *Natural Philosophy*, §§ 825, 826).

Precession and Nutation result from the earth's being not centrobaric.

The distribution of density in the interior of the earth might have been such that the earth would be centrobaric, although the external surface is spheroidal.

The terrestrial constant on which Precession and Nutation depends is $\frac{C-A}{C}$, when A, C are the equatoreal and polar moments of inertia of the earth.

The value of $C-A$ may be determined solely from a knowledge of surface gravity, as determined by pendulum observations; and the interior distribution of density in the earth can be varied in an infinite number of ways subject to the condition of leaving the surface gravity, and consequently the exterior gravity unchanged, and for all these distributions $C-A$ remains the same.

But C will be less or greater according as the mass is more condensed in the central parts, or more nearly homogeneous to within a small distance of the surface.

On the other hand the interior distribution of density in the earth can be varied in an infinite number of ways subject to the condition of keeping $\frac{C-A}{C}$ the same; for instance, by varying the density in any way, keeping the same strata of equal density.

Consequently a comparison of pendulum observations with Precession and Nutation gives us information of the interior distribution of the earth's density, which could not be obtained from pendulum observations or Precession and Nutation separately.

viii. Shew how the determination of the form of an unclosed soap-film which, with the vessels on the rims of which it rests, contains a given quantity of air, may be made to depend upon either the superficial extent, or the curvatures, of the film.

Solve the problem from each of these points of view, and shew that they lead to the same result.

We may solve the problem either by considering the equilibrium of each element of surface, the superficial tension and the difference of pressure on the two sides being constant; or from the consideration that the superficial extent is a minimum subject to the condition of containing a given volume.

If t be the superficial tension, and p the given difference of pressures, the first method gives $\frac{1}{\rho}+\frac{1}{\rho'}=\frac{p}{t}$, a constant, where ρ, ρ' are the principal radii of curvature at any point.

In the second method we have to make

$$U=\iint\sqrt{(1+p^2+q^2)}\,dxdy \text{ a minimum,}$$

subject to the condition that

$$W=\iint z\,dxdy \text{ is constant.}$$

Here $$V=\sqrt{(1+p^2+q^2)}+az,$$

and by the Calculus of Variations, the condition for a minimum is

$$\frac{dM}{dx}+\frac{dN}{dy}=L,$$

or $$\frac{d}{dx}\cdot\frac{p}{\sqrt{(1+p^2+q^2)}}+\frac{d}{dy}\,\frac{q}{\sqrt{(1+p^2+q^2)}}=a,$$

or $$\frac{(1+q^2)r-2pqs+(1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}}=a,$$

or $$\frac{1}{\rho}+\frac{1}{\rho'}=a, \text{ a constant.}$$

9. Find the electrification at any point of an uninfluenced conducting ellipsoid which has a given charge of electricity.

Prove that the capacity of an oblate ellipsoid of revolution is $\dfrac{\sqrt{(a^2-c^2)}}{\cos^{-1}\dfrac{c}{a}}$, where a, c are the equatoreal and polar semi-diameters.

The electrification at any point is proportional to the length of the perpendicular from the centre on the tangent plane at the point, and if it be denoted by λp, we must have $\int \lambda p dS$ equal to the charge Q.

But $\int p dS$ is three times the volume of the ellipsoid, and therefore equal to $4\pi abc$; therefore λ is equal to $\dfrac{Q}{4\pi abc}$.

The potential V at any point in the interior of the conductor is constant, and at the centre of the ellipsoid of revolution

$$V = \int \frac{1}{r} \frac{Qp}{4\pi a^2 c} 2\pi r \sin\theta ds$$

$$= \tfrac{1}{2} \frac{Q}{a^2 c} \int p \sin\theta ds$$

$$= \tfrac{1}{2} \frac{Q}{a^2 c} \int_0^\pi r^2 \sin\theta d\theta,$$

where
$$\frac{1}{r^2} = \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{c^2}.$$

Therefore
$$V = \tfrac{1}{2} Qc \int_0^\pi \frac{\sin\theta d\theta}{a^2\cos^2\theta + c^2\sin^2\theta}$$

$$= \tfrac{1}{2} Qc \int_0^\pi \frac{\sin\theta d\theta}{c^2 + (a^2 - c^2)\cos^2\theta}$$

$$= \tfrac{1}{2} \frac{Q}{\sqrt{(a^2 - c^2)}} \tan^{-1}\left\{\frac{\sqrt{(a^2 - c^2)}}{c} \cos\theta\right\}_\pi^0$$

$$= \frac{Q \tan^{-1} \dfrac{\sqrt{(a^2 - c^2)}}{c}}{\sqrt{(a^2 - c^2)}}$$

$$= \frac{Q \cos^{-1} \dfrac{c}{a}}{\sqrt{(a^2 - c^2)}}.$$

Therefore the capacity $\dfrac{Q}{V}$ is $\dfrac{\sqrt{(a^2 - c^2)}}{\cos^{-1} \dfrac{c}{a}}$.

x. Find the pressure at any point in the interior of a mass of homogeneous incompressible liquid held together by the gravitation of its parts alone.

Employ your expression to find the mutual attraction between two hemispheres of a uniform solid globe.

Take the earth's radius as 4000 miles—suppose its density to be throughout equal to the mean density 5·5—and take the weight of a cubic foot of water at the surface to be 63 lbs. Also suppose the average tensile strength of the earth's materials to be 500 lbs. per square inch. Compare the amounts of the gravitation-attraction and the cohesion between two hemispheres separated by a meridian plane, and calculate the angular velocity of rotation which would just enable inertia to overcome them both.

What would be the radius of a planet, of the earth's mean density, and of the tensile strength above assumed, if gravity and cohesion were equally effective in keeping two hemispheres of it together?

By the ordinary hydrostatical equation, combined with a known theorem in attraction, we have

$$\frac{1}{\rho}\frac{dp}{dr} = -\tfrac{4}{3}\pi\rho r,$$

and if $p = 0$ at the surface where $r = a$,

$$p = \tfrac{2}{3}\pi\rho^2 (a^2 - r^2).$$

Hence, the whole pressure across a diametral section (*i.e.* the whole attraction of two hemispheres for one another) is

$$\int_0^a \tfrac{2}{3}\pi\rho^2 (a^2 - x^2)\, 2\pi x\, dx = \tfrac{1}{3}\pi^2\rho^2 a^4.$$

The centrifugal force tending to split the sphere across a plane section distant b from the centre is

$$\int_b^a x\omega^2 . \pi\rho (a^2 - x^2)\, dx$$

$$= \pi\rho\omega^2 \{\tfrac{1}{2}a^2 (a^2 - b^2) - \tfrac{1}{4}(a^4 - b^4)\}$$

$$= \tfrac{1}{4}\pi\rho\omega^2 (a^2 - b^2)^2.$$

Per unit of area of the section the value of this is $\frac{1}{4}\rho\omega^2(a^2-b^2)$, and is therefore greatest for a diametral plane.

Hence the ratio of gravitation-attraction and centrifugal force is

$$\frac{\frac{1}{3}\pi^2\rho^2a^4}{\frac{1}{4}\pi\rho\omega^2a^4}=\frac{\frac{4}{3}\pi\rho a}{\omega^2 a},$$

the ratio of attraction to centrifugal force on a body at the equator.

This ratio is $\frac{1}{289}$ in the case of the Earth; hence if the Earth revolved 17 times faster, it would just be about to split across a meridian plane.

To reduce to numbers, $\frac{4}{3}\pi\rho^2 a$ the attraction at the surface on a cubic foot of matter of density ρ must be 5.5×63 lbs. weight.

Therefore the whole gravitation between two hemispheres is $\frac{1}{4}\pi\times 5.5\times 63a^3$ lbs. weight.

But the whole cohesion is $\pi a^2\times 500\times 144$ lbs. weight.

Therefore the ratio of gravitation-attraction to cohesion between two hemispheres is

$$\frac{5.5\times 63\times 4000\times 5280}{4\times 500\times 144}=25410.$$

Since the gravitation varies as a^4 and the cohesion as a^2, gravitation and cohesion will be equal for a planet of the same mean density as the Earth if its radius be

$$\frac{4000}{\sqrt{(25410)}}=\frac{4000}{160}\text{ nearly}=25\text{ miles}.$$

xi. State the fundamental phenomena of thermo-electric currents as discovered by Seebeck, Cumming, and Peltier; and shew how Thomson was led, by thermodynamic reasoning from them, to the discovery of the electric convection of heat.

Wires of three different metals A, B, C, having resistances a, b, c, have their ends soldered together at two junctions which are maintained at (different) constant temperatures. If I_a be the strength of the current when A is cut, I_b the strength if B be cut; shew by a rigorous method that the strength of the current in C, when all three wires is continuous, is

$$\frac{a(b+c)I_a+b(a+c)I_b}{ab+bc+ca}.$$

Seebeck discovered that electric currents are established in a closed circuit of two different metals with the junctions at different temperatures.

