# DE RHAM COHOMOLOGY

LILY LI

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## Introduction

This is an expository piece for the final project of Math 215A (Algebraic Topology) in Fall 2020. This writeup discusses the de Rham cohomology, its basic properties, and the de Rham theorem. For the purposes of the assignment, the worked example is the calculation for the cohomology groups of $\mathbb{S}^n$ (2.5), and the carefully-proven theorems are the Poincare Lemma (1.3), the Mayer-Vietoris Theorem (2.3), and the de Rham theorem (3.5). The proofs roughly follow Lee's Smooth Manifolds text [Lee03] and Rudin's Principles of Mathematical Analysis [Rud76].

The de Rham cohomology is a cohomology theory defined for smooth manifolds using differential $k$-forms and the exterior derivative to produce cohomology groups. Intuitively, it measures the extent to which closed forms fail to be exact due to obstructive global structure such as holes in the smooth manifold. Despite the fundamentally different point-of-view it takes in contrast to singular cohomology, de Rham cohomology is isomorphic to singular cohomology with $\mathbb{R}$ coefficients for all smooth manifolds. In particular, it behaves as expected on small open balls and is compatible with gluing open sets, as described more rigorously in future sections.

We will follow this structure of reasoning: In the first section, we establish background on differential forms. We define de Rham cohomology in section 2 and present basic facts about the de Rham cohomology groups in section 3. In section 4, we develop and prove the Mayer-Vietoris theorem analogous to singular cohomology, making rigorous the intuition about gluing compatibility. Using the Mayer-Vietoris sequence, we compute the cohomology of spheres in section 5, and section 6 proves the isomorphism between singular and de Rham cohomology.

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## 1. Background on Differential Forms

First we define differential forms and state basic results that are necessary in discussing the de Rham cohomology. For fear of going down rabbit holes, this recap will be brief at the risk of being terse. Then, we prove a version of the Poincare lemma in detail for use in future sections.

**Differential $k$-forms.** Given a smooth manifold $M$, let $T_pM$ denote the tangent space of $M$ at $p$, and let $T_p^*M$ be the cotangent space dual to $T_pM$. Then, let $\Lambda^k T^*M$ be the bundle of covariant alternating

$k$-tensors on $M$ defined by

$$\Lambda^k T^* M = \bigsqcup_{p \in M} \Lambda^k (T_p^* M).$$

A section of this vector bundle is called a **differential $k$-form**. In other words, a differential $k$-form is a continuous choice of alternating $k$-tensor at each point of $M$. We denote the vector space of smooth $k$-forms by $\Omega^k(M)$.

In a smooth chart, we can write a differential $k$-form locally as

$$\sum_I \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_I \omega_I dx^I$$

where each $I$ is a multi-index consisting of ordered indices $(i_1, ..., i_k)$, and $\omega_I$ are continuous functions defined on the coordinate patch. This amounts to saying that $dx^I$ forms a basis for the cotangent bundle at each point $p \in M$. Unpacking this definition more, a 0-form is a smooth function on $M$, a 1-form is a linear combination of the basis tangent covectors $dx^i$ at each point of $M$ and can be written as $\sum_i \omega_i dx^i$ on a coordinate patch.

A basic operation on the $k$-forms is the wedge product, which is defined point-wise and inherited from wedge products of covectors: For $\omega, \eta$ a $k$-form and $l$-form, respectively:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\omega_p \otimes \eta_p)$$

**Pullback of a form.** As with any covariant tensor field, we can pull back a differential $k$-form along a smooth function. Given a smooth map $F : M \to N$ and a differential form $\omega$ on $N$, the pullback along $F$ gives a differential form on $M$. We can describe it by its action on tangent vectors:

$$(F^*\omega)_p(v_1, ..., v_k) = \omega_{F(p)}(dF_p(v_1), ..., dF_p(v_k))$$

where $dF_p$ is the differential at $p$. It turns out that the pullback is linear and behaves well with wedge products: $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$. For the sake of brevity, we will not reproduce the derivations here.

**Exterior derivative.** Exterior derivatives generalize the differential of a function to $k$-forms, and it replaces the boundary map in singular cohomology as the map that gives cohomology groups.

On patches of $\mathbb{R}^n$, define the **exterior derivative** of a differential $k$-form $\omega$ as

$$d\omega = d(\sum_I \omega_I dx^I) = \sum_I d\omega_I \wedge dx^I$$

where $d\omega_I$ is the differential of the function $\omega_I$:

$$d\omega_I = \sum_i \frac{\partial \omega_I}{\partial x^i} dx^i$$

The exterior derivatives sends a $k$-form to a $(k+1)$-form. In other words, for $U$ an open subset of $\mathbb{R}^n$, $d : \Omega^k(U) \to \Omega^{k+1}(U)$. The exterior derivative satisfies the following properties stated without proof.

**Proposition 1.1.** *Exterior derivatives satisfy the following properties:*

*(1) The exterior derivative operator $d$ is linear over $\mathbb{R}$.*
*(2) $d \circ d = 0$.*
*(3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.*
*(4) $d$ commutes with pullbacks: $F^*(d\omega) = d(F^*\omega)$*
*(5) For a smooth manifold $M$, there exists unique operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying the conditions given in (1)-(4).*

To define the de Rham cohomology, we also need closed and exact forms.

