MATHEMATICAL TRIPOS

1908–1912

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IN THE

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MATHEMATICAL TRIPOS

PART I

1908

(*New Regulations*)

THURSDAY, *June* 4. 9—12.

1. Shew that equiangular triangles have their corresponding sides proportional.

Two straight lines AB, $A'B'$ intersect in O and the circles AOA', BOB' intersect in O and C: prove that CAB, $CA'B'$ are similar triangles.

2. Shew that the locus in a plane of a variable point P, which is such that the ratio of its distances from two fixed points A and B in the plane is constant, is a circle.

Any point P on the locus is joined to a fixed point on AB, and the joining line meets the circle APB again in Q: shew that the locus of Q is a circle in which the ratio of AQ to QB is constant.

3. Shew that the points of contact of tangent lines to a sphere from an external point lie on a small circle of the sphere. What is the theorem for a system of parallel tangent lines?

Shew that in a chosen direction two common tangent lines to two spheres can be drawn, provided the lines through the centres in the direction are at a distance apart intermediate between the difference and the sum of the radii.

4. Shew that $x - y - 2z$ is a factor of the expression

$$2(x^3 + y^3 + z^3) + (y^2z + z^2x + x^2y) - 5(z^2y + x^2z + y^2x) - 2xyz,$$

and write down the other two factors.

5. Shew that there are in general two values of λ for which

$$ax^2 + 2bx + c - \lambda(a'x^2 + 2b'x + c')$$

is a square and deduce that $ax^2 + 2bx + c$, $a'x^2 + 2b'x + c'$ may be put in the forms

$$p(x - \alpha)^2 + q(x - \beta)^2, \quad r(x - \alpha)^2 + s(x - \beta)^2.$$

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Determine the function

$$(ax^2 + 2bx + c)/(a'x^2 + 2b'x + c')$$

which has turning values 2 and 3 when $x = 1$ and $x = -1$ respectively and has the value 2·5 when $x = 0$.

6. Write down the expansion of $(1 - 2hx + x^2)^{-\frac{1}{2}}$ as far as the term involving x^4.

Two circles, each of radius a, are in parallel planes which are at right angles to the line joining the centres and at a distance $2d$ apart: the distances of any point on the line of centres from the circumferences of the circles and from the point midway between the centres are r_1, r_2 and x respectively. Find the relation between a and d which will render $r_1^{-3} + r_2^{-3}$ independent of x, if x^4 and higher powers of x be neglected.

7. Shew how to find the remaining side and angles of a triangle ABC, when a, b, and A are given, discussing any ambiguity that may arise.

A man at the bottom of a hill observes an object, half a mile distant, at the same level as himself. He then walks 200 yards up the hill and observes that the angle of depression of the object is 2° 30′ and that the direction to it makes an angle of 75° with the direction to his starting point. Find to the nearest minute the angle which his path makes with the horizontal.

8. Prove that the radius of the inscribed circle of a triangle is given by

$$r = a \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C / \cos \tfrac{1}{2}A.$$

The two equal sides of an isosceles triangle are given in length: prove that when the radius of the inscribed circle is a maximum, the angle between the equal sides is 76°, to the nearest degree.

9. Find the angle between the straight lines

$$ax^2 + 2hxy + by^2 = 0$$

and prove that for all values of m the angle between the lines

$$(a + 2hm + bm^2)\,x^2 + 2\,[(b - a)\,m - (m^2 - 1)\,h]\,xy + (am^2 - 2hm + b)\,y^2 = 0$$

is the same.

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10. Prove that by a proper choice of axes the equations of two circles can be written

$$x^2 + y^2 + 2k_1x + c = 0 \text{ and } x^2 + y^2 + 2k_2x + c = 0.$$

Shew that the circles are orthogonal if $k_1k_2 = c$ and that this can only happen for real circles if c is negative.

11. Shew how to find the maxima and minima values of a function of a single variable from the variation of its differential coefficient, and apply the method to prove that $\frac{1}{8}(35\cos^4 x - 30\cos^2 x + 3)$ ranges in value between unity and $-\frac{3}{7}$ and has also $\frac{3}{8}$ as a maximum value.

THURSDAY, *June* 4. 2—5.

1. Trace the variations in value of $\cos x$, and determine the series of angles which have a given cosine.

Shew that the graph of $\cos px + \cos qx$ lies between those of $2\cos\frac{p-q}{2}x$ and $-2\cos\frac{p-q}{2}x$, touching each in turn. What general conclusion can be drawn as to the amplitudes of the variations of $\cos px + \cos qx$, when $p - q$ is small compared with $p + q$?

2. Prove that for angles of any magnitude

$$\sin(A + B) = \sin A \cos B + \sin B \cos A.$$

Shew that, if α be the exterior angle of a regular polygon of n sides,

$$\sum_{r=0}^{r=(n-1)} \sin(A + r\alpha) = 0,$$

and prove

$$\sin A = \sin(36° + A) - \sin(36° - A) - \sin(72° + A) + \sin(72° - A).$$

3. Prove that in any triangle

$$(b + c)\tan\frac{B - C}{2} = (b - c)\tan\frac{B + C}{2}.$$

Shew that, if an angle α be divided into two parts so that the ratio of the tangents of the parts is λ, the difference x between the parts is given by

$$\sin x = \frac{\lambda - 1}{\lambda + 1}\sin\alpha.$$

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4. Prove that the chords of intersection of a parabola and a circle are in pairs equally inclined to the axis of the parabola.

Construct the position of the axis of a parabola, given the direction of the axis and three points on the curve.

5. Explain how the axes are chosen when the equations of two straight lines in space are reduced to

$$y = 0,\ z = 0 \ \text{ and } \ y = x \tan \alpha,\ z = c.$$

Pairs of planes are drawn through a fixed line at right angles to one another: shew that they intersect a second fixed line in pairs of points in involution, determining the constant of the involution.

6. Find the form of equation which represents a sphere referred to rectangular axes; and shew that

$$x = \frac{a\lambda}{1 + \lambda^2 + \mu^2},\quad y = \frac{a\mu}{1 + \lambda^2 + \mu^2},\quad z = \frac{a}{1 + \lambda^2 + \mu^2}$$

give, as λ and μ vary, the coordinates of points on a sphere of which a is the length of the diameter.

7. Prove that the locus of the middle points of a system of parallel chords of an ellipsoid is a plane through the centre of the ellipsoid.

Shew that it is possible to get an unlimited number of sets of three lines through the centre, such that the plane through any two bisects all chords parallel to the third.

8. For a curve defined by the equation $p = f(\psi)$ prove that the projection of the radius vector on the tangent is $\frac{dp}{d\psi}$ and that the radius of curvature is $p + \frac{d^2p}{d\psi^2}$.

Prove that the p, r equation of the curve

$$p = a \sin n\psi \ \text{ is } \ r^2 = a^2n^2 + (1 - n^2)\, p^2,$$

and that the p, r equation of the locus of centres of curvature is $r^2 = a^2n^4 + (1 - n^2)\, p^2$.

9. Integrate the functions

$$\frac{x^2 + 2x - 1}{x^2 (x - 1)^2},\qquad \frac{1}{\sqrt{x(x - 2)}},\qquad \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}.$$

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Shew that, if $c > a > 0$

$$\int_0^a \frac{\sqrt{a^2 - x^2}\,dx}{c^2 - x^2} = \pi\left(c - \sqrt{c^2 - a^2}\right)/2c.$$

10. Prove that the area bounded by a closed curve is $\frac{1}{2}\int(xdy - ydx)$ taken round the curve. What area does this integral represent when taken over a portion of a curve which is not closed?

Prove that the area of a loop of the curve

$$x = a \sin 2t, \quad y = a \sin t \text{ is } \tfrac{4}{3}a^2.$$

FRIDAY, *June* 5. 9—12.

1. Discuss the evidence on which the principle of the independence of forces is established.

2. Find the acceleration of a point moving with constant speed in a circle.

Prove, by considering a point on the circumference of a circle rolling uniformly along a straight line, that the radius of curvature of a cycloid at any point is twice the length of the line joining that point to the point of contact of the generating circle with the base.

3. A body is projected from a given point with velocity u, so as to pass through another point which is at a horizontal distance d from the point of projection and at a height h above it. Find an equation to determine the angles at which the body may be projected.

A shot has a range d on the horizontal plane when the angle of elevation is θ and just reaches the base of a vertical target of height $2a$, where $a = d \tan \alpha$: shew that with the same initial velocity and elevation $\alpha + \theta$ it will strike the target at a depth $a \sin^2 \theta \sec^2 (\alpha + \theta)$ below the centre.

4. Find the velocities after impact of two spherical balls of given mass, which impinge directly with given velocities.

Two equal small spheres of steel are suspended by fine strings and constrained to swing in the same vertical plane as simple pendulums of equal length. In equilibrium they are just in contact. They are pulled apart in opposite directions until their centres are each 20 cms. from their equilibrium positions and are then simultaneously released.

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After the hundredth impact each is observed to swing so that its centre is 10 cms. from its equilibrium position. Calculate the coefficient of restitution for steel.

5. Two masses m and m' are connected by a light string passing over a pulley on frictionless bearings; the radius of the pulley is a and its moment of inertia I. Determine the acceleration of the masses when the string does not slip.

Find also the difference of the tensions in the string on the two sides of the pulley.

6. On what evidence does the Law of Inverse Squares for Electrostatics depend?

Two pith balls hung from the same point by silk threads 100 cms. in length are charged to the same potential. The mass of each ball is ·1 gm., and each radius is ·5 cm. The balls come to rest with their centres 8 cms. apart. Find approximately the potential of each ball.

7. Prove that the total normal electric induction over any closed surface is equal to 4π times the total charge of electricity inside the closed surface.

Find the electric intensity between two infinite parallel plates, one being charged with surface density σ, the other being connected to earth. If d be the distance between the plates, find the capacity per unit area of the plates.

Explain the guard ring as applied to parallel plate condensers.

8. Describe the construction of a tangent galvanometer and the adjustments necessary for using it.

Two galvanometers are wound respectively with 2 turns of wire of resistance ·1 ohm per turn and 20 turns of wire of resistance 1 ohm per turn. The coils can be regarded as all having the same radii and their centres at the centre of the suspended magnet. Which galvanometer will shew the greatest deflection when connected to a cell whose internal resistance is 1 ohm?

9. Draw a diagram shewing the course of the rays by which the eye sees an image of an object in a plane mirror.

Explain how, with a point source of light, a plane mirror and a vertical rod, a horizontal shadow can be thrown on a vertical screen.

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10. Describe an experimental method of verifying the laws of refraction of light.

Light is incident on one face of a prism, of which the refracting angle is 60°, in a plane perpendicular to the edge of the prism. The index of refraction of the glass is 1·5. Find the range of angles of incidence for which light is transmitted through the prism without internal reflexion.

11. Prove that, if f be the focal length of a spherical mirror and x, y the distances from the principal focus of an object and image formed by direct reflexion, $xy = f^2$.

Shew that, if a convergent lens be placed at a distance greater than its focal length from a convex mirror, there are two positions of a source of light for each of which the image, formed after the light has passed twice through the lens, is coincident with the object.

Shew that for one position the image is inverted and for the other it is erect.

FRIDAY, *June* 5. 2—5.

1. Prove that triangles, which have one angle of the one equal to one angle of the other, are to one another in the ratio compounded of the ratios of the sides about those angles.

Through D, the middle point of the base AB of the triangle ABC, the line $A'B'$ is drawn to meet the sides in A' and B' respectively, and $A'B'$ is divided internally at D in the ratio $\lambda : \mu$; prove that the areas of ABC and $A'B'C$ are in the ratio $4\lambda\mu : (\lambda + \mu)^2$.

2. If lines drawn from A, B, C, the vertices of a triangle, meet the opposite sides in X, Y, Z and are concurrent, shew that $BX \, . \, CY \, . \, AZ = XC \, . \, YA \, . \, ZB$.

If X remains fixed and Y and Z have any positions for which AX, BY and CZ are concurrent, shew that, if P is the point of concurrence, BY/PY bears a constant ratio to CZ/PZ.

3. Shew that the cross-ratio of the range in which a variable line meets a pencil of four fixed lines through a point is constant; and extend the result to the case of the range formed by the intersections of a variable line with four fixed planes which have a common line of intersection.

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4. Prove that the polar reciprocal of a conic with regard to the focus is a circle.

Determine the number of conics which have a given focus and pass through three given points.

5. In the equation $x^2 - x - 2 = \epsilon x^3$ the quantity ϵ is small: shew that $-1 + \frac{1}{3}\epsilon - \frac{8}{27}\epsilon^2$ is an approximation to a root, and determine the corresponding approximation to the root which is near 2. Write down an approximation to the third and large root.

6. Prove that the arithmetic mean of any number of positive quantities is greater than their geometric mean.

The sum of n positive integers is N, where n and N are assigned numbers. Shew that the greatest value of the product of the n integers is $q^{n-r}(q+1)^r$, where q is the quotient and r the remainder in dividing N by n.

7. Assuming the expansion of $\log(1+x)$, shew that

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \dots\dots\dots\right\}.$$

Calculate $\log_e 10$ to four places of decimals.

8. Determine the nth differential coefficients of

$$\log(1+x),\ \sin x \text{ and } e^x \sin x.$$

Shew that the nth differential coefficient of $\sin x \sinh x$ vanishes for the value $x = 0$ unless n be of the form $4m+2$, when its value is $(-1)^m 2^{2m+1}$.

9. Explain the process of integration by parts, and illustrate its repeated application by determining

$$\int x^4 \sin nx\, dx.$$

Shew that the integrals of $u\dfrac{d^n v}{dx^n}$ and $v\dfrac{d^n u}{dx^n}$ can both be found if one of them can be, u and v being known functions of x admitting of successive differentiation.

10. Find the solution of (i) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 10x = 0$ for which $x = a$ and $\dfrac{dx}{dt} = v$ when $t = 0$, and that of (ii) $\dfrac{d^2x}{dt^2} + n^2 x = \cos pt$, $(n^2 \neq p^2)$, for which $x = 0$ and $\dfrac{dx}{dt} = 0$ when $t = 0$.

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SATURDAY, *June* 6. 9—12.

1. Prove that, in general, a system of forces acting in one plane on a rigid body can be reduced to a single force.

If six forces, of relative magnitude 1, 2, 3, 4, 5 and 6, act along the sides of a regular hexagon taken in order, shew that the single equivalent force is of relative magnitude 6 and that it acts along a line parallel to the force 5, at a distance from the centre of the hexagon $3\frac{1}{2}$ times the distance of a side from the centre.

2. The beam of a balance is of mass m and length $2l$: when the beam is in equilibrium, the centre of gravity of the beam and the plane of the outer knife edges (from which the scale pans are suspended) are respectively at depths h and d below the central knife edge. If each scale pan with its load has mass M, shew that an additional mass μ placed on one pan produces a deflection ϑ given by

$$\tan\vartheta = \frac{\mu l}{mh + (\mu + 2M)\, d}.$$

What is the condition that the sensitiveness of the balance may be independent of the load?

3. Define the Coefficient of Friction and describe an experimental method of determining this constant for a pair of surfaces.

