Convexification of Bilinear Matrix Inequalities via Conic and Parabolic Relaxations

Mohsen Kheirandishfard, Fariba Zohrizadeh, Muhammad Adil, and Ramtin Madani

***Abstract*— This paper develops convex relaxation methods to solve optimization problems with piecewise linear objective function and a bilinear matrix inequality (BMI) constraint. As an alternative to the state-of-the-art semidefinite programming (SDP) and second-order cone programming (SOCP) relaxations, we propose a computationally efficient *parabolic relaxation*, which only relies on convex quadratic constraints to transform BMI problems into polynomial-time solvable surrogates. To ensure the recovery of feasible and near-optimal points, we adopt initial points for constructing penalty terms, which can be incorporated into the objective function of either SDP or parabolic relaxations. We prove that the penalized relaxations are guaranteed to provide feasible points for the original BMI problem, if the initial point lies within an analytical bound from the feasible set. Then, we generalize the penalized relaxations to a sequential scheme which starts from an arbitrary initial point (not necessarily feasible) and solves a sequence of relaxations to find feasible and near-optimal points. We theoretically prove that if certain assumptions hold, the proposed scheme is guaranteed to generate a convergent sequence of points whose objective values monotonically improve. Moreover, we evaluate the effectiveness of the sequential scheme on the problems of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controller design. Numerical results on benchmark instances from COMPl$_\text{e}$ib demonstrate that the proposed approach achieves comparable performance to the existing methods.**

***Index Terms*— Computational methods, LMIs, Optimization, Optimization algorithms, Optimal control**

I. Introduction

OPTIMIZATION problems with matrix inequality constraints are widely used in different areas of control [3]–[5]. As a special case, the class of problems involving linear matrix inequality (LMI) constraints are efficiently solvable up to any desired accuracy via interior-point based methods [6]–[9]. Despite a variety of control applications in robust control [10]–[13], controller design [14]–[20], affine fuzzy system design [21], [22], stability of fractional-order systems [23], optimization problems with bilinear matrix inequality (BMI) are generally computationally prohibitive and NP-hard in general [24], [25]. Great efforts have been devoted on solving special classes of these problems [26]–[28] and multiple solvers such as, LMIRank [29], PENLAB [30], PENBMI [31] are developed for solving BMIs of moderate size. However, an efficient algorithm with theoretical guarantees is still lacking [32].

In [33], [34], alternating minimization (AM)-based algorithms are proposed which divide variables into two blocks that are then alternately optimized until convergence. Although AM-based methods enjoy simple implementation and perform satisfactorily in many cases, they are not guaranteed to converge to a feasible point. Another approach for solving a BMI optimization problem is convex relaxation which reduces the problem into a convex surrogate whose solution approximates the solution of the original BMI problem. Solving the resulting problem in a sequential manner can further improve the quality of solutions [12], [35]–[39]. In [36], [40], BMI problems are cast as sequences of semidefinite programming (SDP) relaxations. In [41] a difference-of-convex (DC) decomposition framework is employed to construct a sequence of SDP relaxations, whose solutions are guaranteed to converge to a (sub)-optimal point of the original BMI. In [37], [38], [42] rank-constrained formulations with nuclear norm penalties and bound-tightening methods are used to tackle BMI optimization problems.

In [13], [43]–[45], branch-and-bound (BB) techniques are investigated, which involve the use of additional inequalities and variables to cast a BMI optimization problem as sequence of LMIs whose solutions converge to a globally optimal point. BB methods are generally computationally prohibitive and their applicability is restricted to moderate-sized problems. A novel global method has been recently presented in [46] which first transforms a given BMI optimization into an unconstrained problem with a fewer number of variables and then employs a hybrid multi-objective optimization technique to solve the resulting problem.

The class of BMI optimization problems can be seen as a special case of polynomial optimization problems (POPs). Therefore, off-the-shelf methods for solving POPs are applicable to BMI optimization problems as well [47], [48]. The most notable example is Lasserre's hierarchy of LMI relaxations [49], based on which several software packages have been developed [50]–[52].