**Definition 1.1** (Closed and Exact Forms)**.** A smooth differential $k$-form $\omega \in \Omega^k(M)$ is **closed** if $d\omega = 0$, and it is **exact** if there is a smooth $(k-1)$-form $\eta \in \Omega^{k-1}(M)$ such that $d\eta = \omega$.

1.1. **Poincare Lemma.** The Poincare Lemma is arguably one of the most fundamental building blocks of understanding de Rham cohomology. Intuitively, it describes closed and exact forms on convex subsets of Euclidean space, which describes the local behavior of these forms on coordinate patches of smooth manifolds. It establishes local exactness of closed forms.

In this section, we will prove the Poincare lemma for convex subsets of $\mathbb{R}^n$, which allows us to prove the de Rham Theorem later on. This will be done in much less generality than most textbooks, but it will be just enough for our purposes.

**Proposition 1.2.** *Let $E \in \mathbb{R}^n$ be a convex open subset, $f$ a continuously differentiable function on $E$, and $1 \leq p \leq n$ an integer such that for all $p < j \leq n$ and $x \in E$,*

$$(D_j F)(x) = 0.$$

*Then, there exits some continuously differentiable function $F$ on $E$ such that*

$$(D_p F)(x) = f(x), \ (D_j F)(x) = 0$$

*In other words, we're able to find $F$ with prescribed derivative $f(x)$ with respect to the $p$-th coordinate while keeping the derivative trivial at all coordinates above $p$.*

*Proof.* We will follow a proof from [Rud76]. First, split the coordinates for $x$ and write $x = (x', x_p, x'')$, where $x' = (x_1, \cdots, x_{p-1})$, and $x'' = (x_{p+1}, \cdots, x_n)$. Let $V \subset \mathbb{R}^p$ be projection of $E$ onto the first $p$ coordinates, or more rigorously, the set of coordinates $(x', x_p') \in \mathbb{R}^p$ where $(x', x_p, x'') \in E$ for some $x''$. Since $E$ is convex, and the partial derivatives with respect to all coordinates above $p$ are 0, we conclude that $f$ does not depend on those coordinates and rewrite $f$ only as a function of the first $p$ coordinates. Indeed,

$$f(x) = g(x', x_p).$$

If $p = 1$, $V \in \mathbb{R}$ is a line segment (possibly unbounded). Inspired by the Fundamental Theorem of Calculus, we choose some $c \in V$ and define

$$F(x) = \int_c^{x_1} g(t)dt.$$

We can check that indeed, $D_{x_1} F(x) = g(x_1) = f(x)$.

If $p > 1$, let $U \subset \mathbb{R}^{p-1}$ be the set $V$ projected onto the first $p-1$ coordinates, or more rigorously the set of $x' \in \mathbb{R}^{p-1}$ such that $(x', x_p) \in V$ for some $x_p$. Then $U$ is a convex open subset of $\mathbb{R}^{p-1}$, and we can construct a smooth function $h$ on $U$ such that $(x', h(x')) \in V$ for each $x' \in U$. In other words, $V$ contains the graph of $h(U)$. Define

$$F(x) = \int_{h(x')}^{x_p} g(x', t)dt.$$

Again, we can check that indeed, $D_p F(x) = f(x)$, and $F$ does not depend on any $x_j$ beyond $p$, so $D_j F(x) = 0$ for all $p < j \leq n$. This concludes the construction and the proof.

□

The same lemma holds for smooth functions with the same proof. Now we're ready to give a proof of the Poincare Lemma, or at least a limited version. Most proofs rely on the existence of a homotopy operator by a construction using Lie derivatives and interior multiplication, but we don't need the most general form of the lemma for our narrow purposes of proving the de Rham theorem. For a nice change from pursuing maximal generality, we will only prove the Poincare Lemma for convex subsets of $\mathbb{R}^n$ following Rudin. Unless otherwise noted, we're always speaking of smooth $k$-forms.

**Theorem 1.3** (Poincare Lemma)**.** *On a convex subset $E \subset \mathbb{R}^n$, every closed $k$-form is exact. In other words, if $k \geq 1$, and $\omega \in \Omega^k(E)$ is a smooth $k$-form where $d\omega = 0$, then there exists a $(k-1)$-form $\lambda \in \Omega^{k-1}(E)$ such that $\omega = d\lambda$.*

*Proof.* We will roughly follow a proof from [Rud76].

By our previous assertions, every differential $k$-form can be written in a standard form as

$$\omega = \sum_I f_I dx^I.$$

Let $Y_p$ denote the set of $k$-forms on $E$ whose standard presentation does not involve any $dx^j$ where $j > p$. We will proceed by induction on $p$.

In the base case, $p = 1$, we write $\omega = f(x)dx^1$ for some function $f(x)$, and $d\omega = 0$. By the previous proposition, we can find a smooth $F(x)$ such that $D_1F = f, D_j = 0$ otherwise, so $dF(x) = (D_1F)(x)dx^1 = f(x)dx^1 = \omega$. Since $F \in \Omega^0(E)$m we've shown the base case.