A string is hung over a horizontal cylindrical rod. One end supports 10 grams the other 100 grams when the string is on the point of slipping. Shew that the coefficient of friction between the string and the rod is ·733 approximately, taking $\log_{10} e = {}$·4343.

4. Prove that in the case of a liquid at rest under gravity the surfaces of equal density are horizontal.

A hemispherical bowl whose mass is 100 grams is placed with its rim downwards on a horizontal plane which it fits closely. Water is poured into the bowl through a hole in the curved surface. Find the height in centimetres at which the water must be in the bowl in order that the bowl may be lifted and the water begin to escape between the plane and the bowl.

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5. Assuming that the air is at the same temperature everywhere, shew that at a height h above the ground the pressure is $p_0 e^{-\frac{g\rho_0}{p_0}h}$, where p_0, ρ_0 are the values of the pressure and density of the air at the ground.

The density of mercury is 13·6 and that of air at 760 mm. pressure is ·001293. Find the reading of the barometer at the top of a building 30 metres high, when the reading at the bottom is 760 mm.

6. If a body moves in an ellipse about a centre of force in a focus, shew that the law of attraction is that of the inverse square.

The eccentricity of the earth's orbit round the Sun is 1/60; prove that the earth's distance from the Sun exceeds the length of the semi-axis major of the orbit during about 2 days more than half the year.

7. Shew that the time of oscillation of a compound pendulum is $2\pi\sqrt{\frac{I}{Mgh}}$, where I is the moment of inertia of the pendulum about its axis of support, M its mass and h the distance of its centre of inertia from the axis of support.

Shew that there are three other axes of support, parallel to the original axis and intersecting the line from the centre of inertia perpendicular to the original axis, for which the time of oscillation is the same as that about the original axis.

What is the practical application of this result?

8. State Ohm's Law.

Point out what in this statement is a physical law and what is definition.

Two cells whose E.M.F.'s are 2 and 3 volts and internal resistances 2 ohms are connected in parallel with a resistance of 1 ohm. Find the currents through the external resistance and each cell.

9. Shew that in free space the electrostatic potential cannot have an absolute maximum or minimum value.

Shew that, if one of a set of conductors has a positive charge, all the others being at zero potential, the total quantity of negative electrification is not greater than the quantity of positive electrification on the first conductor.

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10. Find the focal length of a thin double convex lens in terms of the radii of curvature of its surfaces and the refractive index of the glass.

A thin convergent lens is found to give an inverted image of an object the same size as the object when this is 20 cms. from the lens. An image formed by reflexion from the second surface of the lens is found to be coincident with the object when the object is 6 cms. from the lens. Calculate the radius of curvature of the second surface of the lens, and, assuming $\mu = 1\cdot5$, find the radius of curvature of the first surface.

SATURDAY, *June* 6. 2—5.

1. Prove that between every pair of consecutive real roots of the integral algebraic equation $f(x)=0$, there is an odd number of real roots of the derived equation $f'(x) = 0$.

Determine for what range of numerical values of k the roots of $2x^3 - 9x^2 + 12x - k = 0$ are all real.

2. Prove that, if the central radius to a point on an ellipse makes an angle $\tan^{-1}\left(\frac{b}{a}\tan\phi\right)$ with the major axis, the normal at the point makes an angle $\tan^{-1}\left(\frac{a}{b}\tan\phi\right)$ with the same axis.

Determine the greatest value of the difference of these angles and shew that for a meridian of the Earth in which $b:a::293:294$ the greatest difference is between 11 and 12 minutes.

3. Expand $\log\cos x$ and $\log\frac{\sin x}{x}$ in ascending powers of x, each to the fourth power: and verify that to the same power the formula

$$\log\sin x = \log x - \frac{1}{45}\log\cos x + \frac{64}{45}\log\cos\frac{x}{2}$$

is true.

4. Find an expression for the length of an arc of the curve $y=f(x)$.

Shew that the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the points of intersection of the parabola and $3y = 8x$ is $a(\log 2 + \frac{15}{16})$.

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5. The area of a triangle, S, is expressed as $\frac{1}{2}bc \sin A$; verify geometrically the values of the partial differential coefficients

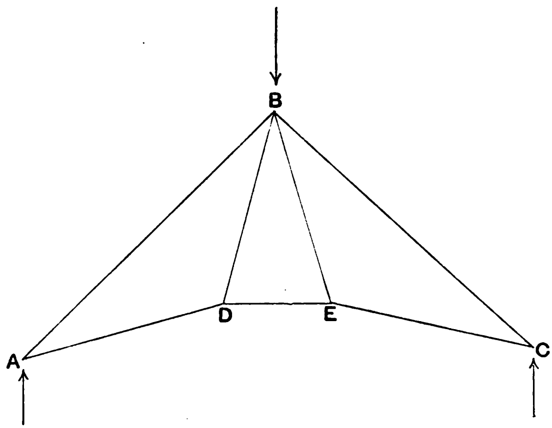
$$\frac{\partial S}{\partial b}, \quad \frac{\partial S}{\partial c} \quad \text{and} \quad \frac{\partial S}{\partial A}.$$

The area of a triangle is determined by measurements which give $b = 125$ feet, $c = 160$ feet, $A = 57^\circ 35'$. Another set of measurements give $b = 125{\cdot}5$ feet, $c = 161$ feet, $A = 57^\circ 25'$. Find the percentage difference between the second determination of the area and the first.

6. Establish the existence of a centre of a system of parallel forces and shew that the centre of gravity is the same point as the centre of mass of a body.

Find the centre of mass of the quadrant of a sphere.

7. The figure shews a roof truss, which is loaded at B with a weight W and is supported at A and C. The angle



ABC is a right angle and is trisected by BD and BE; the angles A and C are each 30° and $BA = BC$. Draw a force diagram to shew the tension or compression in each member.

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8. Explain what is meant by Centre of Pressure; and shew how to find the depth of the centre of pressure of a plane surface immersed in a fluid in a vertical plane.

A door 6 feet high and 3 feet wide separates two water-tight compartments. The door is hinged at the top and bottom of one edge and fastened by a lock at the middle point of the opposite edge. There is water in one compartment up to a height of 4 feet above the top of the door and the pressure on the door is borne by the hinges and lock. Find the stress on each hinge.

9. The total resistance to a car with the brakes on is $\frac{1}{16}$ of the weight of the car: shew that the car, when running at 8 miles per hour, can be stopped in about $11\frac{1}{2}$ yards.

Shew also that, if the ordinary resistance is $\frac{1}{64}$ of the weight, each stoppage by the brakes and recovery of the previous speed of 8 miles per hour adds to the work of traction an amount approximately equal to 181 foot-pounds per cwt. of the car mass.

10. A particle moves in a straight line with an acceleration directed towards a fixed point in the line and equal to μ times the distance of the particle from the point. Shew that the motion is periodic with period $2\pi/\sqrt{\mu}$.

A body is attached to one end of an inextensible string and the other end moves in a vertical line with simple harmonic motion of amplitude a, making n complete oscillations per second. Shew that the string will not remain tight during the motion unless $n^2 < g/4\pi^2 a$.

1909

(*New Regulations*)

THURSDAY, *June* 3. 9—12.

1. Prove that the angles in the same segment of a circle are equal to one another.

A circle touches a fixed line at the fixed point A and a second circle touches a parallel fixed line at a fixed point B: shew that, if the two circles touch one another, the point of contact is on the line AB or on the circle of which AB is a diameter.

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2. Shew that the chords of intersection of a fixed circle with the circles of a given coaxal system pass through a fixed point; give also a construction for the two circles of the system which touch the fixed circle when these circles are real.

3. Find the necessary and sufficient conditions that $ax^2 + 2bx + c$ may be positive for all real values of x: when these conditions are satisfied, determine the additional necessary and sufficient condition that

$$ax^2 + 2bx + c - \lambda(x-p)^2$$

may also be positive for all real values of x.

4. Prove that, if $1 + y = (1 + x)^n$, and y^r, where r is a positive integer, be expanded in ascending powers of x, the coefficient of x^t in the expansion is

$$\sum_{s=0}^{s=r-1} (-1)^s \begin{Bmatrix} r \\ s \end{Bmatrix} \times \begin{Bmatrix} (r-s)\,n \\ t \end{Bmatrix},$$

where $\begin{Bmatrix} m \\ p \end{Bmatrix}$ represents the coefficient of z^p in the expansion of $(1 + z)^m$.

5. Express $\sin 3x$ in terms of $\sin x$ and $\cos 3x$ in terms of $\cos x$.

Trace the graph of $\sin x - \frac{1}{2}\sin 3x$.

6. Three sides of a plane quadrilateral figure traversed in order have lengths a_1, a_2, a_3 respectively and the external angles or changes of direction at the two corners are θ_{12}, θ_{23} respectively: shew that the area of the quadrilateral is

$$\tfrac{1}{2}(a_1a_2 \sin\theta_{12} + a_2a_3 \sin\theta_{23} + a_1a_3 \sin\theta_{13}),$$

where $$\theta_{13} = \theta_{12} + \theta_{23}.$$

7. Interpret the meaning of the constants in the following forms of the equation of a straight line referred to rectangular axes,

$$y = mx + c, \quad x/h + y/k - 1 = 0, \quad x\cos\alpha + y\sin\alpha - p = 0.$$

Through a fixed point (β, γ) lines P_1Q_1 and P_2Q_2 are drawn intersecting the axes of coordinates respectively in P_1, Q_1 and P_2, Q_2, and making angles θ_1 and θ_2 with the axis of x. Prove that the condition that P_1Q_2 may be parallel to P_2Q_1 is

$$\tan\theta_1 \tan\theta_2 = \gamma^2/\beta^2.$$

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8. Define the polar of a point with regard to an ellipse, and find that of the point (h, k) with regard to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Shew that the equations of a pair of lines, which are at right angles and each of which passes through the pole of the other, may be written

$$lx + my + n = 0, \qquad n(mx - ly) + lm(a^2 - b^2) = 0.$$

Shew also that the product of the distances of such a pair of lines from the centre depends only on their direction and cannot exceed $(a^2 - b^2)/2$.

9. Find the equations of the normal at any point of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.$$

Prove that the normals drawn at points in a plane section, which is perpendicular to an axis, intersect the three planes of the axes in an ellipse, a line, and a line respectively.

10. Prove that, if ψ be the angle the radius vector of a plane curve makes with the tangent,

$$\frac{dr}{ds} = \cos\psi, \qquad r\frac{d\theta}{ds} = \sin\psi, \qquad \frac{d^2r}{ds^2} = \frac{\sin^2\psi}{r} - \frac{\sin\psi}{\rho},$$

where ρ is the radius of curvature.

Discuss the conditions for a maximum or minimum value of the radius vector.

THURSDAY, *June* 3. 2—5.

1. Prove from the successive differential coefficients that, if θ be the circular measure of a positive angle, $\theta - \sin\theta$ is positive and increases with θ but remains less than $\frac{\theta^3}{6}$.

Explain the rule given in the logarithmic tables: "For small angles of n minutes of arc, $\log \sin n' = \log n + \bar{4}{\cdot}4637$."

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2. The top of a hill is observed from two stations A and B on the same level; A is south of the hill, and B is north-east of A. If the angles of elevation from A and B are $9° 30'$ and $7° 30'$, find the compass bearing of B from the hill.

3. If I is the incentre and P the orthocentre of a triangle ABC, find expressions for the lengths of AI and AP in terms of the circumradius and the angles of the triangle.

If O is the circumcentre of the triangle, prove that OI is parallel to BC, if

$$\cos B + \cos C = 1.$$

4. Shew how to describe a circle about a triangle.

The lengths of the sides of a triangle are a, b, c and points are taken in them at distances p, q, r respectively from the middle points measured in definite sense: shew that the lines at right angles to the sides through these points are concurrent, provided $ap + bq + cr = 0$.

5. State and prove the harmonic property of a complete quadrilateral and its three diagonals.

State the form which the property assumes when two of the sides of the quadrilateral are parallel.

6. Prove that two straight lines which are not coplanar possess a common perpendicular, and that the length of this perpendicular is the shortest distance between the two lines.

The common perpendicular of two straight lines APP' and BQQ' is AB, also M and M' are the middle points of PQ and $P'Q'$ respectively. Prove that either the common perpendicular of MM' and AB bisects AB or MM' bisects AB.

7. Shew that parallel plane sections of an ellipsoid are similar ellipses with their centres on a straight line.

8. A curve touches the axis of x at the origin O, and P is a point (x, y) on the curve. Prove that ρ, the radius of curvature at the origin, is the limit of $\frac{1}{2} x^2/y$ as P moves towards O.

If the length of the chord OP is c, prove that the limit of $(c - x)/x^3$ is $\dfrac{1}{8\rho^2}$.

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9. Integrate

$$\frac{1}{1+x+x^2},\ x \sin x \sin 2x,\ \frac{1}{1+e\cos\theta}\ (0<e<1).$$

Employ the substitution $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$ to evaluate $\int_0^\theta \frac{1}{(1+e\cos\theta)^2}\,d\theta$.

10. Shew how to solve the differential equation $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x.

Solve the differential equation $av\frac{dv}{dx} + v^2 + bx = 0$ and shew that, if $v^2 = \frac{1}{2}ab$ when $x = 0$, then $v = 0$ when $x = \frac{1}{2}a$.

FRIDAY, *June* 4. 9—12.

1. Establish the kinematical formula

$$s = ut + \tfrac{1}{2}ft^2.$$

Two particles A and B are attached to the ends of a light inextensible string passing over a smooth fixed pulley. In the ensuing motion A suffers no collision, but B strikes the ground, the coefficient of elasticity being e. Prove that, if $e<\frac{1}{2}$, B ceases to rebound before the string is again taut.

2. A heavy particle of mass m moves with angular velocity ω in a horizontal circle, being attached to a fixed point by a light inextensible string of length l; determine the inclination of the string to the vertical and the tension or the string.

3. A light elastic string of natural length l has one extremity fixed at a point A and the other attached to a stone the weight of which in equilibrium would extend the string to a length l': shew that, if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{(l'^2 - l^2)}$ below the equilibrium position.

4. Prove that the angular momentum of a system of particles moving in a plane about an axis perpendicular to the plane is constant, when the particles are under no forces but the stresses between them.

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A uniform rod AB is falling in a vertical plane and the end A is suddenly held fixed at an instant when the rod is horizontal and the vertical components of the velocities A and B are v_1 downwards and v_2 upwards. Prove that the rod will begin to rise round the end A if $v_1 < 2v_2$.

5. The total normal induction across a closed surface of one sheet in a field of electric force is zero, also the surface encloses all points of the field at which there is any distribution of electricity and is not a surface of zero potential; prove that it encloses points of positive and points of negative and points of zero potential.

6. The surfaces of a condenser are two concentric spheres of radii a and b $(a < b)$, of which the outer surface is put to earth; shew that its capacity is $ab/(b-a)$.

Three concentric thin spherical shells are of radii a, b, c $(a < b < c)$, the first and third are connected by a fine wire through a small hole in the second, and the second is connected to earth through a small hole in the third. Shew that the capacity of the condenser so formed is

$$\frac{ab}{b-a} + \frac{c^2}{c-b}.$$

7. The "universal shunt galvanometer" is arranged as follows: between two points A and B there are two wires, one of which contains the coil of the galvanometer, and the other admits of a connection being made at a variable point C. If a current enters at A and leaves at C shew that the fraction of this current measured by the galvanometer is proportional to the resistance of AC.