Cumming found that the order of certain metals in the thermoelectric scale is different at high and low temperatures, so that for a certain temperature two metals may be neutral to each other.

Peltier discovered that when a current of electricity crosses the junction of two metals, the junction is heated when the current is in one direction and cooled when it is in the other direction; if, when the junction was heated, the thermoelectric current was in a certain direction, then the passage of a current from an extraneous source in that direction cooled the junction, while a reversal of the current heated the junction.

From these facts Thomson argued as follows:

Suppose a circuit of two metals in which the temperature of the hotter junction is that of the neutral point, there is no reversible thermal effect produced at the hotter junction, for two metals at the neutral temperature behave as if they were the same; and at the colder junction there is, by Peltier's principle, an evolution of heat, while there is also a current produced.

Hence, the only place where the heat can disappear so as to account for the current and the evolution of heat at the cold junction is in the metals, so that a current from hot to cold must cool one metal, or a current from cold to hot must cool the other metal, or both these effects may take place.

Let α, β, γ, λ be the potentials of the wires and the solder at one junction P; α', β', γ, λ' at the other Q (fig. 81);

and let x, y, z be the currents in the wires supposed to be going from P to Q.

The electromotive force in the wire A is $(\alpha' - \lambda') - (\alpha - \lambda)$, and by Ohm's law this is equal to the product of the resistance into the current.

Therefore $$(\alpha' - \lambda') - (\alpha - \lambda) = ax,$$

or $$\alpha' - \alpha - ax = \lambda' - \lambda$$
$$= \beta' - \beta - by = \gamma' - \gamma - cz,$$

by symmetry, with $$x + y + z = 0.$$

Similarly $$\alpha' - \alpha + aI_b = \gamma' - \gamma - cI_b,$$
$$\beta' - \beta + bI_a = \gamma' - \gamma - cI_a.$$

Therefore $$cz - ax = (c + a) I_b,$$
$$cz - by = (b + c) I_a;$$

therefore

$$(ca + ab) z - ab (x + y) = a (b + c) I_a + b (c + a) I_b,$$

or $$z = \frac{a (b + c) I_a + b (c + a) I_b}{bc + ca + ab}.$$

FRIDAY, *Jan.* 22, 1875. 9 to 12.

PROF. TAIT.	Greek numbers.
MR. FREEMAN	Roman numbers.
MR. GREENHILL.	Arabic numbers.

1. FROM the consideration of a conic and its director circle, prove that the condition that it may be possible to circumscribe quadrilaterals to a conic S such that the ends of two diagonals shall lie on another conic S', is

$$\Theta^3 - 4\Theta\Theta'\Delta + 8\Delta^2\Delta' = 0,$$

where $k^3\Delta + k^2\Theta + k\Theta' + \Delta'$ is the discriminant of $kS + S'$.

Prove that the two diagonals of any such quadrilateral intersect in a fixed point, that the third diagonal will be a fixed straight line, and that the two tangents drawn from any point of S' to S will divide this straight line in an involution whose double points lie on S'.

Prove also that the points of contact of the tangents drawn from these double points will lie on S'.

Let S represent $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, and let S' represent the director circle $\lambda(x^2 + y^2 - a^2 - b^2)$.

The discriminant of $kS + S'$ is

$$\left(\frac{k}{a^2} + \lambda\right)\left(\frac{k}{b^2} + \lambda\right)\{k + \lambda(a^2 + b^2)\},$$

and therefore $\quad \Delta = \frac{1}{a^2b^2}, \quad \Delta' = \lambda^3(a^2 + b^2),$

$$\Theta = 2\lambda\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \quad \Theta' = \lambda^2\left(3 + \frac{a^2}{b^2} + \frac{b^2}{a^2}\right).$$

Therefore $$\Theta^2 - 4\Theta'\Delta$$

$$= 4\lambda^2 \left\{ \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 - \left(\frac{3}{a^2b^2} + \frac{1}{a^4} + \frac{1}{b^4}\right) \right\}$$

$$= -4\,\frac{\lambda^2}{a^2b^2},$$

and $$\Theta^3 - 4\Theta\Theta'\Delta + 8\Delta^2\Delta'$$

$$= -8\,\frac{\lambda^3}{a^2b^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + 8\,\frac{\lambda^3}{a\,b^4}\,(a^2 + b^2) = 0;$$

and since this is homogeneous in all senses (when Δ, Θ, Θ', Δ' are supposed to be of dimensions 3, 2, 1, 0; 0, 1, 2, 3; or 1, 1, 1, 1), it is the relation expressing the projective relation between the two conics, that quadrilateral can be circumscribed to S, such that the ends of two diagonals lie on S'.

In the case of a conic and its director circle, any such quadrilateral is a rectangle, the two diagonals intersect in the centre, and the third diagonal is at infinity; therefore in general the two diagonals intersect in a fixed point, and the third diagonal is a fixed straight line.

Any two tangents from a point on the director circle to the conic are at right angles, that is, divide the distance between the circular points harmonically, and the circular points are the points in which the line at infinity meets the director circle; therefore in general the two tangents drawn from any point on S' to S divide the third diagonal in an involution, the double points of which are the points in which the third diagonal meets S'.

At the points of intersection of a conic and its director circle

$$\frac{x^2}{a^2} + \frac{a^2 + b^2 - x^2}{b^2} = 1, \text{ or } x^2 = \frac{a^4}{a^2 - b^2},$$

that is, they are the points where the directrices meet the conic, or the points of contact of the tangents drawn from the foci, which are also the points of contact of the tangents drawn to the conic from the circular points; hence, in general, the tangents drawn to S from the double points lie on S'.

2. Prove that the normals drawn at different points of a small portion of a surface pass through two focal lines at right angles to each other.

Deduce Gauss' measure of curvature at any point of a surface.

If the part of a screw surface of uniform pitch be taken which is formed by one complete revolution of a generating line and the axis of this part of the surface be bent into a circle, prove that this part of the surface supposed inextensible will assume the form of a surface generated by the revolution of a catenary about its directrix.

Normals along a line of curvature ultimately intersect; hence if C, C' be the centres of principal curvature at the point O of a surface (fig. 82), and if a small square $PQRS$ formed by lines of curvature be drawn enclosing O, then the normal planes through PQ and RS will ultimately pass through C, and the normal planes through PS and QR will ultimately pass through C', and these planes will pass through the two focal lines AB, $A'B'$, which are parallel to the lines of curvature at O; and therefore any normal drawn at a point near O will pass through these two focal lines.

If $P'Q'R'S'$ be the small square cut out on the unit sphere by the normals parallel to the normals to the surface along $PQRS$, then $lt\dfrac{P'S'}{PS}=\dfrac{1}{\rho}$ and $lt\dfrac{P'Q'}{PQ}=\dfrac{1}{\rho'}$, where ρ, ρ' are the principal radii of curvature at O.

Therefore $lt\dfrac{\text{area}\,P'Q'R'S'}{\text{area}\,PQRS}=\dfrac{1}{\rho\rho'}$, which is Gauss' measure of curvature at the point O.

A screw surface of uniform pitch is generated by the motion of a straight line which intersects at right angles a fixed axis, about which it twists with an angular velocity which bears a constant ratio, $\dfrac{1}{c}$ suppose, to the velocity of the point of intersection with the axis; and therefore when the generating line has made a complete revolution, the point of intersection with the axis will have moved through a distance $2\pi c$, and a point on the generating line at a distance σ from the axis will have described a helix of length

$$2\pi \sqrt{(\sigma^2 + c^2)}.$$

When this length $2\pi c$ of the axis is bent into a circle of radius c, the helix will also be bent into a circle of radius $\sqrt{(\sigma^2 + c^2)}$, and the generating lines will be bent into meridian curves on a surface of revolution.

If y be the distance of a point from the axis of revolution, then $y = \sqrt{(\sigma^2 + c^2)}$, and the meridian curve is therefore a catenary, of which the axis of revolution is the directrix.

In the screw surface if the axis be taken as the axis of z, then the equation of the surface is

$$\frac{y}{x} = \tan \frac{z}{c};$$

and therefore $$p = -\frac{cy}{x^2 + y^2},\ q = \frac{cx}{x^2 + y^2},$$

$$r = \frac{2cxy}{(x^2 + y^2)^2},\quad s = -c\frac{x^2 - y^2}{(x^2 + y^2)^2},\quad t = -\frac{2cxy}{(x^2 + y^2)^2},$$

and $$rt - s^2 = -\frac{c^2}{(x^2 + y^2)^2}.$$

Therefore Gauss' measure of curvature

$$= \frac{rt - s^2}{(1 + p^2 + q^2)^2} = -\frac{c^2}{(x^2 + y^2 + c^2)^2}.$$

In the catenary of revolution

$$\rho = -\rho' = \frac{\sigma^2 + c^2}{c};$$

and therefore at corresponding points of the two surfaces Gauss' measure of curvature is the same.

a. Show how to find y as a function of x so that the value of the integral

$$\int_a^b f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right) dx$$

may be a maximum or minimum, the form of f being given.