Let $p > 1$. As the induction hypothesis, we assume that closed $k$-forms in $Y_{p-1}$ are exact. Take a closed $k$-form $\omega \in Y_p$, and we can write

$$d\omega = d(\sum_I f_I dx^I) = \sum_I \sum_{j=1}^{n} (D_j f_I)(x) dx^j \wedge dx^I = 0.$$

Consider the $(k+1)$-indices that result from this expression. Given a fixed $j > p$, two different $I$ indices $I_1 \neq I_2$ will give distinct $(k+1)$-indices $(I_1, j) \neq (I_2, j)$. We only are concerned with $j > p$ because we will see that all the smalle indices are covered by our induction hypothesis. This way, no cancellation will occur in the expression for $d\omega$ concerning the higher indices $j > p$, and we conclude that the coefficient for each term will be 0:

$$(D_j f_I)(x) = 0 \ x \in E, p < j \leq n,$$

which looks cooked up for applying the prervious lemma.

Gathering the problematic terms, terms in $dF$ that contain $dx^p$, we can rewrite $d\omega$ as follows:

$$d\omega = \alpha + \sum_{I'} f_I(x) dx^{I'} \wedge dx^p$$

where $\alpha \in Y_{p-1}$, and $I'$ is an increasing $k-1$ index containing indices in $\{1, ..., p-1\}$, and $I = (I', p)$. By the previous proposition, we obtain a collection of functions $F_I$ on $E$ with the following property

$$D_p F_I = f_I, \ \ D_j F_I = 0 \ \ (p < j \leq n).$$

Using this collection of $F_I$, define a new $(k-1)$-form $\beta$ as follows:

$$\beta = \sum_{I'} F_I(x) dx^{I'}$$

Then use the $d\beta$ to define the following:

$$\gamma = \omega - (-1)^{k-1} d\beta,$$

where we use the $(-1)^{k-1}$ coefficient to exchange the position of $x^j$ to the back from the differential. Writing $d\beta$ in standard summation notation, we obtain:

$$\gamma = \omega - \sum_{I'} \sum_{j=1}^{p} (D_j F_I)(x) dx^{I'} \wedge dx^j$$

Collecting terms with $dx^p$ again and performing cancellation, we obtain

$$\gamma = \alpha - \sum_{I'} \sum_{j=1}^{p-1} (D_j F_I)(x) dx^{I'} \wedge dx^j$$

Since $\alpha \in Y_{p-1}$, and the summation portion has no $dx^p$ terms, it's clear that $\gamma \in Y_{p-1}$. Since $d\omega = d(d\beta) = 0$, $d\gamma = 0$. By the induction hypothesis, we have $\gamma = d\eta$ forr some $\eta \in \Omega^{k-1}(E)$. Now our construction is complete, as we can define one last $(k-1)$-form:

$$\lambda = \eta + (-1)^{k-1}\beta$$
$$d\lambda = d\eta + (-1)^{k-1} d\beta = \gamma + (-1)^{k-1} d\beta = \omega.$$

$\square$

Although this proof seems like proof-by-magic in classic Rudin fashion, it does give us the Poincare Lemma on convex subsets of Euclidean space, which will be used later on.

## 2. De Rham Cohomology

On a smooth manifold $M$, every point is part of a coordinate patch diffeomorphic to an open ball in $\mathbb{R}^n$, which is convex. Then, the Poincare Lemma gives us the intuition that that on a local neighborhood, every closed form is exact. However, this is not globally true since while the manifold locally looks like $\mathbb{R}^n$, it may have certain global topological obstructions, and we capture this by the following definition of de Rham cohomology groups.

**Definition 2.1.** Let $M$ be a smooth manifold, and let $p \in \mathbb{Z}$ be nonnegative. The exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is a linear map with $d^2 = 0$, so its kernel and image are linear subspaces. Define

$$\mathcal{Z}^p(M) = \ker(d : \Omega^p(M) \to \Omega^{p+1}(M))$$
$$\mathcal{B}^p(M) = \mathrm{Im}(d : \Omega^{p-1}(M) \to \Omega^p(M)).$$

And define the **pth de Rham cohomology group** as

$$H^p_{dR}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}.$$

Intuitively, this measures the extent to which closed forms fail to be exact, as it's quotienting closed $p$-forms by exact $p$-forms. An immediate consequence of this definition is that $H^p_{dR}(M) = 0$ for $p > \dim(M)$ since $\Omega^p(M) = 0$. Additionally, $H^p_{dR}(M) = 0$ if and only if every closed $p$-form is exact on $M$.

### 2.1. Basic Properties.

In this subsection, we state basic properties of de Rham cohomology, such as functoriality, homotopy invariance, etc. Most properties will be stated without a complete proof.

**Proposition 2.1.** *Given a smooth map $F : M \to N$ between smooth manifolds, the pullback $F^* : \Omega^p(N) \to \Omega^p(M)$ takes closed forms to closed forms and exact forms to exact ones. Thus, it descends to a linear map between the cohomology groups.*

*Proof.* If $\omega$ is closed, then $d(F^*\omega) = F^*(d\omega) = 0$. So $F^*\omega$ is also closed. If $\omega = d\eta$ is exact, then $F^*\omega = F^*(d\eta) = d(F^*\eta)$, so $F^*\omega$ is exact.