8. State the law of refraction of light, explaining what is meant by the 'critical angle' for two media.

Shew that the angular breadth of the field of view in the principal plane of a prism of given angle is less the greater the refractive index of the material supposed optically denser than air.

9. A thin pencil diverging from a point in air at distance u in front of a block of glass, of refractive index μ, after direct refraction into the glass diverges from a point at

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distance v in front of the glass. The surface of the glass is concave and of radius r. Prove the formula

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}.$$

The cross sections of the refracted pencil and of the prolongation of the incident pencil by a plane at distance t behind the interface, as described above, are circles of radii ρ' and ρ. Prove that

$$\frac{\rho' - \rho}{\rho} = \frac{(\mu - 1)\, t\, (u - r)}{\mu r\, (t + u)}.$$

10. Determine the relative positions of the object and image in direct refraction through a thin convergent lens.

A long-sighted eye, when unaccommodated, is such that a pencil of rays converging to a point distant p cm. behind the front of the eye is brought to a focus on the retina: determine the focal length of the lens which placed close to the eye will enable it to see without accommodation an object placed q cm. from the eye front. In what case will a lens slightly too weak in power be made effective by slight advance from the eye?

FRIDAY, *June* 4. 2—5.

1. Shew that

$$a(x^3 + y^3 + z^3) + b\,(y^2z + z^2y + z^2x + x^2z + x^2y + y^2x) + cxyz$$

is divisible by $x + y + z$, if $3a - 3b + c = 0$.

2. Prove that if $b_1, b_2, \ldots b_n$ be all positive, then

$$\mu \leqq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leqq \nu,$$

where μ is the least of the fractions a_1/b_1, a_2/b_2, ... a_n/b_n, and ν is the greatest of them.

Further, prove that, if $a_1, a_2, \ldots a_n$ are also positive,

$$\frac{a_1 + a_2^2 + \ldots + a_n^n}{b_1 + \nu . b_2^2 + \ldots + \nu^{n-1} . b_n^n} \leqq \nu.$$

3. State the exponential theorem and calculate the 10th root of e to six decimal places.

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Shew that, if p is small, an approximation to a root of $x^{2+p}=a^2$ is

$$a\left\{1-\frac{p}{2}\log_e a+\frac{p^2}{8}(2+\log_e a)\log_e a\right\}.$$

4. State the relations which exist between the roots and the coefficients of an equation and deduce that, if α be a real root of the cubic $x^3+px^2+qx+r=0$, of which the coefficients are real, then the other two roots are real if $p^2-4q-2p\alpha-3\alpha^2$ is positive or zero.

5. Investigate the equation of the circle described on AB as diameter, where x_1, y_1 and x_2, y_2 are the coordinates of A and B respectively.

Two circles intersect in the point A, and the line joining the other extremities of the two diameters through A makes an angle θ with the axis of x. Prove that the equation of the radical axis of the circles is

$$(x-x_1)\cos\theta+(y-y_1)\sin\theta=0.$$

6. In a parabola S is the focus, PSQ a focal chord, PM the perpendicular on to the directrix, T the point where the tangent at P meets the directrix. Prove (1) that ST is perpendicular to SP, (2) that SM is perpendicular to PT, (3) that QM passes through the vertex.

Generalise the last theorem by orthogonal projection.

7. Shew that the coordinates of a point on an ellipse may be given in terms of a parameter t by means of the relations

$$\frac{x}{at}=\frac{b-y}{bt^2}=\frac{b+y}{b}.$$

Shew also that if any point P on the ellipse be joined to the extremities of an axis BB', the line through B at right angles to PB meets PB' on a fixed line.

8. Prove that the four middle points of two pairs of opposite edges of a tetrahedron form the corners of a parallelogram. Shew that the area of this parallelogram is greater than that of any parallel section.

9. The area S of a triangle on a given base c is expressed in terms of c and the angles A and B; prove that

$$\frac{\partial S}{\partial A}=\frac{b^2}{2},\quad \frac{\partial S}{\partial B}=\frac{a^2}{2},\quad \frac{\partial^2 S}{\partial A\,\partial B}=\frac{2S}{\sin^2 C}.$$

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Shew that, in determining the position of the vertex when the base line is accurately known and the base angles are subject to small limits of error $\pm\alpha$, $\pm\beta$ respectively, the approximate small area at any point of which the vertex may lie is

$$4\alpha\beta\frac{\partial^2 S}{\partial A\partial B}.$$

10. Shew that, when $f(x)$ is of the form $A+Bx+Cx^2$,

$$\int_0^1 f(x)\,dx = \tfrac{1}{6}\{f(0)+4f(\tfrac{1}{2})+f(1)\}.$$

Shew also that, if $\phi(x)$ be any polynomial of the fifth degree,

$$\int_0^1 \phi(x)\,dx = \tfrac{1}{18}\{5\phi(\alpha)+8\phi(\tfrac{1}{2})+5\phi(\beta)\}$$

where α and β are the roots of $x^2-x+\frac{1}{10}=0$.

SATURDAY, *June* 5. 9—12.

1. Find necessary and sufficient conditions for the equilibrium of a system of forces acting on a rigid body in one plane.

The moments of such a system (not in equilibrium) about three collinear points A, B, C, in the plane are G_1, G_2, G_3. Prove that

$$G_1 . BC + G_2 . CA + G_3 . AB = 0.$$

2. If a body is in equilibrium under a system of forces in a plane, prove that the virtual work for any small displacement is zero.

Two rods AB, BC are smoothly jointed at B, the end A is fixed and AB can turn about it, and a pin at C is free to move in a straight slot, the direction of which passes through A. Prove that a force F acting at C along the slot and a couple Fd acting on the rod AB will keep the system in equilibrium, where d is the length of the line drawn from A perpendicular to AC to meet BC.

3. State the laws of friction, including those of limiting friction.

The weight of a train is 400 tons, the part of the weight of the engine supported by the driving wheels is 30 tons, and

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the coefficient of friction between the driving wheels and the rails is ·16. Prove that at the end of a minute after starting on the flat the velocity will be less than 15·8 miles per hour.

4. Investigate the conditions under which a body floats in equilibrium in a liquid.

A cylinder of wood, with two parallel plane ends cut obliquely to the generating lines, is in equilibrium with one end resting on the bottom of a vessel. Water is slowly poured into the vessel and percolates between the end of the cylinder and the bottom of the vessel. Shew that the cylinder will tilt before it floats.

5. Prove that, assuming the same temperature at all heights, the pressure of the atmosphere at a height z is given by

$$p = \Pi e^{-\frac{gz}{k}},$$

where Π is the pressure at the ground, and the pressure and density of the air are connected by the formula $p = k\rho$.

The barometer falls from 30 to 29 inches as a balloon rises 900 feet from the surface, prove that k/g is approximately 26,500 feet.

6. Prove that when a body describes a path round a centre of force the radius vector of the path sweeps out equal areas in equal times.

Prove that the earth's velocity of approach to the sun, when the earth in its orbit is at one extremity of the latus rectum through the sun, is approximately $18\frac{1}{2}$ miles per minute, taking the eccentricity of the earth's orbit as 1/60 and 93,000,000 miles as the semi-axis major of the earth's orbit.

7. A body swings about a fixed horizontal axis through an angle α on either side of its position of equilibrium. Prove that for different values of α the maximum angular velocity of the body is proportional to $\sin(\alpha/2)$.

A body of mass M makes complete revolutions about a fixed horizontal axis, about which its moment of inertia is I, and the centre of gravity of the body is at a distance c from

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the axis. If the greatest and least angular velocities are $\frac{1}{2}$ °/$_\circ$ greater and $\frac{1}{2}$°/$_\circ$ less than ω, prove that $\omega = \sqrt{\dfrac{200\,Mgc}{I}}$.

8. Prove that a tube of electric force cannot terminate in empty space, and that at the terminations of such a tube on two conductors the charges are equal and opposite.

9. Find the rate of loss of electric energy when a given current flows in a wire of given resistance.

A current is led to and from an electric lamp at a distance of 40 feet by a pair of wires, each of resistance 13 ohms per 1000 yards. If the difference of potential across the lamp is 25 volts and the current supplies energy to the lamp at the rate of 30 watts, at what rate in watts is energy lost in the leads? (A watt is the rate at which work is done by a current of one ampère working through one volt.)

10. Shew that direct refraction through a plate, of thickness t and refractive index μ, gives an image nearer than the object by $t\left(1 - \dfrac{1}{\mu}\right)$.

How far must a mark be placed in front of a plate so that its image by reflection at the front surface may coincide with that by refraction of a mark on the same normal on the back surface? What practical application can be made of this result?

SATURDAY, *June* 5. 2—5.

1. Prove that the product of the lengths of the perpendiculars from the foci of an ellipse on to any tangent is equal to the square on half the minor axis.

Two ellipses have a common focus S and collinear major axes. Prove that their common tangents intersect the line of the major axes at a distance

$$(b_1^2 \,.\, SH - b_2^2 \,.\, SH_1)/(b_1^2 - b_2^2)$$

from S, where H_1 and H_2 are the other foci and b_1 and b_2 the minor semi-axes of the ellipses.

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2. A variable quantity which can be represented by a quadratic function of the time assumes the values 19·6, 18·8, 17·1 at the middle of each of the three successive quinquennial periods preceding 1906. Find the value of the quantity in the middle of the year 1906.

3. Prove that $\frac{d}{dx}\sin^{-1}x = \pm\frac{1}{\sqrt{1-x^2}}$ and determine the ranges of values of $\sin^{-1}x$ for which the positive sign, and the ranges for which the negative sign, must be taken.

Prove that $\frac{d}{dx}\sinh^{-1}x = +\frac{1}{\sqrt{1+x^2}}$ without ambiguity of sign.

4. Shew how to determine the maximum values of a function of a single variable.

If the sum of the lengths of the hypothenuse and another side of a right-angled triangle is given, shew that the area of the triangle is a maximum when the angle between those sides is 60°.

5. A point A is inside a circle of radius a, at a distance b from the centre. Prove that the locus of the foot of the perpendicular drawn from A to a tangent to the circle encloses an area $\pi(a^2+\frac{1}{2}b^2)$.

6. Prove that a force can be resolved into two components along two given lines parallel to the direction of the force, when the lines lie in the same plane through the line of the force, and that it can be resolved into three components along three given lines parallel to the direction of the force when the three lines do not lie in one plane.

A body of weight W is supported by two vertical strings attached to points A and B in the body. When the inclination of AB to the upward vertical is θ_1, the tension in the string at A is T_1, and when the inclination of AB is θ_2, the tension in this string is T_2. Shew that the perpendicular distance of the centre of gravity of the body from the line AB is $\frac{(T_1-T_2)AB}{W(\cot\theta_2-\cot\theta_1)}$.

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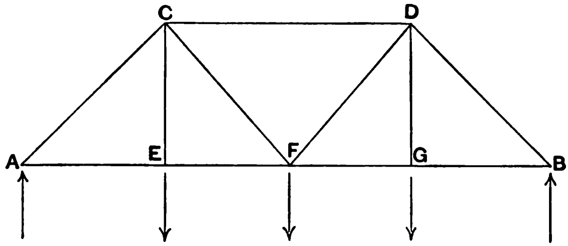
7. The diagram shows a framework of rods, smoothly jointed at A, B, C, D, E, F, G, and each rod is either horizontal or vertical, or is inclined at an angle of 45° to the vertical. Weights of 3, 4, and 7 tons are supported at E, F, and G, and the framework is supported at A and B. Find the stresses in the rods due to this system of loading.



8. Define the pressure at a point in a fluid, proving that it is the same in all directions.

A piston is slowly pushed along a cylinder against water under pressure and to prevent leak the piston is fitted with a leather collar of breadth l which the water pushes against the walls of the cylinder. If μ is the coefficient of friction between the leather and the cylinder and r is the radius of a section of the cylinder, prove that the fraction of the work done which is spent in overcoming friction is

$$\frac{2\mu l}{r + 2\mu l}.$$

9. Two elastic spheres collide directly, one being initially at rest. Determine their velocities after impact.

A large cube, with its edge of length a, moves with velocity v in a direction perpendicular to one face through a medium formed of small spheres, each of mass m and of coefficient of elasticity e. The spheres are initially at rest and their mutual collisions can be neglected. Prove that when the number (n) of spheres per unit volume is large enough, the cube is practically subjected to a retarding force

$$(1 + e)\, nma^2v^2.$$

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10. A stream of water, 1 square foot in section, flowing at the rate of 16 feet per second, enters a turbine. At what rate, in Horse-Power, does the water deliver energy? (1 cubic foot of water weighs 62·5 lbs.)

What fraction of this energy is used when the water-power drives a shaft, at a speed of 100 revolutions per minute, with a couple of which the moment is 140 with the pound weight and foot as units?

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THURSDAY, *June* 2. 9—12.

1. The side BC of a triangle ABC is bisected at D: shew that

$$AB^2 + AC^2 = 2AD^2 + 2BD^2.$$

A variable chord PQ of a circle subtends a right angle at a fixed point O, and C is the centre of the circle. Shew that the locus of the feet of the perpendiculars from O and C to the line PQ is a circle whose centre is the middle point of OC.

2. Prove that if the cross ratios of the two ranges $ABCD$ and $AB_1C_1D_1$, having the point A in common, are equal, the lines BB_1, CC_1, DD_1 are concurrent.

AB and CD intersect in U, AC and BD in V, UV intersects AD and BC in F and G, BF intersects AC in L. Prove that the range $ALVC$ is harmonic, and that LG, CF, and AU are concurrent.

3. Explain methods of verifying that two rational integral functions of x are identical.

Prove the following identity

$$16\,(x-2)\,(x-4)\,(x-6)$$
$$= (x-1)\,(x-3)\,(x-5)\,(x-7)\left\{\frac{5}{x-1} + \frac{3}{x-3} + \frac{3}{x-5} + \frac{5}{x-7}\right\}.$$

4. Shew by means of the expansions of $\log_e(1+x)$ and e^x in powers of x that

$$(1+x)^{(1+x)} = 1 + x + x^2 + \tfrac{1}{2}x^3, \text{ approximately,}$$

where x is small.

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5. For any triangle ABC prove that

$$\frac{b \sec B + c \sec C}{\tan B + \tan C} = \frac{c \sec C + a \sec A}{\tan C + \tan A} = \frac{a \sec A + b \sec B}{\tan A + \tan B}.$$

At a point O on a horizontal plane the angles of elevation of two points P and Q on the side of a hill are found to be 38° and 25°; the distance of A, the foot of the hill, from O is 500 yds., and the distance AQ is 320 yds., the whole figure lying in a vertical plane. Prove that the distance PQ is 329 yds., approximately, and find the slope of the hill.

6. The periodic function

$$y = a_1 \sin \frac{2\pi x}{X} + a_2 \sin \frac{4\pi x}{X} + a_3 \sin \frac{6\pi x}{X}$$

has its graph drawn for a complete period, that is from $x = 0$ to $x = X$. The graph is cut into three portions at the ordinates $x = \frac{X}{3}$ and $x = \frac{2X}{3}$, and the three portions are superposed. Shew that the equation to the graph which is the sum of the superposed portions is

$$y = 3a_3 \sin \frac{6\pi x}{X}.$$

7. The equation to a straight line being given in the form $lx + my = 1$, write down (i) the lengths of the intercepts on the coordinates axes, (ii) the length of the perpendicular from the origin on the line, (iii) the tangent of the angle which the line makes with Ox, (iv) the equation to a straight line parallel to the given line and passing through the point h, k.