If a point move in a plane with velocity always proportional to the curvature of its path, show that the brachistochrone of continuous curvature between any two given points is a complete cycloid.

Prove that in the ordinary gravitation brachistochrone (which is also a cycloid) the velocity is inversely as the curvature of the path, and state the connexion between the two results.

Take the intrinsic equation of the curve between ρ and ψ, ψ being measured from the line joining the points.

Then if $v = \dfrac{c^2}{\rho}$, we must have

$$t = \int \frac{ds}{v} = \int \rho \frac{ds}{c^2} = \int \frac{\rho^2}{c^2} d\psi,$$

a minimum, subject to the condition that $\int \rho \cos\psi \, d\psi = a$, the distance between the given points.

We must therefore make the variation of $\int_{\psi_1}^{\psi_2} V d\psi$ due to the variation of ρ and of the limits vanish, where

$$V = \rho^2 - \lambda \rho \cos\psi.$$

Now
$$\delta \int_{\psi_1}^{\psi_2} V d\psi$$

$$= V_2 d\psi_2 - V_1 d\psi_1 + \int_{\psi_1}^{\psi_2} \frac{dV}{d\rho} \delta\rho \, d\psi,$$

since V is a function of ρ and ψ only.

Therefore $\dfrac{dV}{d\rho} = 0$, or $2\rho - \lambda \cos\psi = 0$, and the curve is a cycloid.

Also $V_1 = 0$ and $V_2 = 0$; therefore $\rho_1 = 0$ and $\rho_2 = 0$, and the curve must be a complete cycloid.

In the ordinary cycloid, when a gravitation brachistochrone, the velocity is proportional to the square root of the distance from the base of cycloid, that is proportional to the radius of curvature, or inversely proportional to the curvature.

In the two cycloids the velocities are inversely proportional, and therefore the action in one cycloid corresponds to the time in the other; one cycloid will be a brachistochrone for a system of forces, while the other cycloid will be a free path under an associated system of forces.

(Tait and Steele, *Dynamics of a Particle*, §§ 280, 281).

β. Show that if any portions of a frictionless liquid have motion differentially rotational, that property remains associated with those portions of the liquid; and that the spaces occupied by such portions, unless they terminate in the free surface, are necessarily doubly-connected spaces.

(Helmholtz, "Vortex Motion," *Phil. Mag.*, 1867; Thomson, "Vortex Motion," *Trans. R. S. E.*, vol. 25).

When the motion of any portion of a frictionless liquid is such that it could not have been produced by fluid pressure transmitted through the portion from the boundary or surrounding liquid, the motion is called rotational.

The ordinary equations of motion of a frictionless liquid are

$$\frac{Du}{dt} = -\frac{dP}{dx}, \quad \frac{Dv}{dt} = -\frac{dP}{dy}, \quad \frac{Dw}{dt} = -\frac{dP}{dz},$$

where
$$P = V + \int \frac{dp}{\rho}.$$

Therefore
$$\frac{D}{dt}(u\delta x + v\delta y + w\delta z)$$

$$= u\frac{D\delta x}{dt} + v\frac{D\delta y}{dt} + w\frac{D\delta z}{dt} + \frac{Du}{dt}dx + \frac{Dv}{dt}dy + \frac{Dw}{dt}dz$$

$$= u\delta u + v\delta v + w\delta w - \frac{dP}{dx}dx - \frac{dP}{dy}dy - \frac{dP}{dz}dz$$

$$= \delta\left\{\tfrac{1}{2}(u^2 + v^2 + w^2) - P\right\}.$$

Hence by integration round any closed line

$$\frac{D}{dt}\int(udx + vdy + wdz) = 0.$$

Therefore, if the line integral of the tangential component velocity along any closed curve be called the circulation in that line, the circulation in a closed line moving with the fluid is constant during the motion in a frictionless liquid, and if the motion is irrotational, the circulation is zero round all mutually reconcilable paths.

By analogy with the rotation of a rigid body, the component rotation of the fluid in any plane at any point is defined as the circulation round any infinitesimal area in the plane enclosing the point divided by twice the area.

The circulation round the element $dydz$ at the point (xyz) is

$$vdy + \left(w + \frac{dw}{dy}dy\right)dz - \left(v + \frac{dv}{dz}dz\right)dy - wdz$$
$$= \left(\frac{dw}{dy} - \frac{dv}{dz}\right)dydz;$$

and, therefore, the component rotation in the plane yz is $\frac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right)$; and, similarly, the component rotation in the plane zx is $\frac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right)$, and in the plane xy is $\frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$.

Since the circulation round any triangular area is the sum of the circulations round the projections on the coordinate planes, the composition of rotations is according to the vector law.

Hence, in any infinitesimal part of the fluid the circulation is zero round every plane curve passing through a certain line, the resultant axis of rotation at that part of the fluid.

But the circulation remains zero in every closed line moving with the fluid for which it was once zero, hence the vortex lines move with the fluid.

A vortex line being endless, a vortex tube bounded by vortex lines is also endless, and therefore forms a doubly connected space, unless it is infinitely long or terminated in the free surface.

5. A rigid body moveable about a fixed point is in stable equilibrium under the action of a potential such that the work

required to move the body through a small angle ω about an axis whose direction cosines with respect to the principal axes at the fixed point are l, m, n is

$$\tfrac{1}{2}\omega^2 (Pl^2 + Qm^2 + Rn^2 + 2Smn + 2Tnl + 2Ulm).$$

Prove that if the body be slightly displaced it will perform oscillations compounded of harmonic oscillations about the common conjugate diameters of the momental ellipsoid at the fixed point

$$Ax^2 + By^2 + Cz^2 = 1,$$

and of the ellipsoid of equal energy

$$Px^2 + Qy^2 + Rz^2 + 2Syz + 2Tzx + 2Uxy = 1.$$

In the particular case when gravity is the acting force, prove that the surface of equal energy is a right circular cylinder the axis of which passes through the centre of gravity, and hence determine the independent motions of the body.

If (xyz) be the coordinates at the time t of the point of the body originally at (abc), the principal axes of the body in the position of equilibrium being taken as coordinate axes, and if

$$\lambda = l\omega, \quad \mu = m\omega, \quad \nu = n\omega,$$

then since the displacement is small, to the first order we have

$$x = a + b\nu - c\mu,$$
$$y = b + c\lambda - a\nu,$$
$$z = c + a\mu - b\lambda;$$

and, therefore, $$\frac{d^2x}{dt^2} = b\frac{d^2\nu}{dt} - c\frac{d^2\mu}{dt^2}\ldots,$$

and $$\int\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right)dm$$

$$= \int\left\{(b+c\lambda-a\nu)\left(a\frac{d^2\mu}{dt^2} - b\frac{d^2\lambda}{dt^2}\right) - (c+a\mu-b\lambda)\left(c\frac{d^2\lambda}{dt^2} - a\frac{d^2\nu}{dt^2}\right)\right\}dm$$

$$= \int(b^2 + c^2)\frac{d^2\lambda}{dt^2}\,dm = A\frac{d^2\lambda}{dt^2}.$$

If U be the work required to perform the displacement from the position of equilibrium

$$U=\tfrac{1}{2}(P\lambda^2+Q\mu^2+R\nu^2+2S\mu\nu+2T\nu\lambda+2U\lambda\mu),$$

and the moment of the impressed forces about the axis of x will be

$$-\frac{dU}{d\lambda}=-(P\lambda+U\mu+T\nu).$$

Therefore the equation of motion will be

$$A\frac{d^2\lambda}{dt^2}+P\lambda+U\mu+T\nu=0,$$

$$B\frac{d^2\mu}{dt^2}+U\lambda+Q\mu+S\nu=0,$$

$$C\frac{d^2\nu}{dt^2}+T\lambda+S\mu+R\nu=0.$$

The motion will consist of a simple harmonic oscillation about an axis whose direction cosines are (f, g, h) if these equations are satisfied by

$$\lambda=f\cos(pt+q),\quad \mu=g\cos(pt+q),\quad \nu=h\cos(pt+q),$$

if

$$Pf+Ug+Th=Afp^2,$$

$$Uf+Qg+Sh=Bgp^2,$$

$$Tf+Sg+Rh=Chp^2;$$

or

$$\frac{Pf+Ug+Th}{Af}=\frac{Uf+Qg+Sh}{Bg}=\frac{Tf+Sg+Rh}{Ch}=p^2;$$

that is, if (fgh) be the direction cosines of a common conjugate diameter of the momental ellipsoid and the ellipsoid of equal energy.