For well-definedness, take two cohomologous $p$-forms, $\omega, \omega'$ such that $\omega - \omega' = d\eta$ for some $\eta$. Then

$$[F^*\omega] = [F^*\omega' + F^*(d\eta)] = [F^*\omega' + d(F^*\eta)] = [F^*\omega'].$$

Note all of these depend on the property that the exterior derivative commutes with pullbacks. □

**Proposition 2.2.** *Given this first basic property, we will state the following ones without proof, sincluding functorirality, cohomology of* 0*-manifolds, cohomology of disjoint unions, and homotopy invariance.*

(1) Functoriality: For nonnegative integer $p \in \mathbb{Z}$, de Rham cohomology assigning manifolds to their $p$-th cohomology group and maps between smooth manifolds to their induced cohomology maps defines a contravariant functor from the category of topological spaces to the category of real vector spaces.
(2) If $\{M_j\}$ is a countable collection of smooth manifolds and $M = \bigsqcup_k M_j$. For each $p \in \mathbb{Z}$, the inclusion maps $M_j \hookrightarrow M$ induce an isomorphism from $H^p_{dR}(M)$ to the direct product $\prod_j H^p_{dR}(M_j)$.
(3) If $M$ is a 0-manifold, which is realized as a disjoint union of 1-point spaces, then $H^0_{dR}$ is a direct product of 1-dimensional vector spaces of constant functions, one copy for each component of $M$. All other cohomology groups are trivial.
(4) If $M \simeq N$ are homotopy equivalent manifolds, then for every $p \in \mathbb{Z}$, any smooth homotopy equivalence $F : M \to N$ induces an isomorphism on the cohomology groups $H^p_{dR}(M) \cong H^p_{dR}(N)$.

At this point, using singular cohomology as a prototype, we see that de Rham cohomology behaves as expected for each of the basic scenarios such as disjoint unions, 1-point spaces, etc. There's already a hint that this new cohomology theory resembles singular cohomology.

### 2.2. Mayer Vietoris Theorem.

In this section we will prove a basic expected property from this new cohomology theory, that de Rham cohomology is compatible with gluing pieces of the manifold together, just like singular cohomology.

Let $M$ be a smooth manifold with or without boundary, and $U, V$ open subsets of $M$ whose union is $M$. There are natural inclusions from $U, V$ into $M$ and $U \cap V$ into $U$ and $V$. We can name the inclusions write them in the following diagram.

$$\begin{tikzcd}
 & U \arrow[dr, hook, "k"] & \\
U \cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & M \\
 & V \arrow[ur, hook, "l"'] &
\end{tikzcd}$$

Since these inclusions are smooth, they induce a diagram of pullbacks on differential forms. These pullbacks are restrictions.

$$\begin{tikzcd}
 & \Omega^p(U) \arrow[dr, "i^*"] & \\
\Omega^p(M) \arrow[ur, "k^*"] \arrow[dr, "l^*"'] & & \Omega^p(U \cap V) \\
 & \Omega^p(V) \arrow[ur, "j^*"'] &
\end{tikzcd}$$

In particular, we can write the following sequence of maps:

$$0 \to \Omega^p(M) \xrightarrow{k^* \oplus l^*} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{i^* - j^*} \Omega^p(U \cap V) \to 0 \tag{1}$$

**Theorem 2.3** (Mayer-Vietoris Theorem)**.** *Let $M$ be a smooth manifold with or without boundary, and $U, V$ open subsets of $M$ whose union is $M$. For each $p$, there is a linear map $\delta : H^p_{dR}(U \cap V) \to H^{p+1}_{dR}$ such that the following sequence is exact.*

$$\cdots \xrightarrow{\delta} H^p_{dR}(M) \xrightarrow{k^* \oplus l^*} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{i^* - j^*} H^p_{dR}(U \cap V) \xrightarrow{\delta} H^{p+1}_{dR}(M) \xrightarrow{k^* \oplus l^*} \cdots$$

The proof of the theorem depends on a lemma in homological algebra, the Zigzag Lemma. Recall that a sequence of $R$-modules $A^i$ and $R$-linear maps form a cochain complex if the sequence

$$\cdots \to A^{p-1} \xrightarrow{d} A^p \xrightarrow{d} A^{p+1} \to \cdots$$

satisfies $d^2 = 0$. In particular, we can write a cochain complex of differential forms

$$0 \to \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

A cochain map from $A^* \to B^*$ is a collection of linear maps $F : A^p \to B^p$ that commute with $d$, and a sequence of complexes

$$0 \to A^* \xrightarrow{F} B^* \xrightarrow{G} C^* \to 0$$

is exact if each constituent sequence

$$0 \to A^p \xrightarrow{F} B^p \xrightarrow{G} C^p \to 0$$

is exact for $p \in \mathbb{Z}$.

**Theorem 2.4** (The Zigzag Lemma)**.** *Given a short exact sequences of complexes described above, there are linear maps $\delta : H^p(C^*) \to H^{p+1}(A^*)$ called connecting homomorphisms so that the following is a long exact sequence:*

$$\cdots \xrightarrow{\delta} H^p(A^*) \xrightarrow{F^*} H^p(B^*) \xrightarrow{G^*} H^p(C^*) \xrightarrow{\delta} H^{p+1}(A^*) \xrightarrow{F^*} H^{p+1}(B^*) \xrightarrow{G^*} \cdots$$

*Proof.* The proof is a wild diagram chase. Checking the well-definedness of $\delta$ is especially laborious. I spent a lot of time chasing and found the process not to be instructive. It's omitted. □

We are now equipped to prove the Mayer-Vietoris theorem.