Also shew that the expression $lx + my - 1$, where x, y are the coordinates of any point P in the plane xy, is positive or negative according as P is on one side or the other of the given line.

8. Investigate the relation between the eccentric angles of the extremities of conjugate diameters of an ellipse.

PCP' and DCD' are conjugate diameters of an ellipse, and ϕ is the eccentric angle of P. Prove that $\frac{1}{2}\pi - 3\phi$ is the eccentric angle of the point where the circle $PP'D$ again cuts the ellipse.

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9. Find the condition that the line

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}$$

may touch the ellipsoid, $x^2/a^2+y^2/b^2+z^2/c^2=1$.

Prove that the locus of the points of contact of parallel tangents to an ellipsoid is plane.

10. Trace the curve $y^2=x(x-1)(2-x)$; and shew that the tangents at the points on the curve given by $x=1+\frac{1}{\sqrt{3}}$ are parallel to the axis of x.

THURSDAY, *June* 2. 2—5.

1. The length of a side of a regular polygon of n sides is $2l$, and the areas of the polygon and of the inscribed and circumscribed circles are A, A_1, A_2, prove that

$$A_2-A_1=\pi l^2, \quad n^2l^2A_1=\pi A^2.$$

2. Three points A, B, C lie in a straight line, and AB is to BC as m to n. Through A, B, and C are drawn parallel straight lines AX, BY, and CZ. A point P moves on AX and a point R moves on CZ so that at any time t the distance AP is equal to $a_1+a_2\sin(nt+\alpha)$ and the distance CR is equal to $c_1+c_2\sin(nt+\gamma)$, and the straight line PR cuts BY in Q. Express the distance BQ in a similar form.

3. Find expressions for $\sin^2 A$ and $\sin^4 A$ in terms of the cosines of multiples of A.

If $\sin\phi=\lambda\sin\theta$, where λ is a fraction so small that λ^6 and higher powers may be neglected, find an expression for $\cos\phi$ in terms of $\cos 2\theta$ and $\cos 4\theta$.

4. Prove that the inverse of a circle is either a circle or a straight line.

A, B, C are three collinear points in order. Obtain the inverse of the theorem, $AB+BC=AC$, where any point O, not collinear with ABC, is the centre of inversion. Investigate whether additional information can be obtained by inverting the new theorem about another coplanar point S.

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5. A and A' are the extremities of an axis of an ellipse, and PN is the ordinate drawn from any point P of the ellipse to AA'. Shew that the ratio $PN^2 : AN . A'N$ is constant.

Shew that the locus of the middle points of focal chords of an ellipse is a similar ellipse.

6. Prove that if a line is perpendicular to two distinct lines which intersect it and lie in a certain plane, it is perpendicular to every line lying in that plane.

Prove that two circles, whose centres are A and B and radii are a and b, lying in different planes, are both sections of the same sphere if A and B lie in a plane normal to the line of intersection of the planes and also

$$AP^2 - BP^2 = a^2 - b^2,$$

where P is any point on this line of intersection.

7. Prove that if (lmn) and $(l'm'n')$ are the direction cosines of two lines inclined at an angle θ,

$$\cos\theta = ll' + mm' + nn'.$$

Three diagonals of a rectangular parallelopiped make with the fourth diagonal acute angles θ, ϕ, ψ. Prove that, provided the sum of the squares of the two shorter edges exceed the square of the longest edge,

$$\cos\theta + \cos\phi + \cos\psi = 1.$$

Explain the reason for the introduction of the condition.

8. The perpendicular to the tangent to a curve being denoted by p, and the angle this perpendicular makes with a fixed line by ϕ, shew that the radius of curvature at any point of the curve is given by the equation

$$\rho = p + d^2p/d\phi^2.$$

Obtain the relation between p and ϕ for the curve given by

$$x = a\cos^3\phi, \quad y = a\sin^3\phi$$

and shew that $\rho = \frac{3}{2}a\sin 2\phi$.

9. Evaluate the following integrals:

$$\int\sqrt{(a^2 - x^2)}\,dx, \quad \int\sec x\,dx, \quad \int\frac{dx}{2 - 3x + x^2}, \quad \int\frac{dx}{\sqrt{(2 - 3x + x^2)}}.$$

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10. Solve the differential equation $\ddot{x} + 2k\dot{x} + n^2x = 0$, when $k^2 < n^2$.

In the case of a pendulum making small oscillations, the time of a complete oscillation being 2 secs. and the angular retardation due to the air being taken as $\cdot 04 \times$(angular velocity of pendulum), shew that an amplitude of 1° will in 10 complete oscillations be reduced to about $40'$.

[Take $\log_{10} e = \cdot 4343$.]

FRIDAY, *June* 3. 9—12.

1. Shew how to apply the properties of the force diagram and the funicular polygon to determine the magnitude and line of action of the resultant of a system of parallel forces acting on a rigid body in one plane.

2. A mass m is acted on by a constant force of P lbs. weight, under which in t seconds it moves a distance x feet, and acquires a velocity v feet per second. Shew that

$$x = g\frac{t^2}{2}\frac{P}{m} = \frac{v^2}{2g}\frac{m}{P}.$$

A horse pulls a wagon of 10 tons from rest against a constant resistance of 50 lbs. weight. The pull exerted is initially 200 lbs. weight, and decreases uniformly with the distance covered until it falls to 50 lbs. weight at a distance of 167 feet from the start. Shew that the resulting velocity of the wagon is very nearly 6 feet per second.

3. A mass $2M$ is fixed to one end of a fine string, which passes over a smooth fixed pulley, and to the other end of the string is attached a smooth pulley. Over this second pulley a fine string passes, to the ends of which are fixed masses $(M+m)$ and $(M-m)$. Determine the motion, neglecting the masses and moments of inertia of the pulleys.

4. State the law of conservation of linear momentum, as applicable to a system of particles in a plane, and deduce it from Newton's Laws of Motion.

Two particles, of masses m and M, lying on a smooth horizontal plane, are connected by a light elastic rod, of length l and modulus of elasticity λ. The particle m is struck by a blow B in the direction of the rod. Determine the subsequent motion, on the assumption that the mass of the

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rod may be neglected, and that consequently the stress in it is the same at all points, and shew that, if its length at any time t is $l-x$, then x is given by the equation

$$\frac{Mm\dot{x}^2}{M+m}+\frac{\lambda x^2}{l}=\frac{MB^2}{m(M+m)}.$$

5. Define electric potential and shew that the potential at any external point due to a charge E freely distributed on a conducting sphere is equal to $\frac{E}{d}$ where d is the distance of the point from the centre of the sphere.

Shew that in an electric field due to a single point-charge the mean value of the potential over the surface of a sphere, which does not enclose the point, is equal to the value of the potential at the centre of the sphere.

6. An electric system consists of a closed conducting shell at potential V, completely surrounded by another closed conducting shell at zero potential. Shew that the inner surface of the inner shell and the outer surface of the outer shell are free from electrification, and that the system exerts no action on points external to the outer shell.

If the opposed surfaces of the shells are concentric spheres of radii R and $R+\rho$ respectively, shew that the densities of the electrification on these surfaces are respectively

$$\frac{V(R+\rho)}{4\pi R\rho} \text{ and } \frac{-VR}{4\pi(R+\rho)\rho}.$$

7. A tangent galvanometer has two concentric and coplanar coils of radii r_1 and r_2 respectively, and the coils are wound with equal lengths of the same wire. If a given potential difference is applied, first to coil 1 giving a deflection α_1 and then to coil 2 giving a deflection α_2, shew that

$$\frac{\tan\alpha_1}{\tan\alpha_2}=\frac{r_2^2}{r_1^2}.$$

The coils are now connected in parallel and the same potential difference is applied. When the connections are such that the currents in the coils circulate in the same

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direction the deflection is β_1, and when the currents circulate in opposite directions the deflection is β_2; shew that

$$\frac{\tan\beta_1}{\tan\beta_2}=\frac{r_2^2+r_1^2}{r_2^2-r_1^2}.$$

8. A prism is placed between the eye and an object so that light from the object falls on the eye after refraction in a principal plane of the prism. The prism is placed, initially, so that the light falls on it normally, and it is then rotated slowly about an axis parallel to its refracting edge. Shew that the image appears first to approach the object and then to recede from it, and that the image is brightest when nearest the object.

9. An object is seen by direct vision through a sphere of radius r and refractive index μ. If the distances of the object and its image from the centre of the sphere be u and v respectively, shew that

$$\frac{1}{u}-\frac{1}{v}=2\frac{\mu-1}{\mu r}.$$

If $r=\frac{1}{2}$ inch and $\mu=\frac{3}{2}$ and if the eye be placed at a distance of $\frac{3}{4}$ inch from the centre of the sphere, find where the object must be placed in order that the image may be 12 inches from the eye.

10. Define the magnifying power of a telescope, and explain how the breadth of the eye-ring may be utilized to obtain the magnifying power of an astronomical telescope, giving the proof of the theory for a telescope of two thin lenses.

FRIDAY, *June* 3. 2—5.

1. Shew that the system of equations

$$\begin{aligned}y^2+yz+z^2&=1+x(x+y+z),\\ z^2+zx+x^2&=1+y(x+y+z),\\ x^2+xy+y^2&=1+z(x+y+z),\end{aligned}$$

is equivalent to two independent equations.

2. If y be given in terms of the real quantity x by the equation

$$y=\frac{x^2+2x+c}{x^2+4x+3c},$$

shew that y can have any real value if $0<c<1$; and draw a figure showing the form of the graph in this case.

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3. Find the first three terms of the expansion of $\dfrac{(100-x)^{\frac{1}{2}}}{(10-x)}$ in ascending powers of x by the binomial theorem. Indicate on the same diagram by rough sketches the graphs of the sum of these three terms and of the original expression, and point out the parts of the graphs for which the inclusion of a larger number of terms would make the approximation better.

4. Prove that $\log_a x = \log_a b \times \log_b x$.

Prove that when x is large

$$\sqrt{\{x(x+1)\}} \times \log_{10} \frac{1+x}{x}$$

differs from $\log_{10} e$ by $\dfrac{\log_{10} e}{24 \times x^2}$, approximately. Assuming $x = 10$, apply this formula to calculate by the aid of your tables $\log_{10} e$, and estimate the number of places of decimals to which the result is correct.

5. Shew that, if any two circles cut orthogonally, the extremities of any diameter of either are conjugate points with respect to the other, and deduce that the polars of a fixed point with respect to a system of coaxal circles are concurrent.

6. Find the equations of the tangent and the normal at the point (x', y') on the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

Any tangent to a hyperbola cuts the asymptotes in L and L', and cuts the tangents at the vertices in M and M'. Prove that $(LML'M')$ is a harmonic range.

7. A sphere of radius r touches the three coordinate planes; find the equations to the circle in which the sphere is cut by the plane $z = \dfrac{r}{2}$.

Also find the equations to the planes which touch the sphere and which have their intercepts on the axes OX, OY, OZ respectively, in the ratio of $2:3:5$.

8. Prove that the volume of a variable tetrahedron with a pair of opposite edges on fixed non-intersecting lines is proportional to the product of the lengths of those edges.

Explain various methods of dividing a tetrahedron into four tetrahedrons of equal volume.

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9. Determine the constants A, B, C so that the curve

$$y = A(a^2 - x^2) + B \sin \frac{\pi x}{a} + C \cos \frac{\pi x}{2a}$$

may cut the axis of y at an angle β, at a distance k from the origin, and that this point may be a point of inflexion.

10. The coordinates x, y of any point of a closed curve being given in terms of a single parameter, shew that the area of the curve may be found by integrating the expression $\frac{1}{2}(x\,dy - y\,dx)$ round the contour of the curve.

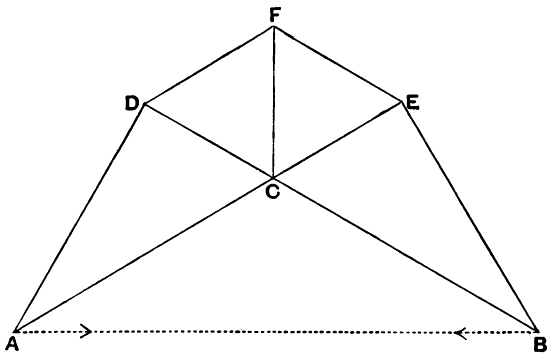
Trace the form of the curve given by the equations

$$x(1 + t^2) = 1 - t^2, \qquad y(1 + t^2) = 2t$$

as t increases from $-\infty$ to $+\infty$, and apply the above method of integration to shew that its area is π.

Saturday, *June* 4. 9—12.

1. In the jointed frame shewn in the figure DCF and ECF are equilateral triangles, ACE and BCD are straight, and the angles ADC and BEC are right angles. Equal forces are applied at A and at B in the line AB. Draw a force diagram for the frame, and state which members are in compression and which in tension.

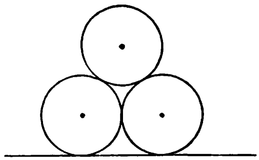


2. State the principle of Virtual Work.

Two smooth rods AB, AC are fixed in a vertical plane at right angles to one another, and a heavy uniform rod BC can slide with its ends B, C on the rods AB, AC, respectively. Shew that the rod BC can rest in equilibrium at an inclination to the vertical double that of either of the fixed rods.

3. State the laws of friction, and of limiting friction.

Three equal rough cylinders are in contact with one another, and with a rough horizontal plane, as in the figure



all surfaces in contact being supposed equally rough. Shew that, if the coefficient of friction be equal to $(2-\sqrt{3})$, the upper cylinder will be on the point of slipping between the lower cylinders, which will be on the point of rolling on the horizontal plane.

4. Determine the conditions under which a body floats in equilibrium, immersed partly in one liquid and partly in another.

A cylindrical vessel (A), the area of whose cross section is a cm.^{2}, is placed with its base on a horizontal table. An iron cylinder (B), whose height is H cm., and sp. gr. 7·5, and the area of whose cross section is β cm.^{2} rests with its axis vertical on the bottom of A. Mercury (sp. gr. 13·5) is now poured into A to a depth h cm. Shew that B will not rise so long as $5H > 9h$. Water is now poured into A until B is immersed. Shew that B will have risen a height $(1-\beta/a)(h-13H/25)$ cm., provided that this expression is positive.

5. A vessel containing heavy liquid under gravity is constrained to move with uniform acceleration in a straight line. Prove that the free surface is a plane.

A barometer is suspended freely from the roof of a railway carriage which is at rest on a slope of inclination β. If the carriage be now allowed to run freely down the slope, shew that the barometric reading is increased in the ratio $\sec\beta : 1$, but that if the barometer be fixed at right angles to the floor of the carriage, then there will be no alteration in the barometric reading when the carriage moves.

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6. Shew that if a particle moving in a circle of radius a has at any instant velocity v, then the acceleration in the direction of the radius is equal to $\frac{v^2}{a}$.