Any arbitrary displacement will give rise to oscillations compounded of simple harmonic oscillations about the three common conjugate diameters of the two ellipsoids as normal axes, and the period of each will be proportional to the ratio of the common conjugate diameters of the ellipsoids.

In the case of gravity, no energy is required to turn the body about the vertical and the same amount of energy is required to turn the body through the same angle about any horizontal axis; hence the ellipsoid of equal energy becomes a right circular cylinder with its axis vertical.

The normal axes are therefore the vertical and the common conjugate diameter of the sections of the cylinder and the momental ellipsoid made by the plane which is conjugate to the vertical, and any small oscillation will be compounded of harmonic oscillations about these common conjugate diameters.

vi. Explain the variation of the inclination and the irregularity in the motion of the moon's node expressed by the second of the following terms in the moon's latitude

$$k.\sin(g\theta - \gamma) + \tfrac{3}{8}mk.\sin\{(2 - 2m - g)\theta - 2\beta + \gamma\}.$$

(Godfray, *Lunar Theory*, § 80).

7. It is found by observation that the mean annual precession is about 50″. Hence prove that the deviation of the instantaneous axis and of the axis of resultant moment of momentum of the earth from the axis of figure is less than 0″·01.

Neglecting this deviation prove by considering the motion of the axis of resultant moment of momentum the equations

$$\frac{d\psi}{dt} = \frac{3n'^2}{2n} \cdot \frac{C-A}{C} \cos\theta\,(1 - \cos 2l),$$

$$\frac{d\theta}{dt} = -\frac{3n'^2}{2n} \cdot \frac{C-A}{C} \sin\theta \sin 2l,$$

where ψ is the precession, θ the obliquity of the ecliptic, $\frac{3n'^2}{2}(C-A)\sin 2\delta$ the moment of the sun's attraction, δ the declination, l the longitude of the sun, and Cn the angular momentum of the earth.

Prove that if the mass of the sun be divided into two equal parts, and placed on the polar line of the ecliptic at distances from the centre of the earth equal to the mean distance of the sun, these two parts supposed repulsive will produce twice the mean solar precession, the centre of the earth being supposed fixed.

In the same manner if the mass of the moon be uniformly and equally distributed over two thin rings, the central lines of which are the circles which are the intersections of the polar lines of the lunar orbits with a sphere concentric with the earth of radius equal to the mean distance of the moon, then these two rings supposed repulsive will produce twice the mean lunar precession.

If α be the inclination of the instantaneous axis to the axis of figure, and if ω be the obliquity of the ecliptic, then

$$\frac{\sin\omega}{\sin\alpha}=\frac{366\times 360\times 60\times 60}{50}.$$

The mean value of ω is about a quarter of a right angle, and with this approximation

$$\sin\omega=\tfrac{1}{2}\sqrt{\left\{1+\frac{1}{\sqrt{(2)}}\right\}}-\tfrac{1}{2}\sqrt{\left\{1-\frac{1}{\sqrt{(2)}}\right\}}$$

$$=\frac{1}{2\,(\sqrt{2})}=\cdot 35 \text{ about.}$$

Since the angle α is small, if it be expressed in seconds,

$$\alpha=\frac{360\times 60\times 60}{2\pi}\sin\alpha$$

$$=\frac{50\sin\omega}{2\pi\times 366}$$

$$=\frac{50\times 7\times\cdot 35}{44\times 366}\ \left(\text{taking } \pi=\tfrac{22}{7}\right)$$

$$=\frac{122\cdot 5}{16104}=\cdot 008 \text{ about.}$$

The earth being a spheroid of revolution flattened at the poles, its momental ellipsoid is an oblate ellipsoid of revolution, and therefore the axis of resultant moment of momentum lies between the instantaneous axis and the axis of figure, and the deviations of these three axes are less than $0''\cdot01$, and therefore quite insensible.

If on the celestial sphere (fig. 83) Z be the pole of the ecliptic, C the pole of the earth, S the direction of the sun, and if SN be drawn perpendicular to the equator, then the axis of the couple due to the sun's attraction is directed to g, a point 90° behind N.

In the figure, the eye is supposed to be at the centre of the earth, and to be looking at the *concave* side of the celestial sphere.

If OG represent the axis of resultant moment of momentum, then the velocity of G is equal to the impressed couple in magnitude and direction; and therefore since OG may be taken as coincident with the axis of figure,

$$\begin{aligned} Cn \sin\theta \frac{d\psi}{dt} &= 3n'^2 (C-A) \sin\delta \cos\delta \sin Æ \\ &= 3n'^2 (C-A) \sin\theta \sin l \cos\delta \sin Æ \\ &= 3n'^2 (C-A) \sin\theta \cos\theta \sin^2 l \\ &= \tfrac{3}{2} n'^2 (C-A) \sin\theta \cos\theta (1-\cos 2l), \end{aligned}$$

or $$\frac{d\psi}{dt} = \frac{3n'^2}{2n} \frac{C-A}{C} \cos\theta (1-\cos 2l),$$

and $$\begin{aligned} Cn \frac{d\theta}{dt} &= -3n'^2 (C-A) \sin\delta \cos\delta \cos Æ \\ &= -3n'^2 (C-A) \sin\delta \cos l \\ &= -3n'^2 (C-A) \sin\theta \sin l \cos l, \end{aligned}$$

or $$\frac{d\theta}{dt} = -\frac{3n'^2}{2n} \frac{C-A}{C} \sin\theta \sin 2l.$$

If the sun be divided into two halves and placed on the polar line of the ecliptic at the mean distance of the sun from the earth, the moment of these two masses, supposed repulsive, will be about $O\Upsilon$, and of magnitude

$$\tfrac{3}{4} n'^2 (C-A) \sin 2\theta,$$

and the equations of motion will become

$$\frac{d\psi}{dt} = \frac{3n'^2}{n} \frac{C-A}{C} \cos\theta \text{ and } \frac{d\theta}{dt} = 0.$$

Hence $\theta = \omega$, the mean value of the obliquity of the ecliptic, and the precession is twice the mean solar precession.

In treating of the lunar precession and nutation, the fortnightly fluctuations due to the position of the moon in its orbit are neglected, and the fluctuations due to the change of position of the moon's orbit only are considered.

Hence, if in a similar way the moon be replaced by two masses of repelling matter, each of half the mass of the moon placed in the polar line of the moon's orbit, the precession and nutation generated will be twice the lunar precession and nutation, and if these masses be uniformly distributed over the circular rings, they will produce twice the mean lunar precession.

viii. Prove that the equation for the variation of the longitude of perihelion of a disturbed planet is

$$\frac{d\varpi}{dt} = \frac{na\sqrt{(1-e^2)}}{\mu e} \frac{dR}{de} + \frac{na \tan\frac{1}{2}i}{\mu\sqrt{(1-e^2)}} \frac{dR}{di}.$$

(Cheyne, *Planetary Theory*, § 29).

ix. Investigate the differential equation for the longitudinal vibration of an elastic rod to which no forces are applied except at the ends.

Determine the solution for the case of a rod free at both ends; state the conditions for the existence of a node at the middle point, and hence deduce the periods of the component tones of a rod fixed at one end and free at the other.

(Donkin, *Acoustics*, § 149—151, 153, 157, 158).

γ. Point out the fundamental distinction between the rotation of the plane of polarization of light produced by turpentine or quartz, and that produced by a transparent solid or liquid in the magnetic field.

Show that the experimental facts are represented by equations of the form

$$\frac{d^2\xi}{dt^2}=a^2\frac{d^2\xi}{dz^2}+e\frac{d^{r+s}\eta}{dt^r dz^s}, \text{ and } \frac{d^2\eta}{dt^2}=a^2\frac{d^2\eta}{dz^2}-e\frac{d^{r+s}\xi}{dt^r dz^s},$$

where, for turpentine or quartz, r is even and s odd; and for the body in the magnetic field r is odd and s even. [Here ξ and η are displacements perpendicular to each other, and to z, the direction of the ray.]

In turpentine and quartz, where the rotation of the plane of polarization depends on the nature of the medium, the reversal of a ray reverses the direction of rotation of the plane of polarization.

In the magnetic field, the rotation of the plane of polarization remains the same when the ray is reversed.

Any incident plane polarized ray can be resolved into two equal rays, circularly polarized in opposite directions, and it is found by experiment, that in media which produce rotatory polarization, the two rays similarly polarized in opposite directions are propagated with different velocities, and therefore with different wave lengths, since the time of vibration is the same.