*Proof.* We need to show that sequence (1) is a short exact sequence for each $p$. We know that the exterior derivative commutes with the pullback maps, so exactness of (1) would give us short exact sequence of cochain maps, which would induce the desired Mayer-Vietoris Sequence by the Zigzag Lemma. In other words, we'd reduced the problem to proving that (1) is exact for every $p$ at each group in the sequence.

To show exactness at $\Omega^p(M)$, we need to show that $k^* \oplus l^*$ is injective. Suppose $(k^* \oplus l^*)\sigma = 0$ for some $\sigma \in \Omega^p(M)$. Since $k, l$ are both natural inclusions into $M$, the induced maps $k^*, l^*$ are restrictions of $\sigma$ to

$U, V$. $(k^* \oplus l^*)\sigma = (\sigma|_U, \sigma|_V) = (0,0)$. Since $U \cup V = M$, this means $\sigma = 0$. Therefore, $k^* \oplus l^*$ has trivial kernel and is injective.

To show exactness at $\Omega^p(U) \oplus \Omega^p(V)$, we need to show that $\mathrm{Im}(k^* \oplus l^*) = \ker(i^* - j^*)$. We can see that

$$(i^* - j^*)(k^* \oplus l^*)(\sigma) = (i^* - j^*)(\sigma|_U, \sigma|_V) = \sigma|_{U\cap V} - \sigma|_{U \cap V} = 0$$

This means that $\mathrm{Im}(k^* \oplus l^*) \subset \ker(i^* - j^*)$. As for the other direction, suppose $(\eta, \eta') \in \Omega^p(U) \oplus \Omega^p(V)$ and $(\eta, \eta') \in \ker(i^* - j^*)$, then $\eta|_{U\cap V} = \eta'|_{U\cap V}$. Since $\eta$ and $\eta'$ agrees on the intersection, there's a globally defined smooth $p$-form $\sigma$ on $M$ given by $\eta$ on $U$ and $\eta'$ on $V$. Then, it's clear that $(\eta, \eta') = (k^* \oplus l^*)\sigma$, which gives $\ker(i^* - j^*) \subset \mathrm{Im}(k^* \oplus l^*)$. Thus, $\ker(i^* - j^*) \subset \mathrm{Im}(k^* \oplus l^*)$.

To show exactness at $\Omega^p(U \cap V)$, we need to show that $i^* - j^*$ is surjective. Take $\omega \in \Omega^p(U \cap V)$, we need to find $(\eta, \eta') \in \Omega^p(U) \oplus \Omega^p(V)$ such that $\omega = (i^* - j^*)(\eta, \eta') = i^*\eta - j^*\eta' = \eta|_{U\cap V} - \eta'|_{U\cap V}$. Let $\{\phi, \psi\}$ be a partition of unity subordinate to the covering $\{U, V\}$ of $M$. Define $\eta \in \Omega^p(U)$ to be $\psi\omega$ on $U \cap V$ and 0 elsewhere, and define $\eta' \in \Omega^p(V)$ to be $-\phi\omega$ on $U \cap V$ and 0 elsewhere. These are smooth by construction, and we have the following:

$$(i^* - j^*)(\eta, \eta') = \eta|_{U\cap V} - \eta'|_{U\cap V} = \psi\omega - (-\phi\omega) = (\phi + \psi)(\omega) = \omega$$

Therefore, we've shown exactness at $\Omega^p(U \cap V)$ by construction the desired $(\eta, \eta')$. This concludes the proof. □

We call this sequence the Mayer-Vietoris Sequence.

2.3. **Examples of Calculating de Rham Cohomology.** In this section , we will do an elementary calculation using the Mayer-Vietoris theorem. We will compute the de Rham cohomology groups of spheres.

**Theorem 2.5** (Cohomology of Spheres)**.** *For $n \geq 1$, the de Rham cohomology of spheres $\mathbb{S}^n$ is*

$$H^p_{dR}(\mathbb{S}^n) = \begin{cases} \mathbb{R}, & \text{if } p = 0, n \\ 0, & \text{otherwise} \end{cases}$$

*Proof.* If $p = 0$, there are no $(-1)$-forms to speak of, which means $\mathcal{B}^0(\mathbb{S}^n) = 0$. Given some closed 0-form $f$, which is any smooth functions on $M$ with the property $df = 0$. Since $\mathbb{S}^n$ is a connected space, $f$ must be constant, so $\mathcal{Z}^0(\mathbb{S}^n)$ is the space of constant functions on $\mathbb{S}^n$. So $H^0_{dR}(\mathbb{S}^n) = \mathcal{Z}^0(\mathbb{S}^n) = \mathbb{R}$.