A heavy particle hanging by a light inextensible string of length a is started into motion with a horizontal velocity v. Shew that the string will not remain tight throughout the motion if v^2 be less than $5ga$.

7. Define angular momentum and explain the principle of the conservation of angular momentum.

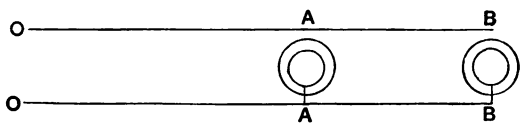
A flywheel whose moment of inertia is I is keyed to an axle of radius r and is rotating with angular velocity ω. As it rotates it winds up on the axle a light inextensible string which is attached to a mass M resting on the ground below. Shew that at the instant the string becomes tight the angular velocity of the flywheel is reduced in the ratio of I to $I+Mr^2$, and that the kinetic energy of the system is reduced in the same ratio.

8. Explain the method of representing the properties of an electric field by means of tubes of force, and illustrate by a sketch shewing generally the distribution of tubes in a field due to a charged sphere placed in front of an infinite conducting plane.

9. A number of linear conductors, all of the same material, have lengths l_1, l_2, etc. and cross-sectional areas α_1, α_2, etc. respectively. If R_1 be their combined resistance when placed in series and R_2 their combined resistance when placed in parallel shew that

$$\frac{R_1}{R_2} = \Sigma\left(\frac{l}{\alpha}\right)\Sigma\left(\frac{\alpha}{l}\right).$$

The sketch shews a pair of conductors through which energy is supplied from OO to motors at AA and BB, AA taking a current of 100 amperes and BB a current of 80 amperes. The length of OA is 300 metres and that of AB



150 metres, and the specific resistance of the material of the conductors is $1\cdot56 \times 10^{-6}$ ohms per cm. cube. If the current density in the conductors is 100 amperes per square cm. find the cross-sectional areas of OA and of AB. Find also the potential difference at AA and at BB if a potential difference of 240 volts is maintained at OO.

10. Prove the formula $PF \,.\, F'Q = f^2$, for the case of a single thin lens, where F and F' are the foci, P and Q are two conjugate points on the axis, and f is the focal length.

It is desired to concentrate the sun's rays at a point on the axis of a convergent thin lens, at a distance c from the lens on the same side as the sun. Prove that this can be done by the aid of a plane mirror placed at a certain distance behind the lens, provided that f does not lie between c and $2c$.

SATURDAY, *June* 4. 2—5.

1. Prove that the segment of any tangent to a conic intercepted between the curve and a directrix subtends a right angle at the corresponding focus.

Find the locus of the focus of a variable conic with a given line as the corresponding directrix, and touching another given line at a given point. Indicate the parts of the locus which correspond respectively to an ellipse, a parabola, or a hyperbola.

2. Having given that, when $x < 1$

$$\log_e (1 + x) = x - x^2/2 + x^3/3 - \ldots,$$

and that $\log_{10} 2\cdot3758 = \cdot3758099$, and $\log_{10} e = \cdot4343$, shew that an approximate solution of the equation

$$x = 100 \log_{10} x \quad \text{is} \quad x = 237\cdot58121.$$

3. Shew that the area of a segment of a circle of height h, bounded by a chord of length c is $\frac{2}{3}hc$ approximately, if powers of h/c higher than the first be neglected.

4. Define a maximum value of a function of one variable. When only one maximum value exists, is it always the greatest value of the function? Illustrate by a diagram.

OC and OD are two fixed lines inclined at an angle ω, and A is a fixed point within the angle COD. A variable line PQ has its ends P and Q on OC and OD, and the angle OPQ

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has a constant value α. Prove that the maximum value of the area APQ is $\frac{1}{8}c^2 \sin\omega \operatorname{cosec}\alpha \operatorname{cosec}(\alpha+\omega)$, where c is the perpendicular from O on the line through A parallel to PQ.

5. Shew that, when $f(x)$ is of the form $A + Bx + Cx^2$,

$$3\int_0^2 f(x)\,dx = f(0) + 4f(1) + f(2),$$

and deduce that an approximate value of the integral

$$\int_0^{2n} f(x)\,dx, \text{ whatever be the form of } f(x), \text{ is}$$

$$\tfrac{1}{3}\{f(0)+f(2n)+2[f(2)+f(4)+\ldots+f(2n-2)] \\ +4[f(1)+f(3)+\ldots+f(2n-1)]\}.$$

The specific heat of water (s, in joules) at temperature t° being given by the following table:

$t=$	0°	10°	20°	30°	40°	50°
$s=$	4·219	4·195	4·181	4·174	4·173	4·174

$t=$	60°	70°	80°	90°	100°
$s=$	4·178	4·184	4·190	4·197	4·205

shew that, to heat 1 gram of water from 0° to 100° requires 418·5 joules, approximately.

[The specific heat of a substance at temperature t° is $\frac{dQ}{dt}$, where Q is the quantity of heat required to heat 1 gram of the substance from some fixed temperature to t°.]

6. The length of the line joining the lowest points of the wheels of a bicycle is a, and the centre of gravity is at a height h above this line and at a distance x in front of its middle point. No account being taken of axle friction or of road resistance to rolling, shew that when the brake is hard on the back wheel the slope of the greatest incline on which

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the bicycle can be held up without slipping back is a, where $\tan a = \frac{\mu}{2}\frac{a-2x}{a-\mu h}$ and μ is the coefficient of friction between the tyres and the ground.

7. Assuming that the tension in a spring is λx when its extension is x, prove that the potential energy of the spring is $\frac{1}{2}\lambda x^2$.

The spring carries a mass m which is oscillating in a vertical straight line about its position of equilibrium. Find expressions for the kinetic energy of the mass, its potential energy relative to its position of equilibrium, and the potential energy of the spring, when the displacement of the mass from its position of equilibrium is x, and shew that the sum of these three quantities is constant, the mass of the spring itself being neglected.

8. Define Work and Power. If a constant force (P dynes) act on a mass (m grams) shew that the rate of working at t seconds from the beginning of motion is $\frac{P^2t}{m}$ ergs per second, and find an expression for the work done in the interval between t and $t+\tau$.

A boat of mass m is moving with a velocity V which varies periodically and which can be expressed, in terms of the time t, as $V = u + v \sin nt$, u and v being constants. The resistance to motion is λV^2. Write down expressions giving the values of the propelling force and of the rate of working, at any instant, and shew that, if $u = 4v$, the work done during a complete period is about $9\frac{1}{2}$ per cent. greater than would be done in the same time if the boat were propelled uniformly at the same average speed.

9. A particle is projected in vacuo with a given velocity in a given direction from a point at a height h above a horizontal plane. Find the time of flight and the horizontal distance traversed by the particle when it strikes the plane.

Shew further that, if the angle of projection be 45°, and the greatest height above the point of projection be h, then the horizontal distance traversed by the particle is $2h(1+\sqrt{2})$.

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10. Shew how to determine the direction and magnitude of the resultant pressure of a heavy fluid on a curved surface with which it is in contact.

A circular cylinder with closed circular ends is filled with liquid, and held in any position. Shew that the resultant pressure on the curved surface bisects the axis at right angles, and determine the magnitude of the resultant pressure.

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THURSDAY, *June* 1. 9—12.

1. ABC is a triangle right-angled at A, and AD is drawn perpendicular to BC. Shew that $AD^2 = BD \cdot DC$.

AP, BQ are parallel tangents to a circle, and a tangent to the circle at any point C cuts them in P and Q respectively. Shew that $CP \cdot CQ$ is independent of the position of the point C.

2. A, P, B are three points on a straight line. Give a geometrical construction for the point Q which is harmonically conjugate to P with regard to A, B.

A, Q, B, P, C are five points in a straight line such that A, P are conjugate with regard to B, C, and C, Q with regard to A, B. Shew that $4AC \cdot QP = 3AP \cdot QC$.

3. A circle cuts orthogonally two fixed non-intersecting circles. Prove that it passes through two fixed points on their line of centres.

Prove also that the two points are inverse points with regard to either of the given circles.

4. Simplify the product

$$\left\{2x + 1 + \frac{x-1}{x+1} - \frac{x+2}{x}\right\} \left\{\frac{1}{2x+1} + \frac{x+1}{x-1} - \frac{x}{x+2}\right\}.$$

Given that x and y are numerically unequal quantities, and that

$$x + y = a + b + c, \quad x(x-a)(x-b)(x-c) = y(y-a)(y-b)(y-c),$$

prove that $x^3 + y^3 = a^3 + b^3 + c^3$.

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5. If the Binomial Expansion of $(1+x)^m$, where m is a positive integer, be written

$$(1+x)^m = 1 + p_1 x + p_2 x^2 + \dots p_m x^m,$$

shew that
$$1 + p_1 + p_2 + \dots p_m = 2^m,$$
$$1 + p_1^2 + p_2^2 + \dots p_m^2 = 2m\,!/(m\,!)^2.$$

Also shew that $p_1 + 2p_2 + 3p_3 + \dots + mp_m = m2^{m-1}$,

$$p_1 + 2^2 p_2 + 3^2 p_3 + \dots + m^2 p_m = m\,(m+1)\,2^{m-2}.$$

6. Shew how to find the remaining side and angles of a triangle in which two sides and the angle opposite to one of them are given. What relation must exist between the given parts in order that two real triangles may be drawn?

At a point P a ship is in line with two lighthouses A and B and it proceeds on a straight course which makes an angle of 16° 20′ with the direction PAB. Observations are made of the angle subtended at the ship by the line AB, and this angle gradually increases and is equal to 90° when the ship is 3·75 miles from P. The distance between the lighthouses is 3 miles; shew that the point P is 3·165 miles from the nearer of them.

7. Find the equation of the circle which circumscribes the triangle whose sides have equations

$$x = 0, \quad y = 0, \quad x/a + y/b = 1.$$

Prove that the tangents to this circle at the angular points meet the opposite sides on the line

$$a^3 x + b^3 y + a^2 b^2 = 0.$$

8. Find the condition that the plane $lx + my + nz = p$ should touch the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$; and prove that the equations of the corresponding normal are

$$\frac{px}{l} - a^2 = \frac{py}{m} - b^2 = \frac{pz}{n} - c^2.$$

Prove that the equations of the normal which is bisected by the plane $x/a^3 + y/2b^3 + z/3c^3 = 0$ are

$$a\,(7x - 6a) = 2b\,(7y - 3b) = 3c\,(7z - 2c).$$

9. Prove that, if $f(x)$ has a derivative $f'(x)$ for all values of x lying between a and b, there must be at least one value ξ of x between a and b such that

$$f(b) - f(a) = (b - a) f'(\xi).$$

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Find ξ, if $f(x) = x^3 - 3x - 1$, $a = -\frac{11}{7}$, $b = \frac{13}{7}$, and illustrate by means of a graph.

10. Shew that the integral $\int \frac{dx}{x\sqrt{(3x^2 + 2x - 1)}}$ is rationalized by the assumption $x = (1 + y^2)/(3 - y^2)$, and hence or otherwise find its value.

Prove that, if m is a positive proper fraction, the value of the above integral when taken between the limits $\frac{1}{3}$ and $\frac{1}{2+m}$ is the same as when taken between the limits

$$\frac{1}{2+m} \text{ and } \frac{1}{m(2+m)}.$$

THURSDAY, *June* 1. 2—5.

1. From a point P of a rectangular hyperbola perpendiculars PM, PM' are drawn to the asymptotes. Shew that $PM \,.\, PM'$ is constant.

The chord QQ' of a rectangular hyperbola is parallel to the tangent at P, and QM, $Q'M'$, PN are drawn perpendicular to either asymptote. Shew that

$$QM \,.\, Q'M' = PN^2.$$

2. Prove that the reciprocal of a circle is a conic with a focus at the centre of reciprocation. Where must the centre of reciprocation be situated that the conic may be a rectangular hyperbola?

Prove by reciprocation or otherwise that a chord of a rectangular hyperbola which subtends a right angle at a focus touches a fixed parabola.

3. Find a rational integral function of x of the second degree which for the values 0, 1, 2 of x takes the respective values $\frac{1}{c}$, $\frac{1}{c+1}$, $\frac{1}{c+2}$; and prove that when $x = c + 2$ its value is $\frac{1}{c+1}$.

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4. Solve the equations $\frac{1}{x} - \frac{2}{y} = \frac{2}{x+1} - \frac{1}{y+1} = m$, when $m = \frac{10}{3}$. Examine the case $m = 1$.

Eliminate a, b, c from the equations

$$x = \frac{a}{b-c}, \quad y = \frac{b}{c-a}, \quad z = \frac{c}{a-b}.$$

5. In any triangle ABC shew that

$$a \cos A + a \cos (B - C) = 2h \sin A,$$

where h is the length of the perpendicular from A to BC.

Hence shew how to solve a triangle in which a, h, and $(B - C)$ are given: and when $a = 100$ cm., $h = 30$ cm., and $B - C = 20°$, shew that $A = 112° \, 44'$.

6. Find an expression for the cosine of the angle between two straight lines whose equations are given.

Prove that the point (x, y) given by the equations

$$x(1+\lambda^2) = a + \lambda(d-b) + \lambda^2 c, \quad y(1+\lambda^2) = d + \lambda(c-a) + \lambda^2 b,$$

lies on the circle which has for diameter the line joining (a, b) and (c, d).

7. Shew that the equation of the chord joining two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, whose eccentric angles are ϕ_1 and ϕ_2, is

$$\frac{x}{a} \cos \frac{\phi_1 + \phi_2}{2} + \frac{y}{b} \sin \frac{\phi_1 + \phi_2}{2} = \cos \frac{\phi_1 - \phi_2}{2}.$$

A point P on the ellipse, whose eccentric angle is α, is joined to the foci S and S', and PS, PS' meet the curve again in Q and Q'. Shew that the equation to QQ' is

$$\frac{x}{a} \cos \alpha \, (1 - e^2) + \frac{y}{b} \sin \alpha \, (1 + e^2) = e^2 - 1,$$

e being the eccentricity of the ellipse.

8. State and establish the usual rule for the differentiation of a quotient of two functions.

Prove that, if $y = (ax+b)/(cx+d)$, $2y'y''' = 3y''^2$; and that, if $a + d = 0$,

$$(y - x) y'' = 2y' (1 + y').$$

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9. Evaluate the integrals

$$\int \frac{2x^2+1}{x(x-1)^2}\,dx\,; \qquad \int 4\cos x \cos 2x \cos 3x\,dx\,; \qquad \int_0^{\pi} \sin^2 x \cos^6 x\,dx.$$

10. Solve completely the equations

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0\,; \qquad \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 5\sin t.$$

Find the solution of the latter equation which satisfies the conditions that $x = 2\dfrac{dx}{dt}$ when $t=0$ and also when $t=\alpha$.

FRIDAY, *June* 2. 9—12.

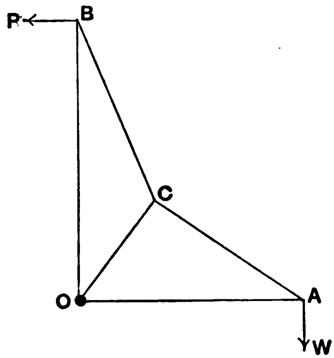
1. Find the resultant of two forces P, Q which act at a point and make an angle α with one another.