Representing the ray which is circularly polarized in one direction by

$$\xi=b\cos 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}\right), \quad \eta=b\sin 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}\right),$$

then $$\frac{d^{r+s}\xi}{dt^r dz^s}= b\left(\frac{2\pi}{T}\right)^r\left(\frac{2\pi}{\lambda}\right)^s\cos 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}+\frac{r+s}{4}\right)$$

$$=-\eta\left(\frac{2\pi}{T}\right)^r\left(\frac{2\pi}{\lambda}\right)^s\sin(r+s)\tfrac{1}{2}\pi,$$

and similarly

$$\frac{d^{r+s}\eta}{dt^r dz^s}=\xi\left(\frac{2\pi}{T}\right)^r\left(\frac{2\pi}{\lambda}\right)^s\sin(r+s)\tfrac{1}{2}\pi.$$

Substituting in either of the differential equations we obtain

$$\frac{4\pi^2}{T^2}=\frac{4\pi^2 a^2}{\lambda^2}-e\left(\frac{2\pi}{T}\right)^r\left(\frac{2\pi}{\lambda}\right)^s\sin(r+s)\tfrac{1}{2}\pi.$$

Writing aT for λ in the coefficient of e, since e is small,

$$\frac{4\pi^2a^2}{\lambda^2} = \frac{4\pi^2}{T^2} - \frac{e}{a^s}\left(\frac{2\pi}{T}\right)^{r+s} \sin(r+s)\tfrac{1}{2}\pi,$$

and to the same order

$$\frac{2\pi a}{\lambda} = \frac{2\pi}{T} - \frac{1}{2}\frac{e}{a^s}\left(\frac{2\pi}{T}\right)^{r+s-1} \sin(r+s)\tfrac{1}{2}\pi.$$

Representing the ray circularly polarized in the opposite direction by

$$\xi = b\cos 2\pi\left(\frac{t}{T} - \frac{z}{\lambda'}\right),\quad \eta = -b\sin 2\pi\left(\frac{t}{T} - \frac{z}{\lambda'}\right),$$

we obtain in a similar way

$$\frac{2\pi a}{\lambda'} = \frac{2\pi}{T} + \frac{1}{2}\frac{e}{a^s}\left(\frac{2\pi}{T}\right)^{r+s-1} \sin(r+s)\tfrac{1}{2}\pi.$$

The resultant vibration at any point will therefore be represented by

$$\xi = b\cos 2\pi\left(\frac{t}{T} - \frac{z}{\lambda}\right) + b\cos 2\pi\left(\frac{t}{T} - \frac{z}{\lambda'}\right),$$

$$\eta = b\sin 2\pi\left(\frac{t}{T} - \frac{z}{\lambda}\right) - b\sin 2\pi\left(\frac{t}{T} - \frac{z}{\lambda'}\right),$$

and therefore $$\frac{\eta}{\xi} = \tan\pi\left(\frac{z}{\lambda'} - \frac{z}{\lambda}\right).$$

Therefore the rate of rotation of the plane of polarization is

$$\frac{\pi a}{\lambda'} - \frac{\pi a}{\lambda} = \frac{1}{2}\frac{e}{a^s}\left(\frac{2\pi}{T}\right)^{r+s-1} \sin(r+s)\tfrac{1}{2}\pi.$$

If $r+s$ is even, $\sin(r+s)\frac{1}{2}\pi$ is zero, and there is no rotatory polarization.

If r is even and s odd, then the rotation of the plane of polarization changes sign with a the velocity of light, representing the effect of turpentine or quartz.

If r is odd and s even, then the rotation of the plane of polarization is unaltered by reversing the ray, representing the state of things in the magnetic field.

δ. Form the equation for the conduction of heat in a bar, on the supposition that the temperature is the same throughout a transverse section, and that the rate of loss by surface radiation and convection is, at each point, directly as the excess of temperature over that of the surrounding medium. Point out the dimensions of the various quantities introduced.

Integrate the equation completely in the two following cases, where the bar is very long, and is supposed to be heated at one end

(*a*) periodically, supposing the conductivity, specific heat, density, &c. unaltered with temperature, and the temperature of each transverse section a periodic function of the time,

(*b*) steadily, supposing the conductivity inversely as the absolute temperature, density, &c. unaltered with temperature, the surrounding medium at absolute zero, and the flow of heat steady.

Let c be the thermal capacity per unit of volume; k, h the coefficients of interior and exterior conductivity per unit of area; v the temperature at the distance x from the origin; A the sectional area; and l the perimeter of the sectional area.

The quantity of heat which enters the element dx of the bar in the time dt from the adjacent parts of the bar is $\frac{d}{dx}\left(Ak\frac{dv}{dx}\right)dxdt$, and the loss of heat from the surface in the same time is $hlvdxdt$.

The increase in the quantity of heat in the element dx in the time dt is $\frac{d}{dt}(Acv)\,dt\,dx$.

Therefore, equating the gain and loss of heat, we obtain the equation

$$\frac{d}{dt}(Acv)-\frac{d}{dx}\left(Ak\frac{dv}{dx}\right)+hlv=0.$$

If $[L]$ be the unit of length, $[T]$ the unit of time, $[\Theta]$ the unit of temperature, and $[H]$ the unit of heat, the dimensions of k will be $\frac{[H]}{[LT\Theta]}$, of h will be $\frac{[H]}{[L^2T\Theta]}$, and of c will be $\frac{[H]}{[L^3\Theta]}$ (Maxwell, *Theory of Heat*, Ch. XVIII.).

(*a*) If A, c, h, k, l be constant, the differential equation may be written

$$\frac{dv}{dt} = K\frac{d^2v}{dx^2} - Hv,$$

where
$$K = \frac{k}{c}, \quad H = \frac{hl}{Ac}.$$

We must assume as the general integral of the equation

$$v = V + \Sigma_1^\infty A_n e^{-p_n x} \cos\left(2\pi n\frac{t}{T} - q_n x + \beta_n\right),$$

where T is the period and V is the mean temperature.
Substituting in the equation we must have

$$K\frac{d^2V}{dx^2} - HV = 0,$$

$$\frac{2\pi n}{T} = 2kp_n q_n,$$

$$K(p_n^2 - q_n^2) - H = 0.$$

These equations determine p_n and q_n, and A_n and β_n are determined from the given circumstances of heating.

(*b*) The differential equation reduces to the form

$$\frac{d}{dx}\left(\frac{1}{v}\frac{dv}{dx}\right) - \frac{v}{a} = 0,$$

where a is a constant.

Let $v = \frac{d\phi}{dx}$, then ϕ is the area of the curve representing the temperatures, and the differential equation becomes

$$\frac{d^2v}{d\phi^2} = \frac{1}{a};$$

and, therefore,

$$v = A + B\phi + \tfrac{1}{2}\frac{\phi^2}{a} = \frac{d\phi}{dx}.$$

Hence ϕ can be found in terms of x and then $v = \dfrac{d\phi}{dx}$.

xii. When a substance is melting at the absolute temperature t under pressure p, if l be the latent heat of fusion, and u, u' the volumes of unit of mass of the substance corresponding to its liquid and solid states, prove that

$$J\frac{l}{t} = (u - u')\frac{dp}{dt},$$

where J is the dynamical equivalent of heat.

Show how J. Thomson's discovery of the dependence of the temperature of fusion of ice on pressure is connected with this relation.

(Briot, *Théorie mécanique de la chaleur*, §§ 127–129).

FRIDAY, *Jan.* 22, 1875. $1\frac{1}{2}$ to 4.

Prof. TAIT. Roman numbers.
Mr. WRIGHT. Arabic numbers.

1. FORM the equation

$$\frac{d^2(r\delta r)}{dt^2} + \frac{\mu}{r^3}(r\delta r) = a\frac{dR}{da} + 2n\int\frac{dR}{d\varepsilon}dt$$

for the perturbation in radius vector of a planet.

Integrate this equation so as to obtain a first approximation to δr.

Explain why this and the similar equations for longitude and latitude cannot be employed with advantage in the calculation of secular variations or of long inequalities.

(Cheyne, *Planetary Theory*, § 112).

ii. Assuming the equation

$$\iiint\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)dx\,dy\,dz = \iint(l\xi + m\eta + n\zeta)\,ds,$$

where l, m, n are the direction-cosines of the outward-drawn normal to the element ds of the surface of a closed space S, throughout which and over whose surface the integrals are taken, prove Green's Theorem; and show how to adapt it to the case in which one of the potentials is many-valued, and S is multiply-connected.

Hence show that the whole exhaustion of potential energy of any number of gravitating particles, originally scattered at infinite distances from each other, is

$$\frac{1}{8\pi}\iiint F^2\,dx\,dy\,dz,$$

where F is the resultant attraction on unit mass at x, y, z, and the integral is taken through all space.

Assume first

$$\xi = U\frac{dV}{dx}, \quad \eta = U\frac{dV}{dy}, \quad \zeta = U\frac{dV}{dz};$$

therefore

$$\iiint\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right)dx\,dy\,dz$$

$$= \iint U\left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right)ds$$

$$- \iiint U\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right)dx\,dy\,dz \quad \text{.....................} \quad (1).$$

Next assume

$$\xi = V\frac{dU}{dx}, \quad \eta = V\frac{dU}{dy}, \quad \zeta = V\frac{dU}{dz},$$

then

$$\iiint\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right)dx\,dy\,dz$$

$$= \iint V\left(l\frac{dU}{dx} + m\frac{dU}{dy} + n\frac{dU}{dz}\right)ds$$

$$\iiint V\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}\right)dx\,dy\,dz \quad \text{........................} \quad (2),$$

and comparing (1) and (2), we obtain Green's Theorem.