The rest of the calculation is done by induction on $n$. The base case is $n = 1$. Let $N, S$ be the north and south poles of the circle, respectively. Then let $U = \mathbb{S}^1 - \{N\}$ and $V = \mathbb{S}^1 - \{S\}$. Since $U \cup V = \mathbb{S}^1$, and $U \cap V$ is homotopic to a disjoint union of two points. We can write the Mayer-Vietoris sequence as follows:

$$0 \to H^0_{dR}(\mathbb{S}^1) \to H^0_{dR}(U) \oplus H^0_{dR}(V) \to H^0_{dR}(U \cap V) \to H^1_{dR}(\mathbb{S}^1) \to H^1_{dR}(U) \oplus H^1_{dR}(V) \to H^1_{dR}(U \cap V) \to 0$$

From here, we can substitute in the terms that are known from the basic properties section:

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1_{dR}(\mathbb{S}^1) \to 0 \to 0 \to 0$$

From here, by the Euler characteristic of exact sequences, we know that the alternating sum of the dimensions in this sequence must be 0, so $\dim(H^1_{dR}(\mathbb{S}^1)) = 1 + 2 - 2 = 1$. Thus $H^1_{dR}(\mathbb{S}^1) = \mathbb{R}$.

Suppose the statement holds for $\mathbb{S}^k$, and we seek to prove it for $\mathbb{S}^{k+1}$. Again, let $N, S$ be the north and south poles of the sphere, respectively. Then let $U = \mathbb{S}^{k+1} - \{N\}$ and $V = \mathbb{S}^{k+1} - \{S\}$. As before, both $U$ and $V$ are contractible. The only difference is that now $U \cap V \simeq \mathbb{S}^k$. For each $0 < i < k$, we have a portion of the Mayer-Vietoris sequence

$$0 \to H^i_{dR}(\mathbb{S}^{k+1}) \to 0,$$

which means that $H^i_{dR}(\mathbb{S}^{k+1}) = 0$ for all $0 < i < k$.

As for $i = k$, we can again write the Mayer-Vietoris sequence as follows:

$$0 \to H^k_{dR}(\mathbb{S}^{k+1}) \to H^k_{dR}(U) \oplus H^k_{dR}(V) \to H^k_{dR}(\mathbb{S}^k) \to$$
$$\to H^{k+1}_{dR}(\mathbb{S}^{k+1}) \to H^{k+1}_{dR}(U) \oplus H^{k+1}_{dR}(V) \to H^{k+1}_{dR}(\mathbb{S}^k) \to 0$$

Substituting in the information we have thus far assuming the inductive hypothesis, we obtain:

$$0 \to \mathbb{R} \to H^{k+1}_{dR}(\mathbb{S}^{k+1}) \to 0$$

Thus, we obtain that $H^{k+1}_{dR}(\mathbb{S}^{k+1}) = \mathbb{R}$, as desired.

□

## 3. The De Rham Theorem

In this section, we seek to establish an equivalence between singular cohomology and de Rham cohomology. The crucial connection is that we can integrate differential forms over singular chains and produce real numbers. In particular, given some singular $p$-complex $\sigma$ in $M$ and a $p$-form $\omega$, we would like to integrate the pullback of $\omega$ by $\sigma$ over the simplex $\Delta_p$. In doing this, we can associate a $p$-form with a function taking singular chains to real numbers. We will use Stoke's theorem for chains to obtain the machinery to define a suitable homomorphism, and the naturality of this homomorphism will equip us to prove the isomorphism between singular and de Rham cohomology in full.

There's an immediate issue. We note that the discussion of differential forms gives us a problematic constraint: we can only pull back differential forms along smooth functions, but for singular homology, our simplices as functions embedding a simplex are only guaranteed to be continuous. Therefore we need to work with smooth singular cohomology, which imposes the further condition that all the maps $\Delta_p \to M$ can be smoothly extended to a neighborhood. This necessitates a discussion of smooth singular homology.

Define instead a **smooth $p$-simplex** as a smooth map $\Delta_p \to M$ extendable to a smooth neighborhood of each point. If $p = 0$, the map is always smooth. This way, the smooth $p$-complexes generate the chain group in degree $p$, and we define the **smooth singular homology group** analogously. The expected statement is that smooth singular homology is isomorphic to singular homology by an induced map of the inclusion of smooth singular homology into singular homology, and this turns out to be true. However, the proof is involved, and we will refrain from proving it here. However, going forward, we will treat them as isomorphic. In reality, we may adopt the abuse of speaking of them interchangeably, as the isomorphism gives us the point of view that singular homology can be computed with smooth simplices.

**Proposition 3.1** (Properties of Singular Cohomology)**.** *We assume the following properties of singular cohomology.*

*(1)* One-point space*: The space of a single point has trivial singular cohomology except at $p = 0$, where it is 1-dimensional.*
*(2)* Disjoint unions*: If $\{M_j\}$ is any collection of topological spaces, then the disjoint union $M = \bigsqcup_j M_j$ has cohomology isomorphic to the direct product $\prod_j H^p(M_j;\mathbb{R})$ induced by the inclusion maps $M_j \hookrightarrow M$.*
*(3)* Homotopy invariance*: homotopy equivalent spaces have isomorphic singular cohomology.*

As we seek to establish a connection between (smooth) singular cohomology and de Rham cohomology, we first establish two statements that will aid this discussion.

**Theorem 3.2** (Stokes's Theorem for Chains)**.** *If $c$ is a smooth $p$-chain on a smooth manifold $M$, then*

$$\int_{\partial c} \omega = \int_c d\omega.$$

Another theorem we will use is the Five Lemma.