The resultant of two forces P, Q acting at a certain angle is X, and that of P, R acting at the same angle is also X. The resultant of Q, R, again acting at the same angle, is Y. Prove that

$$P = (X^2 + QR)^{\frac{1}{2}} = \frac{QR\,(Q+R)}{Q^2 + R^2 - Y^2}.$$

Prove also that, if $P+Q+R=0$, $Y=X$.

2. The figure shews a framework of light stiff rods freely jointed together and hinged at a fixed point O. OA is 12 feet, OB is 15 feet and C is one-third of the way up the diagonal of the rectangle on OA, OB. A weight W of 10 cwt., hanging



from A, is supported by a horizontal force P applied at B. Determine by a graphical construction or otherwise the stresses in the three rods which meet in C.

3. A straight line OP of length a is rotating uniformly with angular velocity ω about a fixed point O, and Q is the projection of P on a fixed straight line OA. If θ be the angle which OP makes with OA at any instant, shew that the displacement, the velocity, and the acceleration of Q, in the line OA, are respectively

$$a\cos\theta, \quad \omega a\cos\left(\theta+\frac{\pi}{2}\right), \quad \text{and} \quad \omega^2 a\cos(\theta+\pi).$$

If the displacement OQ be x, express the velocity and the acceleration in terms of x.

Shew that the position of Q for which the time of motion from rest to rest is divided in the ratio of 2 to 1 divides the length of the path, from rest to rest, in the ratio of 3 to 1.

4. A particle of mass m is describing an orbit in a plane under a force directed to a fixed point in the plane and at any instant r, θ are its polar coordinates with the fixed point on the pole, and P is the value of the force. Deduce from the principles of angular momentum and of work respectively that $r^2\dfrac{d\theta}{dt}=h$, a constant, and that $mv\dfrac{dv}{dr}+P=0$.

If the equation to the orbit be $\dfrac{l}{r}=1+e\cos\theta$, shew that v^2 is equal to $\dfrac{h^2}{l}\left(\dfrac{2}{r}-\dfrac{1-e^2}{l}\right)$, and that P is equal to $\dfrac{mh^2}{lr^2}$.

5. If k be the radius of gyration of a heavy body about a horizontal axis GG' which passes through its centre of gravity, find the period of a complete small oscillation of the body about an axis parallel to GG' and at a distance h from it.

Shew that all parallel axes for which the period of oscillation is the same lie on one or other of two cylinders, and that the length of the simple pendulum of the same period is equal to the sum of the radii of these cylinders.

6. Rays from a point P fall on a concave spherical mirror of radius r. Shew that after reflection the rays converge to a

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point Q on the diameter through P, such that if the distances of P and Q from the face of the mirror be u and v respectively then

$$\frac{1}{u}+\frac{1}{v}=\frac{2}{r}.$$

Two spherical mirrors, of the same radius r, one concave and the other convex, are placed on a common axis; their reflecting faces are opposite to one another and the distance between them is equal to $2r$. A small circle of diameter d is drawn on the convex mirror about its pole as centre. Shew that the diameters of the first, second, and third images of the circle are respectively $\frac{1}{3}$, $\frac{1}{11}$, and $\frac{1}{41}$ of d.

7. Express the focal length of a thin lens in terms of the curvatures of its faces and the refractive index of the glass.

An object at a distance 8 inches from the eye is viewed through a convex lens of focal length 3 inches, and the image is 12 inches from the eye. Shew that the linear magnification is 3 and the angular magnification 2.

8. State the conditions of equilibrium of a floating body.

A buoy is formed of a closed hollow circular cone of height a which is weighted so that it floats in stable equilibrium with its axis vertical and vertex downwards. The vertex is at a depth of h feet. Water is now admitted to the inside of the cone through a leak near the vertex, so that no air escapes, and the cone now floats in equilibrium with its vertex at a depth x, and with a depth y of water inside. If b be the height of the water barometer shew that

$$x^3-y^3=h^3 \text{ and } (x-y)(a^3-y^3)=by^3.$$

9. A charge of electricity e is at a point O, whose perpendicular distance from an infinite conducting plane at zero potential is h. Find the surface density of the induced electrification at a point on the plane whose distance from O is r, and shew that the total induced charge on any portion of the plane bounded by a closed curve S is proportional to the solid angle subtended by S at O.

10. Currents are passing in a network of linear conductors under electromotive forces acting in the conductors. Shew

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that for any closed circuit in the network the algebraic sum of the electromotive forces is equal to the algebraic sum of the products of the currents and resistances in the several members of the network which make up the circuit.

A and B are points in a circuit in which a current is passing and the resistances of the two parts of the circuit between A and B are r_1 and r_2 respectively. The difference of potential between A and B is V. A conductor of resistance R is added, joining A and B, the electromotive forces acting in the circuit remaining as before. Shew that the current in the added conductor is $V\Big/\left(R+\dfrac{r_1 r_2}{r_1+r_2}\right)$.

FRIDAY, *June* 2. 2—5.

1. Chords AB, CD of a circle intersect at O. Shew that $OA \,.\, OB = OC \,.\, OD$. Deduce a tangential property of the circle as a limiting form of this proposition when O moves towards, and ultimately lies on, the circumference of the circle.

ACB is a diameter of a circle, and PCP' a chord at right angles to it. A circle is inscribed in the figure bounded by AC, CP, and the smaller arc AP, and touches AB at D. Shew that $BD = BP$, and hence give a construction for this circle.

2. With centres at the vertices of a triangle ABC three circles are drawn so that the circles with B and C as centres intersect at D, D_1, those with C and A as centres intersect at E, E_1, and those with A and B as centres intersect at F, F_1. The three common chords meet in G, a point within all three circles. Shew that a tetrahedron can be formed with the triangle ABC as base and with BCD, CAE, and ABF as inclined faces and that the line joining G to the vertex is perpendicular to the plane ABC.

Give a construction for drawing these circles when it is required that the edge of the tetrahedron opposite AB shall be at right angles to and at a given distance from AB and that the height of the tetrahedron shall be equal to a given length.

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3. An approximate value of a root of a numerical algebraic equation being given, indicate, in general terms, how to obtain a closer approximation.

Shew that the equation $x + x^2 + x^3 + x^4 = 5$ is satisfied, approximately, by $x = 1\cdot0913$.

4. A circle of radius r touches internally a circle of radius R. Two circles, each of radius x, are drawn touching the outer circle internally and the inner circle externally. Shew that the length of the arc of the outer circle between its points of contact with these two circles is $2R\theta$, where

$$1 + \cos\theta = \frac{2xr}{(R-x)(R-r)}.$$

If $R = 2r = 4x$, how many circles of radius x can be placed in the area between the inner and outer circles, so that each touches the outer circle?

5. Shew that, if A, B, C be the angles of a triangle,

$$\cos 2A + \cos 2B + \cos 2C + 4\cos A\cos B\cos C + 1 = 0.$$

Conversely shew that, if the above relation is satisfied, one of the expressions $A \pm B \pm C$ will be an odd multiple of two right angles.

6. Shew that the equation of the polar of the point x', y' with respect to the parabola $y^2 = 4ax$ is $yy' = 2a(x + x')$.

The polar of the point P with respect to the parabola $y^2 = 4ax$ meets the curve in Q, R. Shew that, if P lies on the straight line $Ax + By + C = 0$, then the middle point of QR lies on the parabola $A(y^2 - 4ax) + 2a(Ax + By + C) = 0$.

7. Find the condition that two straight lines in space whose equations are given should intersect one another.

Prove that, if $\alpha + \beta + \gamma = 0$, the line $x + \alpha = y + \beta = z + \gamma$ intersects at right angles each of the lines

$$x = 0,\ y + z = 3\alpha; \quad y = 0,\ z + x = 3\beta; \quad z = 0,\ x + y = 3\gamma;$$

$$x + y + z = 3\lambda, \qquad \frac{\alpha x}{\lambda - \alpha} + \frac{\beta y}{\lambda - \beta} + \frac{\gamma z}{\lambda - \gamma} = 0.$$

8. Determine the radius of curvature of a curve, whose (x, y) equation is given, in the form $\rho = (1 + y'^2)^{3/2}/y''$.

A curve is determined by the property that the tangent to the curve at any point P meets a fixed straight line in T, so

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that the length PT is constant. Shew that the radius of curvature of the curve at P is $PT \cdot TN/PN$, where PN is the perpendicular from P to the fixed straight line.

9. Shew how to determine the real and finite asymptotes of an algebraic curve whose equation is given.

Determine the asymptotes of the curve

$$2x(y-3)^2 = 3y(x-1)^2,$$

and trace the curve.

10. Shew that the length of one quadrant of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is equal to $\frac{3a}{2}$, and find the length of one quadrant of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

SATURDAY, *June* 3. 9—12.

1. Prove that three forces in equilibrium are coplanar, and that their lines of action meet in a point or are parallel.

A square uniform lamina of side $2a$ rests in a vertical plane on two smooth pegs in a horizontal line. Shew that if the sum of the distances of the pegs from the lowest corner is equal to a there is equilibrium.

2. Explain the principle of Virtual Work as applied to a system having one degree of freedom.

An endless string of length $2\pi a + l$, $(6a < l < 8a)$, passes round three equal smooth cylinders of weight W and radius a, having their axes horizontal and parallel, and two of these rest on a horizontal plane, the third lying between them. Prove that the tension of the string is $\frac{1}{2} W(l-4a)/\{l(8a-l)\}^{\frac{1}{2}}$.

3. Attwood's machine consists of two particles, of mass m and m' respectively, attached to the ends of a light string which passes over a pulley. Find the acceleration of the system and the tension of the string, neglecting the inertia of the pulley and the friction of the axle.

Further, find the circumstances of the motion, and the tension of the string, if the machine be placed in a railway truck which is allowed to run freely down a slope of given inclination.

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4. A particle is projected in vacuo with a given velocity in a given direction: find the time of flight and the horizontal range.

A juggler keeps three equal balls, each of radius c, in the air by throwing them up successively with one hand from the same point at equal intervals of time. Shew that the distance through which the hand moves horizontally cannot be less than $6c$, if the velocity of the hand be supposed infinitely great, nor less than $5c$, if the hand be supposed to move with uniform horizontal velocity, and not to rest at the points of turning.

5. State and prove the principle of Conservation of Angular Momentum for a system of particles moving in a plane.

Two equal particles, each of mass m, connected by a fine string of length $2a$, are placed on a smooth horizontal plane, the string between them being straight. One of the particles is struck a blow B, in a direction at right angles to the string. Determine the magnitude and direction of the velocity of either particle at any instant: shew that each particle describes a cycloid: and draw a figure showing the character of the motion of the system in general.

6. A train is drawn by an engine which exerts a constant pull at all speeds. Find the equation of motion at any time, assuming that the total resistance to the motion of the train varies as the square of the velocity.

In the above case, the mass of the engine and train combined is 300 tons, the maximum speed on the level is 60 miles per hour, and the horse power then developed is 1500. Shew that, when climbing a slope of 1 in 100, the maximum speed is nearly 32 miles per hour.

7. Define, and determine the position of, the principal planes of a pair of thin lenses.

If the lenses are respectively convex and concave, each of numerical focal length f, and are at a distance of $\frac{1}{2}f$, give a diagram from which the positions of conjugate foci and the magnification may be found.

8. A plane lamina is immersed in a homogeneous heavy liquid and its plane intersects the surface of the liquid in the

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straight line AB. If the distance of the centre of gravity of the area of the lamina from AB be h, and the distance of its centre of pressure from AB be H, shew that $H-h=I/Ah$, where A is the area of the lamina and I is the moment of inertia of the area about an axis through its centre of gravity, parallel to AB.

A rectangular sluice-gate is placed in the vertical wall of a tank and is free to swing, so that its top edge moves outwards, on a horizontal axis which is distant a from the top edge and b from the bottom edge of the sluice. Shew (1) that however great the depth of water in the tank the sluice will not open unless a be greater than b, and (2) that the sluice will open before it is completely immersed if a be greater than $2b$.

9. Shew that the electrostatic capacity of an insulated spherical conductor is measured by its radius.

A sphere of radius a has its centre at a distance c from an infinite plane conductor at zero potential. Shew that its capacity is, approximately, $a+a^2/2c$, where a/c is regarded as small.

10. A small magnet swings freely in a horizontal plane at the centre of a circular coil which is set with its plane in the magnetic meridian. The coil has n turns of wire, closely wound, and the mean radius of the turns is r. When a current C is passing through the coil the magnet makes an angle ϕ with the magnetic meridian. Shew that $2\pi nC=rH\tan\phi$, where H is the horizontal component of the earth's magnetic field.

Two circular coils A and B are set in planes at right angles to one another; the centres of the coils are coincident and their common diameter is vertical, and the coil A is set with its plane in the magnetic meridian. Under the influence of a current in A a small horizontal magnet at the centre is in equilibrium at an angle of 60° with the meridian. A current is then passed through B such that the deflection of the magnet is reduced to 45°. If the intensities of the fields at the centre produced by these currents separately are respectively H_1 and H_2, find the value of the ratios $H_1 : H_2 : H$.

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SATURDAY, *June* 3. 2—5.

1. A sphere moves in contact with two fixed straight lines which intersect at an angle 2α. Shew that the centre of the sphere describes an ellipse of eccentricity equal to $\cos \alpha$.

If in any position the planes which touch the sphere at the points of contact make each an angle θ with the plane of the fixed straight lines, and if the line of intersection of these planes make an angle ϕ with the same fixed plane, shew that

$$\tan \phi = \tan \theta \sin \alpha.$$

2. Assuming the result

$$\log_e (1 + x) = x - x^2/2 + x^3/3 - \dots \qquad (0 < x < 1)$$

shew that $\frac{1}{2} \log \dfrac{n^2}{n^2 - 1} = \mu + \mu^3/3 + \mu^5/5 + \dots$

when $\mu = 1/(2n^2 - 1)$ and $n > 1$.

Shew that, when $0 < x < 1$

$$x - x^2/2 + x^3/3 - \dots = \frac{x}{1+x} + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1}{3}\left(\frac{x}{x+1}\right)^3 + \dots.$$

3. If D be the middle point of the side BC in a triangle ABC and the angle ADC be θ shew that

$$AB^2 - AC^2 = 2AD \,.\, BC \cos \theta.$$

$ABCD$ is a parallelogram and a straight line drawn parallel to the diagonal BD cuts the sides AB, BC, CD, DA, produced if necessary, in K, L, M, and N respectively. Prove that

$$\frac{KN}{LM} = \frac{AL^2 - AM^2}{CM^2 - CL^2}.$$

4. Determine the stationary values of the function $y \equiv e^{-ax} \sin bx$, where a and b are positive.

Shew that these values form a geometrical progression whose common ratio is $-e^{-\pi a/b}$; and illustrate your results by a figure.

5. Shew that the area swept out by the radius vector of a polar curve, between its angular positions θ_1 and θ_2, is equal to $\displaystyle\int_{\theta_1}^{\theta_2} \frac{r^2 d\theta}{2}$.

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Sketch one loop of the curve $r = a\cos 2\theta$, and find its area. Shew that the arc of a circle of radius $\frac{1}{2}a$, with its centre at the pole, divides the area of the loop into two parts which are in the ratio of $2\pi + 3\sqrt{3}$ to $4\pi - 3\sqrt{3}$.