If V be many-valued and S multiply-connected, the quantities $\frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz}$ will have definite values at all points within S; the expressions in equation (1) and on the left-hand side of equation (2) will have definite values, but the expression on the right-hand side of equation (2) will be many-valued.

If the space S has n cycles, it must be rendered a-cyclic by drawing n diaphragms.

If S_1 be a diaphragm and k_1 its cyclic constant, then to the right-hand side of equation (2) a series of terms of the form

$$k_1 \iint \left(l \frac{dU}{dx} + m \frac{dU}{dy} + n \frac{dU}{dz} \right) ds_1$$

must be added to make Green's theorem determinate.

(Thomson and Tait, *Natural Philosophy*, §§ 548, 549).

iii. Write down the equations of motion of a connected system in terms of generalized coordinates, and point out the meanings of the various terms.

Solve these equations for a number of bodies, each of which has but one degree of freedom, and vibrates (when undisturbed) at a given rate according to the simple harmonic law, the connections being supposed slight.

Apply your result to the case of the small motions (in the magnetic meridian) of two permanent bar-magnets of equal mass suspended each by its extremities, by parallel strings, all four of equal length, from points in a horizontal line. Show as a particular case that if one of the magnets be initially at rest the whole energy will in time be communicated to it.

If the kinetic energy T be expressed in terms of the generalized coordinates q, and their rates of increase per unit of time $\dot{q}$, then Lagrange's equations of motion are of the form

$$\frac{d}{dt} \frac{dT}{d\dot{q}} - \frac{dT}{dq} = F.$$

$\dfrac{dT}{d\dot{q}}$ is the generalized component of momentum, and $\dfrac{dT}{dq}$ is the rate of increase of the kinetic energy per unit of length in the direction of the coordinate q.

For the given system of bodies

$$T = \tfrac{1}{2} ([a, a] \dot{\theta}^2 + 2 [a, b] \dot{\theta}\dot{\phi} + [b, b] \dot{\phi}^2 + \ldots),$$

and the potential energy

$$V = \tfrac{1}{2} ([\alpha, \alpha] \theta^2 + 2 [\alpha, \beta] \theta\phi + [\beta, \beta] \phi^2 + \ldots),$$

and therefore the equations of motion are

$$[a, a]\,\ddot{\theta} + [a, b]\,\ddot{\phi} + \ldots = -[\alpha, \alpha]\,\theta - [\alpha, \beta]\,\phi - \ldots$$

....

Since the connexions are slight, $[a, a]$, &c. and $[\alpha, \alpha]$, &c. are considerable compared with $[a, b]$, &c. and $[\alpha, \beta]$, &c.; also $\dfrac{[\alpha, \alpha]}{[a, a]} = n_1^2$, &c., where n_1, &c. are the angular velocities in the undisturbed harmonic motions.

To solve the equations, assume $\theta = A \sin(\lambda t + B)$, &c., substitute in the equations and form the determinant, the roots of which will give the different values of λ.

Let $2a$ be the distance between the magnets in equilibrium if they were demagnetized; x, ξ their displacements at time t; μ the strength of each pole;

$$T = \tfrac{1}{2}M(\dot{\xi}'^2 + \dot{\eta}'^2),$$

$$V = \frac{\mu}{(2a + x - \xi)} + \tfrac{1}{2}M\frac{g}{l}(\xi^2 + x^2) + \text{\&c.},$$

where l is the length of the strings, and the magnets are supposed so long that only the two contiguous poles act on one another. This simplifies the work, but does not alter the character of the result.

Hence, the equations of motion are

$$\left.\begin{aligned} \frac{d}{dt}(M\dot{x}) &= -\frac{\mu}{(2a + x - \xi)^2} - M\frac{g}{l}x + \text{\&c.} \\ \frac{d}{dt}(M\dot{\xi}) &= \frac{\mu}{(2a + x - \xi)^2} - M\frac{g}{l}\xi + \text{\&c.} \end{aligned}\right\} \ldots\ldots\ldots(1).$$

Adding, $\dfrac{d}{dt}(\dot{x} + \dot{\xi}) = -\dfrac{g}{l}(x + \xi) + \text{\&c.}$

Subtracting,

$$M\frac{d}{dt}(\dot{x} - \dot{\xi}) = \frac{\mu}{2a^2}\left(1 - \frac{x - \xi}{a} + \ldots\right) - M\frac{g}{l}(x - \xi) + \text{\&c.} \quad \ldots(2).$$

Making x and ξ constant in (1), we get their equilibrium values; and measuring x and ξ from these, (2) become

$$\left.\begin{aligned} \frac{d^2}{dt^2}(x+\xi) &= -\frac{g}{l}(x+\xi) \\ \frac{d^2}{dt^2}(x-\xi) &= -\left(\frac{g}{l}+\frac{\mu}{2a^3M}\right)(x-\xi) \end{aligned}\right\} \text{.......(3).}$$

Thus, if $$n^2 = \frac{g}{l},$$

$$n_1^2 = \frac{g}{l} + \frac{\mu}{2a^3M},$$

we have $$\left.\begin{aligned} x+\xi &= A\cos(nt+B) \\ x-\xi &= A_1\cos(n_1t+B_1) \end{aligned}\right\} \text{..............(4).}$$

(It depends upon whether the proximate poles of the magnets attract or repel one another whether n or n_1 is the greater).

Now if at $t=0$ we have $\xi=0$, $\dot{\xi}=0$, we must have at time $\dfrac{2\pi}{n_1-n}t$, $x=0$, $\dot{x}=0$, which is the statement.

In fact, if the magnets be swung as one piece at their equilibrium distance from one another, the time of oscillation will be the same as that of either pendulum when left to itself, since the magnetic force does not vary during this motion.

Again, if the magnets be swung with equal and opposite motions, the centre of inertia is fixed, and the time of oscillation will be the same as if one of the magnets were held fixed and its magnetic strength doubled; it will therefore be shorter or longer than the former period according as the poles presented to one another attract or repel.

Hence, as the small motions can be represented separately by harmonic motions of periods $\dfrac{2\pi}{n}$ and $\dfrac{2\pi}{n_1}$, the period of any complete oscillation produced by superposition of these simple motions will be $\dfrac{2\pi}{n_1-n}$, and therefore at intervals $\dfrac{\pi}{n_1-n}$ the configuration of the magnets will be the same to a spectator who changes the side from which he regards them in successive intervals. Thus, if one magnet were originally at rest, the two will alternately be reduced to rest.

4. Prove that to every surface harmonic of order i there correspond two solid harmonics of degrees i and $-(i+1)$ respectively.

Show that a surface harmonic distribution of density σ_i over a sphere of radius a gives rise to a potential $\dfrac{4\pi r^i \sigma_i}{(2i+1)\,a^{i-1}}$ at all points inside the sphere and a potential $\dfrac{4\pi a^{i+2}\sigma_i}{(2i+1)\,r^{i+1}}$ at all points outside.

Show how to determine the form of a series of spherical harmonics expressing a function which has any arbitrary value over a spherical surface.

(Thomson and Tait, *Natural Philosophy*, Appendix B., *k*, *r*, *s*, § 536).

5. Form the equations for the transverse vibrations of a stretched string. Show how to solve the equations when the initial circumstances are given.

In the case when a stretched string of length l is set in vibration by a transverse displacement h of a point at a distance $\dfrac{l}{m}$ from one end, show that the disturbance at the time t is given by

$$\frac{2hm^2}{(m-1)\,\pi^2}\,\Sigma_1^\infty\,\frac{\sin\dfrac{r\pi}{m}}{r^2}\,\sin\frac{r\pi x}{l}\cos\frac{r\pi at}{l}.$$

(Donkin, *Acoustics*, §§ 99, 100).

6. A pencil of light, which originally came from a single luminous point and is converging to a focus, falls directly on a screen in which is a small hole; prove that the intensity I of illumination at any point ξ, η of a parallel screen which passes through the focus of the pencil is given by the equation

$$I^2=\left(\iint\cos 2\pi\,\frac{x\xi+y\eta}{R\lambda}\,dx\,dy\right)^2+\left(\iint\sin 2\pi\,\frac{x\xi+y\eta}{R\lambda}\,dx\,dy\right)^2,$$

the integrations extending over the area of the hole, R being the distance between the screens.

(Airy, *Undulatory Theory of Optics*, § 80).

vii. Show that the velocity of propagation of oscillatory waves of very small elevation, in a liquid of practically infinite depth and without surface-tension, is $\sqrt{\left(\frac{g\lambda}{2\pi}\right)}$, where λ is the wave-length.