**Proposition 3.3** (The Five Lemma)**.** *if there's a commutative diagram as below*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

*where the horizontal rows are exact, and $f_i$ is an isomorphism for each $i = 1, 2, 4, 5$, then $f_3$ is also an isomorphism.*

Equipped with Stoke's Theorem, we relate the cohomology theories by defining a natural linear map relating $H^p_{dR}(M)$ and $H^p(M)$. First, define integration of smooth $p$-forms over smooth $p$-simplices: the **integral of $\omega$ over $\sigma$** is

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^*\omega$$

Using this, we can define the **de Rham homomorphism** $\ell : H^p_{dR}(M) \to H^p(Ml\mathbb{R})$. Take $[\omega] \in H^p_{dR}(M), [c] \in H_p(M)$, define $\ell[\omega]$ by its action $H_p(M) \to \mathbb{R}$ as

$$\ell[\omega][c] = \int_{c'} \omega$$

where $c'$ is any representative of the homology class $[c]$. It's only a formality to check that this homomorphism is well-defined. For one, the choice of $c'$ does not matter because any two representatives of the same homology class differ by a boundary $\partial b$:

$$\int_{c'} \omega - \int_{c''} \omega = \int_{\partial b} \omega = \int_b d\omega = 0.$$

A basic result following this definition is that the de Rham homomorphism satisfies naturality properties that can be exploited in the de Rham Theorem. They will be stated without proof.

**Proposition 3.4** (Naturality of the de Rham homomorphism)**.** *Let $M$ be a asmooth manifold, and let $p \in \mathbb{Z}$ a nonnegative integer. The de Rham homomorphism $\ell : H^p_{dR}(M) \to H^p(M;\mathbb{R})$ satisfies the following naturality properties:*

*(1) If $F : M \to N$ is a smooth map, then the following diagram commutes.*

$$\begin{array}{ccc} H^p_{dR}(N) & \xrightarrow{F^*} & H^p_{dR}(M) \\ \downarrow \ell & & \downarrow \ell \\ H^p(N;\mathbb{R}) & \xrightarrow{F^*} & H^p(M;\mathbb{R}) \end{array}$$

*(2) Let $U, V$ be open subsets of $M$ whose union is $M$, as in the set up of the Mayer-Vietoris sequence, and let $\delta, \partial^*$ be connecting homomorphisms in the Mayer-Vietoris sequence of de Rham and singular cohomology, respectively. The following diagram commutes.*

$$\begin{array}{ccc} H^{p-1}_{dR}(U \cap V) & \xrightarrow{\delta} & H^p_{dR}(M) \\ \downarrow \ell & & \downarrow \ell \\ H^{p-1}(U \cap V;\mathbb{R}) & \xrightarrow{\partial^*} & H^p(M;\mathbb{R}) \end{array}$$

Before proceeding to prove the de Rham theorem, the main theorem of this project, we establish the final pieces of vocabulary for the proof.

**Definition 3.1.** A smooth manifold $M$ is a **de Rham manifold** if the de Rham homomorphism is an isomorphism for each nonnegative integer $p$. If $M$ has an open cover $\{U_i\}$ such that the subsets $U_i$ and their finite intersections $U_{i_1} \cap \cdot \cap U_{i_k}$ are all de Rham manifolds, we call the cover a **de Rham cover**. If an open de Rham cover $\{U_i\}$ also forms a basis for the topology on $M$, it is a **de Rham basis**.

**Theorem 3.5** (De Rham Theorem)**.** *Let $M$ be a smooth manifold, and let $p \in \mathbb{Z}$ be nonnegative. The de Rham homomorphism $\ell : H^p_{dR}(M)^p(M;\mathbb{R})$ is an isomorphism.*

*Proof.* We will complete the proof first for special cases such as disjoint unions and convex open subsets of $\mathbb{R}^n$, and then we will generalize to all smooth manifolds.

*If $\{M_j\}$ is a countable collection of de Rham manifolds, then their disjoint union is de Rham.* By 3.1 and 2.2, the inclusions $M_j \hookrightarrow \bigsqcup_j M_j$ induce an isomorphism between the cohomologies of the disjiont union and the direct product of the cohomology groups of the manifolds $M_j$. Naturality (3.4) gives that $\ell$ commutes with these isomorphisms.

*Every convex open set of $\mathbb{R}^n$ is de Rham.* Take a convex open subset $U \subset \mathbb{R}^n$. By the Poincare Lemma (1.3), $H^p_{dR}(U) = 0$ for all $p > 0$. Since $U$ is homotopy equivalent to a single point space, by homotopy invariance of singular cohomology (3.1), $H^p(U) = 0$ for all $p > 0$.

On the other hand, if $p = 0$, then $H^0_{dR}(U)$ is one-dimensional consisting of constant functions, and $H^0(U;\mathbb{R}) = \mathrm{Hom}(H_0(U), \mathbb{R})$ is also a one-dimensional space, since $H_0(U)$ is generated by a single 0-simplex, and thus $\mathrm{Hom}(H_0(U), \mathbb{R})$ is determined by the image of that generator. Let $\sigma : \Delta_0 \to M$ be a singular $0-$simplex, and let $f$ be the constant function with value 1, then $\ell[f][\sigma] = \int_{\Delta_0} \sigma^* f = (f \circ \sigma)(0) = 1$. This means the homomorphism is non-zero, so it's an isomorphism. Thus we've proven the statement for any nonnegative $p \in \mathbb{Z}$.

