WEIGHTS AND VERMA MODULES

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Contents

Introduction

This is an expository piece for the final project of Math 261A (Lie Groups) in Spring 2021.

The motivating eventual goal for these concepts is the full classification of all finite dimensional semisimple representations of Lie algebras, but we won't get quite that far in this writeup. To this end, we will discuss one way of obtaining finite dimensional irreducible representations of complex semisimple Lie Algebras: through the lens of Verma modules. There are other ways to do so, of course, but Verma modules have the advantage of being relatively easy to construct and prove the existence of, although the dimension will be problematic since all Verma modules are infinite dimensional.

We will follow this structure of reasoning: In the first section, we establish background on Cartan subalgebras and root systems. We define weights in section 2 and prove some results regarding highest weights, commonly called the Highest Weight Theorems. Then in section 3, we give two constructions of Verma modules. In section 4, we use Verma modules to obtain irreducible finite-dimensional representations of $\mathfrak{g}$.

I'm grateful for Prof. Artamonov for his instruction and empathy in this virtual semester.

1. Background on Semisimple Lie Algebras

We will establish some necessary vocabulary and notation for the future in this section. The definitions and exposition will be brief and as concise as possible at the risk of being terse.

Definition 1.1 (Cartan Subalgebra)**.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a comlex subspace of $\mathfrak{g}$ with the following properties.

(1) For all $H_1, H_2 \in \mathfrak{g}$, $[H_1, H_2] = 0$.
(2) For all $X \in \mathfrak{g}$, if $[H, X] = 0$ for all $H \in \mathfrak{g}$, then $X \in \mathfrak{h}$.
(3) For all $H \in \mathfrak{h}$, ad_H is adiagonalizable.

This definition works even if $\mathfrak{g}$ is not semisimple, but in the case where $\mathfrak{g}$ is semisimple, we are guaranteed the existence of a Cartan subalgebra $\mathfrak{h}$. For a more explicit demonstration of this fact, see section 6.3 in [Hal03]. From this point forward, suppose we've fixed a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$.

Definition 1.2 (Root System)**.** A root of $\mathfrak{g}$ (with respect to the fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$) is a nonzero linear functional α on $\mathfrak{h}$ (in other words, $\alpha \in \mathfrak{h}^*$) such that there exists an element $X \in \mathfrak{g}$ with

$$[H, X] = \alpha(H)X$$

for all $H \in \mathfrak{h}$. The set of all roots is called the root system and denoted R.

Next, we need to introduce some sense of order. This will be used in our discussion of weights in a later section.

Definition 1.3. Let E be a finite dimensional inner product space and $R \subset E$ a root system. Then a base for R is a subset of roots $\Delta \subset R$ where Δ is a basis for E as a vector space nd such that for each $\alpha \in \mathbb{R}$, we have

$$\alpha = \sum n_i \alpha_i$$

where n_i are integers that are either all positive or all negative.

If n_i are positive, we call α a positive root. If n_i are negative, we call α a negative root. Elements of Δ are the positive simple roots.

The existence of a base is a crucial prerequisite to constructing Verma modules, so we present the following theorem without proof. For a proof, see section 8.3 of [Hal03].

Theorem 1.1. *For any root system, a base exists.*

Since $\mathfrak{g}$ can be regarded as an finite dimensional complex inner product space, we now have that the root system for $\mathfrak{g}$ can be written as the union of the positive and negative roots $R^+ \cup R^-$. This gives a partial order on $\mathfrak{h}^*$ by declaring that given two roots λ_1, λ_2, $\lambda_1 < \lambda_2$ if

$$\lambda_2 - \lambda_1 = \sum_{\alpha_i \in \Delta} n_i \alpha_i$$

for non-negative integers n_i.

2. Weights

As mentioned in the introduction, we can study finite-dimensional irreducible representations of a complex semisimple Lie algebra through the lens of highest weights. In particular, we will starrt by defining weights and build towards the "Highest Weight Theorems".

For this section and going forward, we adopt the following notational conventions. Unless otherwise noted, we're working with a finite dimensional representation π of a semisimple Lie algebra $\mathfrak{g}$, fixing a Cartan subalgebra $\mathfrak{h}$ along with a root system R whose base is denoted Δ.