If there be surface-tension T, and if ρ be the density of the liquid, show that the velocity is given in terms of the wave-length by the equation

$$v^2 = \frac{g\lambda}{2\pi} + \frac{2\pi}{\lambda}\frac{T}{\rho}.$$

Hence show that waves are propagated in a liquid mainly by gravity if longer than, and mainly by molecular forces if shorter than, $2\pi\sqrt{\left(\frac{T}{g\rho}\right)}$.

Let ϕ denote the velocity function; taking the axis of x in the undisturbed surface and drawing the axis of y vertically downwards,

$$C + \frac{p}{\rho} = gy - \frac{d\phi}{dt},$$

neglecting the square of the velocity since the motion is small.

The equation of continuity is

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0;$$

and, therefore,

$$\phi = \Sigma A_n e^{-my} \cos(nt - mx),$$

and
$$C + \frac{p}{\rho} = gy + \Sigma n A_n e^{-my} \sin(nt - mx).$$

At the free surface $C\rho + p$ is the excess of the pressure in the liquid over the atmospheric pressure due to the curvature and tension of the surface.

If y_0 denote the vertical displacement of the free surface, the curvature is $\dfrac{d^2y_0}{dx^2}$ approximately; and, therefore, at the free surface

$$\frac{T}{\rho}\frac{d^2y_0}{dx^2} = C + \frac{p}{\rho}$$
$$= gy_0 + \Sigma nA_n \sin(nt - mx) \ldots\ldots\ldots\ldots\ldots (1),$$

putting $y = 0$ in the exponential terms.

Since $$\frac{dy_0}{dt} = \frac{d\phi}{dy}, \text{ when } y = 0,$$
$$= -\Sigma mA_n \cos(nt - mx);$$

therefore $$y_0 = -\Sigma\frac{m}{n}A_n \sin(nt - mx),$$

and $$\frac{d^2y_0}{dx^2} = \Sigma\frac{m^3}{n}A_n \sin(nt - mx).$$

Substituting in equation (1), we must have

$$\frac{m^3T}{n\rho} = -\frac{m}{n}g + n,$$

or $$\frac{n^2}{m^2} = \frac{g}{m} + \frac{mT}{\rho}.$$

But $$n = \frac{2\pi v}{\lambda}, \quad m = \frac{2\pi}{\lambda};$$

therefore $$v^2 = \frac{g\lambda}{2\pi} + \frac{2\pi}{\lambda}\frac{T}{\rho}.$$

At the limit between waves and ripples

$$\frac{dv^2}{d\lambda} = \frac{g}{2\pi} - \frac{2\pi}{\lambda^2}\frac{T}{\rho} = 0,$$

and therefore

$$\lambda = 2\pi\sqrt{\left(\frac{T}{g\rho}\right)} \text{ and } v^2 = 2\sqrt{\left(\frac{gT}{\rho}\right)}.$$

8. If k be the elasticity of volume and n the rigidity of an elastic isotropic substance, prove that Young's modulus of elasticity, that is the longitudinal elasticity of the substance when there is no lateral constraint, is $\frac{9nk}{3k+n}$.

If a cylindrical beam originally straight be bent in a plane, prove that the bending moment across a normal section is $\frac{EI}{\rho}$, where ρ is the radius of curvature of the mean fibre, I the moment of inertia of the normal section about the axis through the centre perpendicular to the plane of flexure, and E is Young's modulus.

If the beam be supported at its ends by two props in the same horizontal line, prove that the deflection of the middle point below the ends is $\frac{5Wa^3}{48EI}$, where W is the weight and $2a$ the length of the beam.

(Thomson and Tait, *Natural Philosophy*, §§ 682, 683).

If the beam be uniformly curved by a properly applied stress-couple at its ends, the fibres of the beam parallel to the axis will be bent into coaxal circles.

If x be the distance of the fibre from the straight line through the centre of inertia of a cross section of the beam perpendicular to the plane of flexure, then $\rho + x$ is the radius of curvature of the fibre when bent, and therefore $\frac{x}{\rho}$ is the longitudinal strain.

If dA be the cross section of the fibre, then neglecting the lateral influence of the adjacent fibres, the tension is $E\frac{x}{\rho}dA$, and the moment of these tensions, balancing the applied couple at one end, is $\frac{E}{\rho}\int x^2 dA = \frac{EI}{\rho}$.

Take the middle point between the props as the origin, the axis of x horizontal, and measure the axis of y vertically downwards.

The deflection of the beam being small, the curvature may be put equal to $-\dfrac{d^2y}{dx^2}$.

The bending moment across the section at a distance x from the origin is

$$\frac{W}{2}(a-x)-\frac{W}{2a}(a-x)\frac{a-x}{2}$$

$$=\frac{W}{4a}(a^2-x^2).$$

Therefore $$EI\frac{d^2y}{dx^2}=\frac{W}{4a}(x^2-a^2).$$

Integrating $$EI\frac{dy}{dx}=\frac{W}{4a}(\tfrac{1}{3}x^3-a^2x),$$

and $$EIy=\frac{W}{4a}\{\tfrac{1}{12}(x^4-a^4)-\tfrac{1}{2}a^2(x^2-a^2)\},$$

since $\dfrac{dy}{dx}=0$ when $x=0$, and $y=0$ when $x=a$.

Putting $x=0$, the deflection of the middle point of the beam is $\dfrac{5Wa^3}{48EI}$.

ix. Show that the potential, α, and the current-function, β, in a uniform conducting plate, satisfy the equations

$$\frac{d^2\alpha}{dx^2}+\frac{d^2\alpha}{dy^2}=0,\quad \frac{d^2\beta}{dx^2}+\frac{d^2\beta}{dy^2}=0.$$

Show that the resistance of the portion bounded by α_1, α_2, β_1, β_2 is as

$$\frac{\alpha_1-\alpha_2}{\beta_1-\beta_2},$$

and that the heat developed in it by resistance is in unit of time as $(\alpha_1-\alpha_2)(\beta_1-\beta_2)$.

As a particular case, show that if electrodes be attached, at any two points, to an infinite plate, the resistance of the plate will be doubled if it be cut down to a circular disc of any radius whatever, provided its edge passes through the centres of both electrodes.

The potential-function α is such that the excess of its value at one point over its value at another point is the electromotive force acting from the first to the second point along a conductor joining the points.

The current-function β is such that the excess of its value at one point over its value at another point is equal to the current which flows across any line joining the points, from right to left to a person at the first point looking at the second point.

Hence, if R denote the resistance of the conducting plate per unit of area,

$$\frac{d\alpha}{dx} = R\frac{d\beta}{dy}, \text{ and } \frac{d\alpha}{dy} = -R\frac{d\beta}{dx};$$

and, therefore,

$$\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} = 0,$$

$$\frac{d^2\beta}{dx^2} + \frac{d^2\beta}{dy^2} = 0.$$

The resistance of a conductor is measured by the electromotive force divided by the current.

In the portion bounded by α_1, α_2, β_1, β_2, the electromotive force is $\alpha_1 - \alpha_2$, and the current is $\beta_1 - \beta_2$; hence, the resistance is $\dfrac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}$.

The heat generated is proportional to the work done by the current, which in unit of time is equal to the product of the current and the electromotive force; hence, the heat generated in the unit of time is proportional to $(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)$.

For a single electrode at the origin the current-function would be $\tan^{-1}\dfrac{y}{x}$; and, therefore, if the coordinates of the electrodes be $(\pm a, 0)$, we must put

$$\beta = \tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a}.$$

The lines of flow are therefore circles passing through the electrodes, and the current in a part of the plate bounded by two lines of flow is proportional to the angle at which the lines of flow intersect at an electrode.

If the plate be cut down into a circular disc passing through the electrodes, half the lines of flow are cut away, consequently the resistance of the remaining part is doubled.

x. Investigate the magnetization of an ellipsoid of soft iron in a uniform magnetic field.

Prove that when, as in iron, the magnetic susceptibility is 30 or upwards, the intensity of magnetization in a sphere, and still more in an oblate ellipsoid of revolution (when the lines of force of the magnetic field are parallel to its axis), is nearly the same as if the susceptibility were infinite; but that in a very long prolate ellipsoid of revolution it is nearly proportional to the susceptibility.

(Maxwell, *Electricity*, §§ 437, 438).

xi. Show that the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + n^2x = n^2P(1 + 2\Sigma_1^\infty \cos ipt),$$

in which i has all positive integral values, and k is less than n, represents cycloidal pendulum motion, with viscous resistance, under the action of an infinite series of equal impulses (in the same direction) succeeding one another at intervals of $\dfrac{2\pi}{p}$.