*If $M$ has a finite de Rham cover, then $M$ is de Rham* Let $M = U_1 \cup U_2 \cup \cdots \cup U_k$ be the de Rham cover, where each $U_i$ is an open subset of $M$ that is de Rham, and any finite intersections are de Rham by definition. We will proceed by induction on $k$. The base case where $k = 1$ is trivially true. Suppose the statement is true for $k$, that any space with a $k$-fold de Rham cover is de Rham. Then, given a manifold $M$ with a $(k+1)$-fold covering $M = U_1 \cup U_2 \cup \cdots \cup U_{k+1}$, we can cover $M$ by two sets, $U = U_1 \cup U_2 \cup \cdots \cup U_k$, and $V = U_{k+1}$. Then we write the following diagram following the discussion of the Mayer-Vietoris sequence for both de Rham and singular cohomology, as well as the naturality of the de Rham homomorphism. When we write $H^*(X)$, we mean $H^*(X; \mathbb{R})$ since the diagram goes out of bo unds if we include it.

$$\begin{tikzcd}
H^{p-1}_{dR}(U) \oplus H^{p-1}_{dR}(V) \arrow[r] \arrow[d] & H^{p-1}_{dR}(U \cap V) \arrow[r] \arrow[d] & H^p_{dR}(M) \arrow[r] \arrow[d] & H^p_{dR}(U) \oplus H^p_{dR}(V) \arrow[r] \arrow[d] & H^p_{dR}(U \cap V) \arrow[d] \\
H^{p-1}(U) \oplus H^{p-1}(V) \arrow[r] & H^{p-1}(U \cap V) \arrow[r] & H^p(M) \arrow[r] & H^p(U) \oplus H^p(V) \arrow[r] & H^p(U \cap V)
\end{tikzcd}$$

In the diagram, all horizontal maps are exact, and the vertical maps are de Rham homomorphisms. The diagram commutes because of the naturality of the de Rham homomorphism (3.4). By the induction hypothesis, $U$ is de Rham because it's covered by $k$ open de Rham subsets, and $V$ is de Rham. Their intersection $U \cap V$ is de Rham because it's covered by $k$ de Rham subsets $\{U_1 \cap V, U_2 \cap V, \cdots, U_k \cap V\}$. Therefore, all the vertical maps already commute except for the middle one. Therefore, by the five lemma, the middle map is also an isomorphism. Thus, a manifold with $(k+1)$-fold de Rham cover is de Rham.

*If $M$ has a de Rham basis, then $M$ is de Rham.* Let $\{U_\alpha\}$ be a de Rham basis for $M$. We will proceed by constructing a de Rham cover of M using this basis and use step 3. At this point in the proof, we understand how the de Rham homomorphism behaves on disjoint unions and finite unions, so the following construction works to divide $M$ with these two properties in mind.

Let $f : M \to \mathbb{R}$ be an exhaustion function , which is a continuous function with the property that every sublevel set $f^{-1}([-\infty, c])$ is compact. The existence of such a function is deferred to the ending section. For each integer $m$, define the following subsets of $M$:

$$A_m = \{q \in M : m \leq f(q) \leq m + 1\}$$

$$A'_m = \{q \in M : m - \frac{1}{2} < f(q) < m + \frac{3}{2}\}$$

For each point $p \in A_m$, there's some $U_p \in \{U_\alpha\}$ in the basis containing $p$ that's also contained within $A'_m$. The collection of all such sets $\{U_p | p \in A_m\}$ is an open cover of $A_m$. By construction, $A_m$ is compact, so the cover reduces to a finite subcover $B_m$. Since $B_m$ is covered by finitely many basis sets and therefore has a de Rham cover, $B_m$ is de Rham. $B_m$ covers all the points in $A_m$, so the collection of all $B_m$ over $m \in Z$ covers all of $M$.

By construction, $B_m \subset A'_m$, so $B_m$ can only have nontrivial intersection with $B_{m+1}$ and $B_{m-1}$. Their intersections $B_m \cap B_{m+1}$ are de Rham because they are covered by $U_\alpha \cap U_\beta$, intersections of the basis sets that are used to construct $B_m$ in the first place. Therefore, we can define two disjoint sets whose union is $M$.

$$U = \bigcup_{m \text{ odd}} B_m, \ V = \bigcup_{m \text{ even}} B_m$$

By our previous discussion, both $U$ and $V$ are disjoint unions of de Rham manifolds, which is de Rham by step 1. Their intersection $U \cap V$ is also de Rham because it's the disjoint union of $B_m \cap B_{m+1}$ for each $m$. Therefore, $M = U \cup V$ is de Rham.

*Every open subset of $\mathbb{R}^n$ is de Rham.* Take an open subset $U \subset \mathbb{R}^n$. Since $\mathbb{R}^n$ has a basis consisting of Euclidean balls, so does $U$. Each ball and their finite intersections are convex and therefore de Rham, so the basis is a de Rham basis. Therefore, $U$ has a de Rham basis and is de Rham by step 4.

*Every smooth manifold is de Rham.* Every smooth manifold has a basis of smooth coordinate domains, each of which is diffeomorphic to an open subset of $\mathbb{R}^n$. Their finite intersections are also diffeomorphic to open subsets of $\mathbb{R}^n$. Therefore, by step 4, every smooth manifold is de Rham.

$\square$

## References

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