6. A uniform beam is placed in a vertical plane, its lower end resting on a rough floor and its upper end against a rough vertical wall. The coefficient of friction is $\mu\,(< \sqrt{2} - 1)$ and the beam makes an angle of 45° with the floor. A string passes horizontally from the lower end of the beam, towards the wall. A tension T_1 in the string is required to prevent the beam from slipping down, and a tension T_2 is required to make it slip up. Shew that

$$\frac{T_1}{T_2} = \frac{1 - 3\mu + \mu^2 + \mu^3}{1 + 3\mu + \mu^2 - \mu^3}.$$

7. A heavy particle is attached by a light string to a fixed point O, and moves so as to describe a circle in a horizontal plane with uniform angular velocity ω. Shew that this plane is at a distance g/ω^2 below the point O.

An elastic string of unstretched length l is extended by an amount λ_1 when it supports a mass m at rest, and is extended by an amount λ_2 when it is rotating as above carrying a particle of the same mass m. Shew that $g\lambda_2/\lambda_1 = \omega^2(l + \lambda_2)$.

8. State Newton's Laws of Motion, and shew that when two masses impinge on one another the sum of their momenta in any direction is unaltered by the impact.

An inelastic pile weighing w tons is driven vertically by a hammer weighing W tons, the hammer having a free fall of h feet. Shew that the energy lost in each blow is $Wwh/(W + w)$ foot-tons.

9. A narrow pencil of rays from a point P falls normally on one face of a thick plate with parallel bounding surfaces. The material of the plate has an index of refraction μ. Shew that the pencil after refraction into and out of the plate diverges from a point distant $t\dfrac{\mu - 1}{\mu}$ from P, where t is the thickness of the plate.

Light falling normally on one face of an isosceles glass prism of refractive index μ is reflected at the base of the prism and emerges normally at the other face. Shew that the

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length of the path in the prism is constant, and that the apparent path of the light is shorter than its real path by $\dfrac{\mu-1}{\mu}$ times this length.

10. Three conductors A, B, C are such that B completely surrounds A, and that C is external to B. B is maintained at zero potential, and given charges are placed upon A and C. Shew that the potential of, and the distribution of the charge on, A are independent of the charge on C, and vice versa.

In the above case, is the potential of A independent of the presence of C in the field?

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THURSDAY, *May* 30. 9—12.

1. Prove that two triangles which have an angle of the one equal to an angle of the other and have the sides about the equal angles proportionals are similar.

P, Q, R are three points on a circle on the same side of a diameter AB. Prove that, if PQ, QR subtend equal angles at C, the foot of the perpendicular from Q on AB, then the triangles PCQ, QCR are similar.

2. Prove that, if TP, TQ are tangents at P and Q to a parabola whose focus is S, the triangles PST, TSQ are similar.

Give a geometrical construction for determining the focus of a parabola when two tangents and their points of contact are known.

3. Find a formula for the number of combinations of n different things r at a time.

Prove that a point can move from a corner of a chess-board to the opposite corner by sixteen displacements, each along a side of a square, in 12870 different ways.

In how many of these ways will it pass through the centre of the board?

4. Prove that

(1) $\log_e(x+1) - \log_e x$

$$= 2\left\{\frac{1}{2x+1} + \frac{1}{3}\frac{1}{(2x+1)^3} + \frac{1}{5}\frac{1}{(2x+1)^5} + \dots\right\},$$

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(2) $\log_e(x+2) - 2\log_e(x+1) + 2\log_e(x-1) - \log_e(x-2)$

$$= 2\left\{\frac{2}{x^3-3x} + \frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3 + \ldots\right\}.$$

Prove, by putting $x = 10$ in (2) or otherwise, that $\log_e 11 = 2{\cdot}397895$, having given $\log_e 2 = {\cdot}6931471$, $\log_e 3 = 1{\cdot}0986123$.

5. Explain what is meant by the circular measure of an angle, and prove that as θ diminishes to zero the limit of $\sin\theta/\theta$ is unity.

Prove that in the triangle whose sides are 31, 56, 64 one of the angles differs from a right angle by rather less than a minute of angle.

6. Observations on the position of a ship are made from a fixed station. At one instant the bearing of the ship is α_1 West of North. Ten minutes later the ship is due North and after a further interval of ten minutes its bearing is α_2 East of North. Assuming that the speed and direction of motion of the ship have not altered, shew that its course is θ East of North, where

$$\tan\theta = \frac{2\sin\alpha_1\sin\alpha_2}{\sin(\alpha_1-\alpha_2)}.$$

7. Evaluate the integrals

$$(1)\ \int (x^2-a^2)^{-\frac{3}{2}}\,dx, \quad (2)\ \int (x^3-1)^{-1}\,dx, \quad (3)\ \int (\sin x)^{-1}\,dx,$$

and prove that

$$\int_0^1 \frac{1}{(1+x)(2+x)\sqrt{x(1-x)}}\,dx = \pi\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right).$$

8. Find the area between the curve whose equation is $y^2(2a-x) = x^3$ and its asymptote.

Shew that the ordinate $x = a$ divides the area into two parts in the ratio of $3\pi - 8$ to $3\pi + 8$.

9. Find the cosine and the sine of the angle between two lines whose direction cosines are given.

O, P, P' are three points on the line $y = 0$, $z = c$, and O', Q, Q' are three points on the line $x = 0$, $z = -c$, O, O' being on the Z axis. Prove that, if PQ is perpendicular to $P'Q'$,

$$OP\,.\,OP' + O'Q\,.\,O'Q' + OO'^2 = 0,$$

OP &c. being taken positive in the directions of the axes.

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10. Shew that the middle points of chords of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ parallel to the line $x/l = y/m = z/n$ lie on a plane; and find its equation.

The middle point of the normal chord at a point P on the ellipsoid, not on a principal plane, lies on the plane $y = 0$; shew that P lies on one of the planes

$$x\,(a^2 - b^2)^{\frac{1}{2}}/a^3 = \pm\, z\,(b^2 - c^2)^{\frac{1}{2}}/c^3.$$

THURSDAY, *May* 30. 2—5.

1. Find the locus of a point from which the tangents drawn to two given circles are equal.

Pairs of points L, L'; M, M'; N, N' are taken on the three sides of a triangle so that L, L', M, M' lie on a circle, as also do M, M', N, N' and N, N', L, L'. Prove that the three circles are the same.

2. Circles are inscribed in a given segment of a circle so as to touch the curve and the bounding chord. Prove (1) that the chords of contact all pass through a fixed point; (2) that all the circles cut one fixed circle orthogonally; (3) that if the circles are inverted from one end of the chord of the segment, the centres of the inverse circles will be collinear.

3. Find the sum s_1 of n consecutive integers beginning with a, also the sum s_2 of their squares; and prove that $ns_2 - s_1^2$ is a quantity which is independent of a.

4. Solve the equations:

(i) $15x + 3y = 15x^2 + 3y^2 = 10$;

(ii) $\dfrac{x}{a^2} = \dfrac{y}{b^2} = \dfrac{z}{c^2} = xyz.$

Prove that the equation

$$\frac{a-b}{x} + \frac{a+b}{x-a^2} - \frac{a+b}{x-b^2} = 0 \text{ has equal roots.}$$

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5. The tangents of two angles are respectively

$$\frac{m - n\cos\alpha}{n\sin\alpha} \text{ and } \frac{n - m\cos\alpha}{m\sin\alpha}.$$

Find the tangent of the sum of the angles.

Find the tangents of the angles which satisfy the equation

$$(2m-1)\cos\theta + (m+2)\sin\theta = 2m+1.$$

6. Find the equation of a circle which touches the axis of y and passes through two given points on the axis of x, on the same side of the origin.

Two circles touch the axis of y and intersect in the points $(1, 0)$, $(2, -1)$. Find their radii, and shew that they both touch the line $y + 2 = 0$.

7. Find the equation of the polar of any point (x', y') with reference to the parabola $y^2 - 4ax = 0$.

Prove that the conic $2y^2 - x^2 = 1$ touches the polar of any point on itself with reference to the parabola $y^2 - x = 0$.

8. Find the first and second differential coefficients of y with regard to x when y is given implicitly in terms of x by the equation $f(x, y) = 0$.

Prove that, if $y^3 - 3ax^2 + x^3 = 0$,

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Shew that the curve given by the above equation is everywhere concave to the axis of x, and that there is a point of inflexion where $x = 3a$.

9. Prove that the equation of the tangent at any point $(4am^2, 8am^3)$ of the curve $x^3 = ay^2$ is $y = 3mx - 4am^3$, and that it meets the curve again at the point $(am^2, -am^3)$.

Shew that, if $9m^2 = 2$, the tangent is also a normal of the curve.

10. Solve the linear equation $\dfrac{dy}{dx} + 2y\tan x = \sin x$, and shew that, if $y = 0$ when $x = \frac{1}{3}\pi$, the maximum value of y is $\frac{1}{8}$.

Find a solution of the equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2e^{3x}$ which shall vanish when $x = 0$ and also when $x = \log_e 2$.

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FRIDAY, *May* 31. 9—12.

1. Fig. 1 illustrates a framework of light rods freely jointed together at their extremities. The framework is supported by vertical reactions at A and B and carries a light platform suspended from C and D. A load W is placed at the centre of the platform. Sketch the force diagram for the left half of the framework. Shew also that, if W is shifted to a position on the platform distant $\frac{1}{3}a$ from EC, the stress in the member DE is reduced to zero.

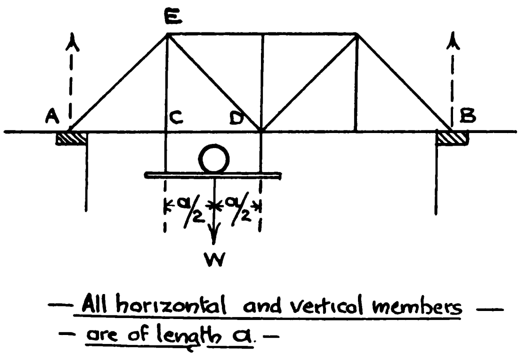

Fig. 1.

2. A given system of coplanar forces acting on a rigid body is to be balanced by two forces, one acting along a given line and the other acting through a given point. Describe some geometrical or analytical process for completely determining these two forces.

A six-wheeled railway truck is urged by a horizontal force of 3 tons against a log of wood fastened transversely to the railway lines as shewn in fig. 2. If the wheels C are just lifted off the rails, determine the loads carried on the wheels A and B.

[The wheels are supposed to rotate in their bearings without appreciable friction.]

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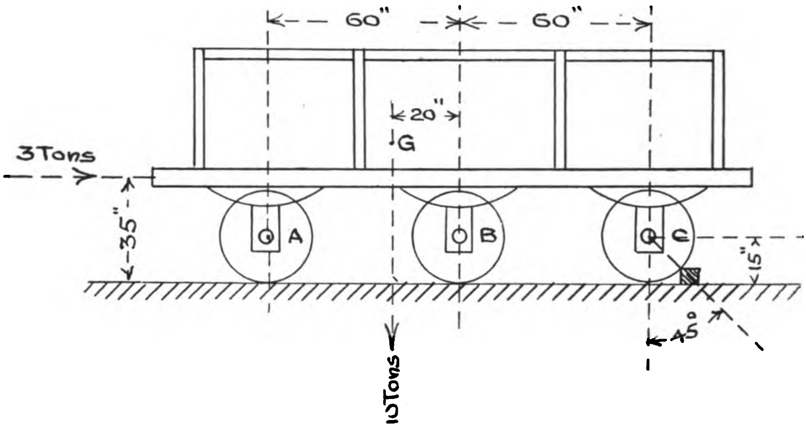

Fig. 2.

3. A particle is describing a circle of radius r at a uniform speed v; shew that it possesses an acceleration v^2/r directed towards the centre of the circle.

The light rods AB, CD shewn in fig. 3 are constrained to move freely in the vertical plane through the axis of the disc to which their lower ends are hinged. The upper ends carry

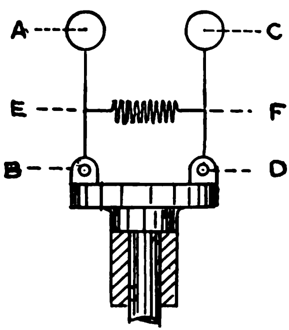

Fig. 3.

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spheres of mass M lbs. E and F, the middle points of AB and CD, are connected by a spring whose strength is such that a pull of P lbs. extends it one inch. In the unstretched condition of the spring the rods are vertical and $EF = AB = CD$. Neglecting the effect of gravity, and treating the spheres as particles, shew that if the disc is rotating about the vertical axis with uniform angular velocity ω and the steady inclination of the rods to the vertical is θ, then

$$\operatorname{cosec} \theta = 12Pg/M\omega^2 - 2.$$

4. If a constant force P acting upon a body mass M for a time t gives it a velocity v, establish the relation $Pt = Mv$.

A tug-boat A of 400 tons is attached to a ship B of 4000 tons by means of a cable fixed to A and passing round a capstan on B, the arrangement being such that a pull of 5 tons causes the cable to slip round the capstan. The propulsive force on the tug-boat exerted by its propeller is 3 tons and it starts off with the cable slack. At the instant when the cable becomes taut the velocity of A is 5 feet per sec., and B is stationary. Prove that the common velocity of A and B when the cable has just ceased slipping is 1 foot per sec. Determine also the time during which slip occurs, and the length of cable which passes over the capstan. [The resistance of the water may be neglected.]

5. A rigid body under the influence of gravity is performing small oscillations about a fixed horizontal axis. Investigate an expression for the period of a small oscillation.

A flywheel is balanced upon a horizontal knife-edge parallel to the axis of the wheel and inside the rim at a distance of 30 inches from the axis of the wheel. The wheel oscillates with a period of 2·6 seconds; find the radius of gyration of the wheel about its axis.

6. Find the depth of the centre of pressure (1) on a triangular lamina immersed vertically in fluid with one side in the surface, (2) on a quadrilateral which has one side in the surface and one side parallel to the surface.

A quadrilateral is immersed vertically having two sides of lengths $2a$, a parallel to the surface at depths h, $2h$ respectively. Shew that the depth of the centre of pressure is $3h/2$.

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7. The surface of a canal is disturbed by ripples moving across from bank to bank, the steepest part of the surface of these ripples having an inclination $\tan^{-1} 0{\cdot}1$. The reflection of a lamp situated on one bank is viewed from a point on the other side exactly opposite the light. The lamp and the point of observation are respectively 50 and 25 feet above water level, and the horizontal distance between them is 100 feet. Find over what length the reflection of the light on the surface of the water extends.

8. Define the terms : electric intensity, electric potential, tube of force.

Prove that the total normal induction over a closed surface is 4π times the charge inside the surface, and deduce that the electric intensity at any point inside a closed conductor which has no internal charge is zero.

9. Explain what is meant by the capacity of a condenser. In the case of a parallel plate condenser how is the capacity affected by variations in the size of the plates and their distance apart?