Integrate this equation; and, by comparing the result with that obtained by treating the problem for each impulse separately from an epoch so distant that the motion has become independent of the initial circumstances, show that

$$\frac{1}{n^2} + 2\Sigma_1^\infty \frac{(n^2 - i^2p^2)\cos ipt + 2ikp\sin ipt}{(n^2 - i^2p^2)^2 + 4i^2p^2k^2}$$

$$= \frac{2\pi}{pn_1}\varepsilon^{-kt}\frac{\left(1 - \varepsilon^{-\frac{2\pi k}{p}}\cos\dfrac{2\pi n_1}{p}\right)\sin n_1t + \varepsilon^{-\frac{2\pi k}{p}}\sin\dfrac{2\pi n_1}{p}\cos n_1t}{1 - 2\varepsilon^{-\frac{2\pi k}{p}}\cos\dfrac{2\pi n_1}{p} + \varepsilon^{-\frac{4\pi k}{p}}},$$

where $n_1 = \sqrt{(n^2 - k^2)}$, and t lies between 0 and $\dfrac{2\pi}{p}$.

The force represented by the series

$$n^2P(1+2\Sigma_1^\infty e^i \cos ipt) = n^2P\frac{1-e^2}{1-2e\cos pt+e^2},$$

and therefore in the limit when $e=1$, the force is zero except when $\cos pt=1$, when the force is infinite.

The momentum generated by the force during an interval of time $\dfrac{2\pi}{p}$

$$=n^2P\int_0^{\frac{2\pi}{p}}(1+2\Sigma_1^\infty \cos ipt)\,dt=\frac{2\pi n^2}{p}P;$$

and, therefore, the first equation represents cycloidal pendulum motion with viscous resistance, under the action of impulses in the same direction, of magnitude $\dfrac{2\pi n^2P}{p}$, succeeding one another at intervals of $\dfrac{2\pi}{p}$.

The symbolical solution of the equation, neglecting the complementary function, which depends on the initial circumstances, gives

$$x=n^2P\frac{(1+2\Sigma_1^\infty \cos ipt)}{\dfrac{d^2}{dt^2}-2k\dfrac{d}{dt}+n^2}$$

$$=n^2P\frac{\left(\dfrac{d^2}{dt^2}-2k\dfrac{d}{dt}+n^2\right)(1+2\Sigma_1^\infty \cos ipt)}{\left(n^2+\dfrac{d^2}{dt^2}\right)^2-4k^2\dfrac{d^2}{dt^2}}$$

$$=n^2P\left\{\frac{1}{n^2}+2\Sigma_1^\infty\frac{(n^2-i^2p^2)\cos ipt+2ipk\sin ipt}{(n^2-i^2p^2)+4i^2p^2k^2}\right\}\text{.....(1)}.$$

If we solve the differential equation

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+n^2x=0,$$

supposing the motion to have been originated at the time $t=0$ by an impulse $\frac{2\pi n^2}{p}P$, the solution will be

$$x=\frac{2\pi n^2}{pn_1}Pe^{-kt}\sin n_1 t.$$

Similarly the motion due to the preceding impulse which took place at the time $-\frac{2\pi}{p}$ will be represented by changing t into $t+\frac{2\pi}{p}$ is the last expression; and so on for all the impulses which have taken place.

The resultant motion due to the infinite series of impulses that have taken place will therefore be represented by putting

$$x=\frac{2\pi n^2}{pn_1}P\Sigma_0^\infty e^{-k\left(t+\frac{2i\pi}{p}\right)}\sin n_1\left(t+\frac{2i\pi}{p}\right)$$

$$=\frac{2\pi n^2}{pn_1}Pe^{-kt}\left(C\sin n_1 t+S\cos n_1 t\right)\ldots\ldots\ldots\ldots(2),$$

where $$C=\Sigma_0^\infty e^{-\frac{2\pi ik}{p}}\cos\frac{2\pi in_1}{p},$$

$$S=\Sigma_0^\infty e^{-\frac{2\pi ik}{p}}\sin\frac{2\pi in_1}{p}.$$

Now $$C+S\sqrt{(-1)}$$

$$=\Sigma_0^\infty e^{-\frac{2\pi i}{p}\{k+n_1\sqrt{(-1)}\}}$$

$$=\frac{1}{1-e^{-\frac{2\pi}{p}\{k+n_1\sqrt{(-1)}\}}}$$

$$=\frac{1}{1-e^{-\frac{2\pi k}{p}}\cos\frac{2\pi n_1}{p}-\sqrt{(-1)}\,e^{-\frac{2\pi k}{p}}\sin\frac{2\pi n_1}{p}}$$

$$=\frac{1-e^{-\frac{2\pi k}{p}}\cos\frac{2\pi n_1}{p}+\sqrt{(-1)}\,e^{-\frac{2\pi k}{p}}\sin\frac{2\pi n_1}{p}}{1-2e^{-\frac{2\pi k}{p}}\cos\frac{2\pi n_1}{p}+e^{-\frac{4\pi k}{p}}},$$

and therefore

$$C=\frac{1-e^{-\frac{2\pi k}{p}}\cos\frac{2\pi n_1}{p}}{1-2e^{-\frac{2\pi k}{p}}\cos\frac{2\pi n_1}{p}+e^{-\frac{4\pi k}{p}}},$$

$$S=\frac{e^{-\frac{2\pi k}{p}}\sin\frac{2\pi n_1}{p}}{1-2e^{-\frac{2\pi k}{p}}\cos\frac{2\pi n_1}{p}+e^{-\frac{4\pi k}{p}}}.$$

Comparing equations (1) and (2) we see that the last equation of the question holds, provided t be restricted to lie between 0 and $\frac{2\pi}{p}$.

The equation might also have been established by proving that the left-hand side is the expansion by Fourier's theorem in a series of sines and cosines of multiples of pt of the right-hand side

$$\frac{2\pi}{pn_1}e^{-kt}(C\sin n_1t+S\cos n_1t).$$

THE END.

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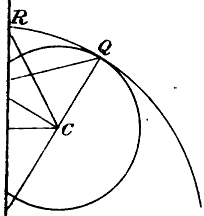
3



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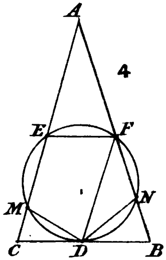

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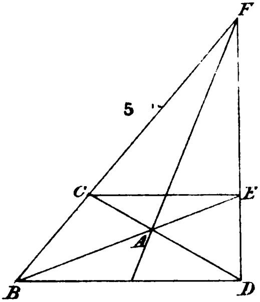

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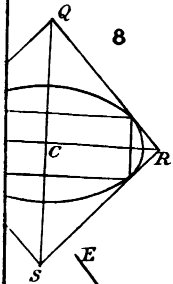

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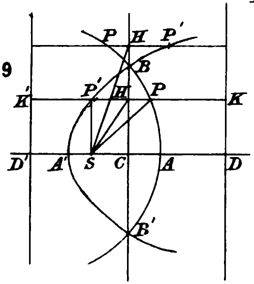

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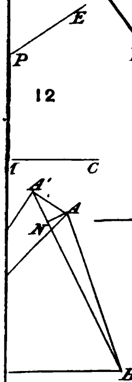

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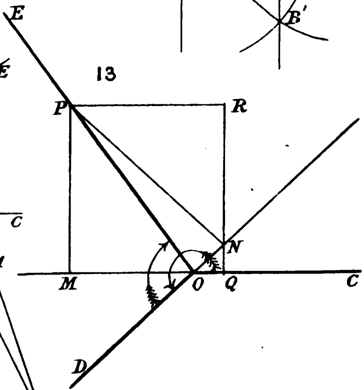

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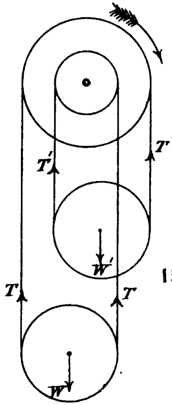

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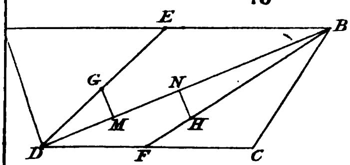

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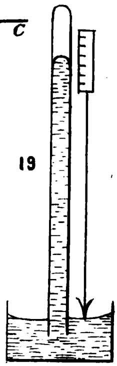

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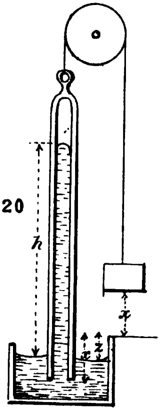
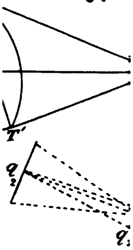
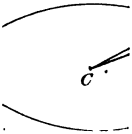
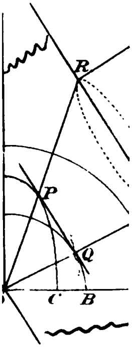


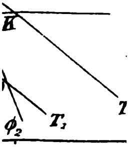
64







76



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