Definition 2.1 (Dominant Integral)**.** An element $\mu \in \mathfrak{h}^*$ is dominant integral if $\mu(H_\alpha)$ is a non-negative integer for each positive simple root α.

Definition 2.2 (Weight)**.** Suppose π is a finite dimensional representation of a Lie algebra $\mathfrak{g}$ on a vector space V. Then an element $\mu \in \mathfrak{h}^*$ is a weight for the representation π if there exists a nonzero vector $v \in V$ such that

$$\pi(H)v = \mu(H)v$$

for every $h \in \mathfrak{h}$. We call the vector v a weight vector for the weight μ.

For each of the previous two definitions, there is a corresponding alternative definition that declares elements of $\mathfrak{h}$ to be weights by replacing the $\mu(H)$ with an inner product $\langle \mu, H \rangle$. These definitions are equivalent, but we adopt the one concerning h^* to be consistent with the discussion of roots and the established partial ordering on $\mathfrak{h}^*$.

Definition 2.3 (Weight Space)**.** Given a weight μ, the collection of all vectors $v \in V$ satisfying the equation above is called the weight space with weight μ. The dimension of the weight space is called the multiplicity of the weight μ.

Since we must talk about highest weight modules, we must first declare an ordering on weights. In section 2, we established a partial order on $\mathfrak{h}^*$, and here, we will follow the same ordering. To make this more precise, we say that two weights $\mu_1 < \mu_2 \in \mathfrak{h}^*$ if

$$\mu_2 - \mu_1 = \sum_{\alpha \in \Delta} n_i \alpha_i$$

where n_i are non-negative integers, and α_i are simple positive roots. If π is a representation, then we say that a weight μ_0 is the highest weight if μ_0 is higher than all other weights.

Now, we are ready to state the Theorems of the Highest Weight.

Theorem 2.1 (Theorems of the Highest Weight)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the following statements hold for its representations.*

1. *Every irreducible representation has a highest weight.*
2. *Two irreducible representations with the same highest weight are equivalent representations.*
3. *The highest weight of any irreducible representation is dominant integral.*
4. *Every dominant integral element is realized as the highest weight of some representation.*

Proof. The proof is ommitted here. For a full proof, see section 20.3 of [Hum72]. □

3. Verma Modules

In this section, we will define Verma modules. The motivation for this is that we'd like to construct an irrerducible finite-dimensional representation for the Lie algebra $\mathfrak{g}$ with a highest weight. Given an element $\mu \in \mathfrak{h}$, we will construct a representation with μ as the highest weight, a Verma module. Note that here, we impose no restrictions on μ: it does not have to be dominant integral for the definition of Verma modules to make sense. We will later restrict our attention to the dominant integral case because it turns out that we can find an invariant subspace that serves as our irreducible finite-dimensional representation.

We will give two approaches to constructing Verma modules, first by an extension of scalars through tensoring with the universal enveloping algebra and then by quotienting the universal enveloping algebra by an ideal.

3.1. First Construction.

Since $\mathfrak{g}$ is a semisimple Lie algebra, as a vector space, it can be written as the Cartan Decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha.$$

Notationally, we write

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$
$$\mathfrak{n}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$$

The literature calls $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ a Borel subalgebra and $\mathfrak{n}^-$ the nilpotent subalgebra.

Now we look to the universal enveloping algebras of these subalgebras. The inclusion $\mathfrak{b} \hookrightarrow \mathfrak{g}$ induces a map of algebras $U(\mathfrak{b}) \to U(\mathfrak{g})$. Similarly, the inclusion $\mathfrak{n}^- \hookrightarrow \mathfrak{g}$ induces a map of algebras $U(\mathfrak{n}^-) \to U(\mathfrak{g})$. By the Poincare-Birkhoff-Witt (PBW) Theorem, these induced maps are injective.

Define F_μ to be the one-dimensional vector space generated by a vector v_0. This will turn out to be the highest weight vector in the construction. To make F_μ into a -representation, we define the following action:

$$\pi(H)v_0 = \mu(H)v_0, H \in \mathfrak{h}$$
$$\pi(X)v_0 = 0, X \in \mathfrak{n}^+$$

In other words, elements of $\mathfrak{h}$ act by $\mu(H)$ whereas elements of $\mathfrak{n}^+$ act by 0. This is thus a $U(\mathfrak{g})$ module. We're now ready to define Verma modules.