A battery is sending current through an external resistance R. The terminals of the battery are connected to a condenser and the charge communicated is found to be Q_1. On repeating the experiment with an infinite external resistance the charge given to the condenser is Q_2. Prove that the internal battery resistance is

$$R(Q_2 - Q_1)/Q_1.$$

10. ABC is the trolley wire of an electric tramway 2000 yards long. B is its middle point and O is the position of the generating station. Current is supplied along three feeders, OA, OB, OC, the resistances of these being 0·35, 0·25, 0·10 ohms respectively. The resistance of the trolley wire is 0·80 ohms. A car situated at a point 500 yards distant from A is taking 100 amps. from the trolley wire. Find the current in each of the three feeders.

If the resistance of the return circuit is negligible, and the potential difference at the generating station is 500 volts, determine the potential difference at the car.

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FRIDAY, *May* 31. 2—5.

1. Define the cross-ratio of a range of four points, and shew that according to the order of the points it may have any one of six values which may be written $\sec^2\alpha$, $\cos^2\alpha$, $\sin^2\alpha$, $\operatorname{cosec}^2\alpha$, $-\tan^2\alpha$, $-\cot^2\alpha$.

A, B, C, D are fixed points on a straight line, and $\{BCAP\}=\{BCPD\}$. Prove that P has one of two positions which are harmonically conjugate with regard to B, C.

2. Prove that, if $\dfrac{x}{l+a}=\dfrac{y}{m+b}=\dfrac{x-y}{a+b}$, each fraction is equal to $\dfrac{y}{l-b}-\dfrac{x}{m-a}$.

Divide $x^3-x+\frac{2}{3}$ by $x+1{\cdot}24$, and shew that the remainder is less than ·0001.

3. Find an expression for the radius of the circle described about a triangle in terms of the lengths of its sides.

A point O is situated on a circle of radius R, and with centre O another circle of radius $\frac{3}{2}R$ is described. Inside the smaller crescent-shaped area intercepted between these circles a circle of radius $\frac{1}{8}R$ is placed. Shew that, if the small circle moves in contact with the original circle of radius R, the length of arc described by its centre in moving from one extreme position to the other is $\frac{7}{12}\pi R$.

4. A, B, C are three positions at the same level and in the same straight line such that $AB=500$ feet, and $BC=1000$ feet. Viewed simultaneously from A, B, C the elevations of an aeroplane are respectively $26^\circ\,34'$, $30^\circ\,0'$, and $33^\circ\,41'$. Prove that the height of the aeroplane at the instant of observation is nearly 1100 feet.

5. Find the equation of a straight line which passes through the point of intersection of two given lines and is perpendicular to a third given line in their plane.

Prove that the point $(-1,\,4)$ is the orthocentre of the triangle which is formed by the lines whose equations are

$$x-y+1=0,\qquad x-2y+4=0,\qquad 9x-3y+1=0.$$

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6. Prove that the line $x\cos\alpha + y\sin\alpha = p$ will touch the ellipse $x^2/a^2 + y^2/b^2 = 1$ if $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.

The perpendiculars on a tangent TT' of an ellipse from the extremities of the minor axis are BT, $B'T'$. Prove that $TT'^2 : 4BT \,.\, B'T' :: BB'^2 : AA'^2$; also that the circles whose centres are B, B' and radii BT, $B'T'$ will cut at a constant angle.

7. Prove that, if

$$S_1 \equiv x^2 + y^2 + z^2 + a_1x + b_1y + c_1z + d_1,$$

and

$$S_2 \equiv x^2 + y^2 + z^2 + a_2x + b_2y + c_2z + d_2,$$

then $S_1 - \lambda S_2 = 0$ represents a family of spheres, each of which passes through the intersection of $S_1 = 0$ and $S_2 = 0$.

Shew that the general equation of all spheres through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ is

$$x^2 + y^2 + z^2 - ax - by - cz - \lambda\,(x/a + y/b + z/c - 1) = 0.$$

Find the value of λ so that this sphere may cut orthogonally the sphere whose equation is

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0.$$

8. Apply Maclaurin's theorem to obtain the first three significant terms in the expansion of $\tan x$ in ascending powers of x.

Deduce a similar expansion for $\tan^2 x$.

9. What is meant by the radius of curvature of a curve? Prove that it is given by the expression

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \Big/ \frac{d^2y}{dx^2}.$$

In the case of a cycloid, where $y = a\,(1 - \cos\theta)$ and $x = a\,(\theta + \sin\theta)$, shew that at a point defined by the angle θ the radius of curvature has the value $4a\cos\frac{1}{2}\theta$.

10. Prove the rule for integration by parts and apply it to find

$$(1)\quad \int e^{ax}\cos bx\,dx, \qquad (2)\quad \int x^3\sin ax\,dx.$$

Find a formula of reduction for $\int (x^2 + a^2)^n\,dx$, and evaluate this integral when $n = \frac{3}{2}$.

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SATURDAY, *June* 1. 9—12.

[*In question* 4, g *in foot-second units has been assumed to be* 32.]

1. A chain of light rods having its ends attached to fixed points carries loads at each of the intermediate joints. If the total load is known and the form of the chain is given, shew how to determine the load carried by each joint.

Fig. 1 illustrates a simple form of suspension bridge in which the roadway consists of two beams PQ, RQ freely hinged together at Q and supported by three vertical tie rods, hung from the chain of light rods $ABCDE$. Prove that for the given form of the chain the tensions in the tie rods must be the same, and shew that this tension amounts to $3\frac{3}{4}$ tons when the beam PQ is loaded in the given manner. [The weight of the beams is to be neglected.]

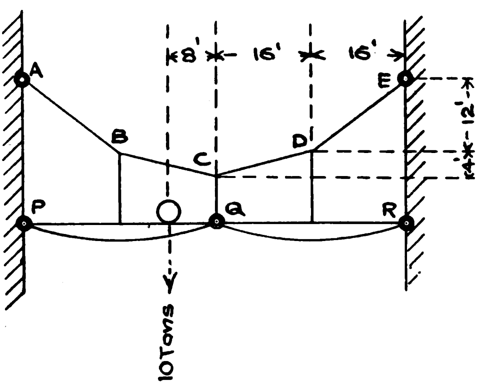

Fig. 1.

2. AB and CD are two levers freely hinged to the fixed points B and D. The upper end of CD and the point E on AB are hinged to the link EF, as shewn in fig. 2. A vertical

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force W applied at F is balanced by a horizontal force applied at A. Use the principle of work to prove that

$$\frac{P}{W}=\frac{EF}{EC}\cdot\frac{BE}{BA}.$$

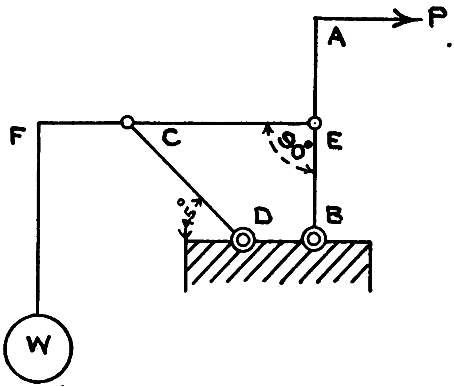

Fig. 2.

3. In reference to a machine what is meant by (*a*) Mechanical Advantage, (*b*) Efficiency?

Fig. 3 on p. 66 illustrates a method of raising a block of stone by means of light wedges C and D inserted between fixed inclines A and B and the underside of the block. The coefficients of friction between the wedges and the block and inclines are $\tan\phi_1$ and $\tan\phi_2$ respectively, and the wedges are driven forward by equal horizontally applied forces; prove that the efficiency of this arrangement for raising the load is

$$\frac{\tan\theta}{\tan\phi_1+\tan(\theta+\phi_2)},$$

where θ is the angle of the wedges.

4. A particle is projected with given velocity V in a given direction under gravity. Shew that it moves in a

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parabola whose directrix is at a height $V^2/2g$ above the point of projection.

A projectile is shot from the ground and grazes the tops of two chimneys at heights of 36 and 64 feet respectively which stand 96 feet horizontally apart. Shew that the initial velocity must be at least 80 feet per second, and that, if the time from chimney to chimney is 5 seconds, the initial velocity is 100 feet per second.

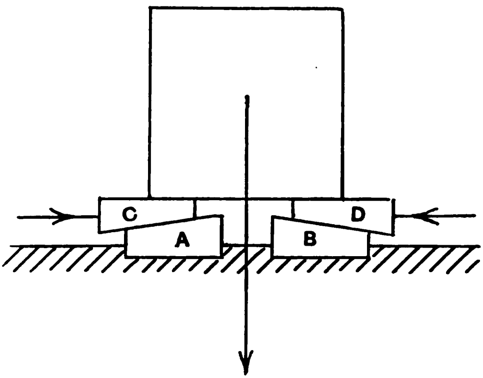

Fig. 3.

5. An inelastic pile of mass m is driven into the ground by a weight of mass M which falls on it. Shew that at each blow a proportion $m/(M+m)$ of the kinetic energy of the moving weight is wasted.

When the clear fall of the weight in each stroke is 10 feet, and $M=7m$, it is found that 15 strokes are necessary to drive the pile 7 inches. Shew that if the weight is doubled 7 strokes only will be necessary.

6. A train can be accelerated by a force of 55 lbs. per ton weight, and when steam is shut off can be braked by a force of 440 lbs. per ton weight. Find the least time between stopping stations 3850 feet apart, the greatest velocity of the train, and the horse-power per ton weight of train necessary for the engine.

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7. A ray of light passes through a prism of angle i and refractive index μ in a principal plane. Find the least value of the deviation produced.

Prove that, if the prism is right angled, and the least and greatest values of the deviation are respectively α and β,

$$\cos\alpha = \mu \cos\beta, \qquad \sin\alpha = \sin^2\beta.$$

Find $\cos\alpha$ and $\cos\beta$ when $\mu = 1\cdot 4$.

8. Find the positions of the principal planes and principal foci of a hemispherical lens whose refractive index is $\frac{3}{2}$.

Two such lenses are placed on the same axis with their curved surfaces in contact. Prove that the focal length of the combination is equal to the radius.

9. Prove that the electrostatic energy of a system of charged conductors is half the sum of the products of the charges and the corresponding potentials.

Three concentric spherical conductors, radii a, b, c ($a < b < c$), have charges E_1, E_2, E_3 respectively. Shew that, if the inner conductor is now connected with the earth, the potentials of the conductors are diminished by amounts inversely proportional to their radii, and that the loss of energy is

$$\frac{a}{2}\left(\frac{E_1}{a} + \frac{E_2}{b} + \frac{E_3}{c}\right)^2.$$

10. Find the potential at any point due to a spherical conductor at zero potential in the presence of an external point charge.

Write down the corresponding potential when the sphere is insulated and has a given charge.

Shew that in this case the distribution on the sphere will be partly negative if

$$E < \frac{ec^2(3f - c)}{f(f-c)^2},$$

where E is the charge on the sphere, c its radius, e the point charge, which is positive, and f its distance from the centre of the sphere.

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SATURDAY, *June* 1. 2—5.

1. Prove that the rectangle contained by the segments of chords of a circle which pass through a fixed point is constant.

A point P on a circle is joined to a point D on a diameter AB. PK is the perpendicular from P on AB, and AM the perpendicular from A on PD. Prove that

$$DP \,.\, MP = BD \,.\, KA.$$

2. Prove that, if a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to the plane containing them.

OA, OB, OC are three straight lines mutually at right angles. AD is the perpendicular from A on BC and OE is the perpendicular from O on AD. Prove that OE is perpendicular to the plane ABC.

3. Prove that between every pair of consecutive real roots of the integral algebraic equation $f(x) = 0$ there is an odd number of real roots of the derived equation $f'(x) = 0$; and hence deduce the condition that the equation $f(x) = 0$ may have equal roots.

Shew that the equation $3x^3 + 5x^2 + 3x + k = 0$ has two imaginary roots, whatever be the value of k.

4. Sketch the form of the curve $y = ae^{-\kappa x} \sin px$ for positive values of x, and prove that it touches the curves $y = \pm ae^{-\kappa x}$ each at an infinite series of points.

Shew that the successive maximum values of y form a series in geometrical progression, and that these maxima lie on the curve $y = a \sin \alpha \epsilon^{-\kappa x}$, where α is the angle between 0 and π whose tangent has the value p/κ.

5. Define a definite integral as the sum of a series, and shew that it can be represented by an area.

Prove without integration that

$$\int_0^\pi f(\sin x)\, dx = 2 \int_0^{\frac{\pi}{2}} f(\sin x)\, dx,$$

and that
$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx > \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{x} dx.$$

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6. Explain and prove the formula $v^2 = u^2 + 2fs$ for a particle moving in a straight line with uniform acceleration.

A particle is projected in a straight line with a certain velocity and a constant acceleration. One second later another particle is projected after it with half the velocity and double the acceleration. When it overtakes the first the velocities are 22 and 31 feet per second respectively. Prove that the distance traversed is 48 feet.

7. Explain what is meant by Simple Harmonic Motion, and obtain an expression for the acceleration at any point of the motion in terms of the displacement and the number of oscillations per second.

A horizontal shelf is given a horizontal simple harmonic motion. The amplitude of the motion is a feet, and n complete oscillations are performed per second. A particle of mass m lbs. is placed on the shelf at the instant when it is at the extremity of its motion. Prove that, if μ is less than $4\pi^2n^2a/g$, slipping between the particle and shelf will occur for a period t, where t is found from the equation

$$\frac{\sin 2\pi nt}{2\pi nt} = \frac{\mu g}{4\pi^2 n^2 a}.$$

Shew that, if for a particular case this value of t is $1/6n$, the distance through which the particle moves relatively to the shelf in this time is

$$\frac{a}{2}\left(1 - \frac{\pi\sqrt{3}}{6}\right).$$

8. A rigid body of mass M is in motion parallel to a plane so that its angular velocity is ω and the velocity of its centre of gravity is v. Shew that its kinetic energy is $\frac{1}{2}Mv^2 + \frac{1}{2}M\kappa^2\omega^2$, where κ is its radius of gyration about an axis through its centre of gravity perpendicular to the plane of motion.

AB and CD are two rods of lengths $2a$ and b respectively, the mass of each rod being m per unit length. The rods are rigidly joined together at right angles at C, the middle point of AB, and the system is free to rotate in a vertical plane about D. If the system is held with AB vertical and then let go, calculate the angular velocity when AB is horizontal.

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9. Prove that the resultant thrust on a solid immersed in fluid is equal to the weight of the fluid displaced.

A vertical tube which can move freely in the direction of its length is closed at the top and allowed to sink into water. The tube is of thin material, is 40 cm. in length, 1·85 sq. cm. in section, and weighs 50 grammes, and the water is so deep that the tube does not touch the bottom. Find approximately the lengths of the parts into which the tube is divided by the water levels inside and outside the tube, assuming that the height of the water barometer is 10 metres and that a cubic centimetre of water weighs one gramme.

10. Prove that, if F, F' are the principal foci of a convex lens whose focal length is f, and P, P' are any two conjugate foci, then $FP \, . \, F'P' = -f^2$, and find the least distance between a small object and its real image.

A thin convex lens, focal length 2 ft., is placed in front of, and parallel to, a plane mirror, so that a real image of the lens is formed by reflection at the mirror and refraction through the lens. Shew that the least distance of the image from the mirror is $(3 + 2\sqrt{2})$ feet.

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