Definition 3.1. The Verma module V_μ is defined as

$$V_\mu = U(\mathfrak{g}) \oplus_{U(\mathfrak{b})} F_\mu.$$

This is a left $U(\mathfrak{g})$ module and thus a representation of $\mathfrak{g}$.

3.2. Second Construction.

The second way to construct Verma modules is by quotienting the universal enveloping algebra $U(\mathfrak{g})$. The rough idea is that if we desire for certain elements to be trivialized, we should form an ideal generated by such elements and quotient by it.

Let $\mu \in \mathfrak{h}^*$. Define a left ideal I_μ to be the left ideal generated by $\mathfrak{b}$ as well as elements of the form $H - \mu(H)1$. Notice that these are precisely the elements we seek to trivialize in the previous construction.

Definition 3.2. The Verma module is the quotient $U(\mathfrak{g})/I_\mu$.

In this definition, we see that the highest weight vector is the image of the coset $1 \in U(\mathfrak{g})$, and the highest weight is μ. These definitions are isomorphic because of the universal property established in the previous section.

3.3. **Quotient Modules.** The advantage to Verma modules is that it's easy to prove that they exist and to construct them. However, we're not quite meeting our goal of getting finite dimensional representations yet: Verma modules are always infinite-dimensional, even if the the highest weight is dominant integral. We will show that this can be fixed by quotienting.

Definition 3.3. Let a Verma module V_μ be given. Let U_μ denote the subspace of V_μ consisting of all vectors v such that the v_0 component of v is 0, and the v_0 component of any

$$\pi_\mu(X^1)\pi_\mu(X^2)\cdots\pi_\mu(X^l)v$$

is 0 for $X_i \in \mathfrak{n}^+$.

Proposition 3.1. *The space $U_\mathfrak{g}$ is invariant under $\mathfrak{g}$'s action.*

Proof. Suppose that $v \in U_\mu$. Let $Z \in \mathfrak{g}$. We must show that $\pi_\mu(Z) \in U_\mu$. To this end, we need to show that

$$\pi_\mu(X^1)\pi_\mu(X^2)\cdots\pi_\mu(X^l)\pi_\mu(Z)v$$

has v_0-component zero. We can write this expression as a linear combination of vector of the form

$$\pi_\mu(Y^1)\cdots\pi_\mu(Y^j)\pi_\mu(H^1)\cdots\pi_\mu(H^k)\pi_\mu(\tilde{X}^1)\cdots\pi_\mu(\tilde{X}^m)v$$

where Y^i's are in $\mathfrak{n}^-$, H^i's are in $\mathfrak{h}$, and $\tilde{X}^i$'s are in $\mathfrak{n}^+$. Since $v \in U_\mu$, the v_0-component of

$$\pi_\mu(\tilde{X}^1)\cdots\pi_\mu(\tilde{X}^m)v$$

is zero. So the above vector is a linear combination of weight vectors that have weight lower than μ. Applying elements of $\mathfrak{h}$ and $\mathfrak{n}^-$ will only lower the weights or keep them the same. Thus, the v_0-component of the above expressions are all zero. This means that $\pi_\mu(Z)v \in U_\mu$. □

We can always quotient a representation by an invariant subspace through endowing the subspace with the same vector space operations and same actions. Because the subspace is an invariant subspace, the action descend unambiguously. This enables us to take the quotient V_μ/U_μ.

It turns out that in the case that μ is dominant integral, V_μ/U_μ is an irreducible representation on $\mathfrak{g}$ and is in fact also finite-dimensional. The proof for the dimension amounts to showing that it is a direct sum of all its weight spaces, and each of its weights have finite multiplicity.

A note about representations we get from quotienting Verma modules is that such a representation doesn't lift to a representation of the corresponding Lie group. I'm not exactly sure why this is, but I saw this mentioned in a few of the sources I consulted. I'm afraid I'll leave this stated without a counterexample or explanation.

References

[Hal03] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations.* Springer-Verlag New York, 2003.
[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag New York, 1972.