The success of sequential frameworks and penalized SDP in solving quadratically constrained quadratic programs is demonstrated in [53]. In [54], it is shown that penalized SDP is able to find the roots of overdetermined systems of polynomial equations. Moreover, the incorporation of penalty terms into the objective of SDP relaxations are proven to be effective for solving non-convex optimization problems in power systems [55]–[58]. These papers show that penalizing certain physical

Parts of this paper have appeared in the conference papers [1] and [2]. Compared with the conference version, the new additions to this paper are detailed proofs and major theoretical results that guarantee the convergence of the proposed algorithm.

quantities in power network optimization problems such as reactive power loss or thermal loss facilitates the recovery of feasible points from convex relaxations. In [37], a sequential framework is introduced for solving BMIs without theoretical guarantees.

A. Contributions

In this work, a novel and general convex relaxation framework, regarded as parabolic relaxation, is introduced for solving the class of problems with piecewise linear objective function and a BMI constraint. The proposed relaxation relies on convex quadratic constraints as opposed to the SDP relaxation that can be computationally expensive.

Since the solutions of the relaxed problems may not be feasible for the original BMI, we choose an initial point to construct a penalty term which can be incorporated into the objective function of the proposed relaxation. This penalty term is compatible with either SDP or parabolic relaxation and can direct them towards finding feasible and near-optimal points of BMI problems. We theoretically prove that if an initial point is feasible or if it lies within a certain analytical bound from the feasible set of a BMI problem, the solution of penalized relaxation is guaranteed to be feasible. We propose a feasibility preserving sequential scheme for solving a BMI problem, whose each round involves solving a penalized convex relaxation. The proposed scheme can be initiated from an arbitrary point which is not necessarily feasible for the original BMI problem. In this sense, it is advantageous over many existing algorithms as it requires no ad-hoc heuristics and approximation techniques to recover a feasible initial point. We assess the effectiveness of this sequential scheme on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ static output-feedback controller design problems. For control plants from COMPl$_\text{e}$ib [59], the proposed scheme achieves comparable results on two centralized and fully decentralized scenarios, in comparison to two existing methods.

B. Notations

Throughout the paper, the scalars, vectors, and matrices are respectively shown by italic letters, lower-case bold letters, and upper-case bold letters. Symbols $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ respectively denote the sets of real scalars, real vectors of size n, and real matrices of size $n \times m$. The sets of real $n \times n$ symmetric matrices and positive semidefinite matrices are shown by $\mathbb{S}_n$ and $\mathbb{S}_n^+$, respectively. For given vector $\boldsymbol{a}$ and matrix $\boldsymbol{A}$, symbols a_i and A_{ij} respectively indicate the i^{th} element of $\boldsymbol{a}$ and $(i,j)^{th}$ element of $\boldsymbol{A}$. Notations $[a]_{i\in\mathcal{I}}$ and $[A]_{ij\in\mathcal{I}}$ respectively shows the sub-vector and sub-matrix corresponding to the set of indices $\mathcal{I}$. Notation $\boldsymbol{A} \succeq 0$ denotes $\boldsymbol{A}$ is positive-semidefinite ($\boldsymbol{A} \succ 0$ indicates positive definite) and $\boldsymbol{A} \preceq 0$ means $\boldsymbol{A}$ is negative-semidefinite ($\boldsymbol{A} \prec 0$ indicates negative definite). For two given matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size, symbol $\langle \boldsymbol{A}, \boldsymbol{B}\rangle = \mathrm{tr}\{\boldsymbol{A}^\top \boldsymbol{B}\}$ shows the inner product between the matrices where $\mathrm{tr}\{.\}$ and $(.)^\top$ respectively denote the trace and transpose operators. Notation $\|.\|_p$ refers to either matrix norm or vector norm depending on the context and $|.|$ indicates either the absolute value operator or the cardinality of a set depending on the context. Operator $\mathrm{diag}(.)$ gets a vector and forms a diagonal matrix with its input on the diagonal. For an arbitrary matrix $\boldsymbol{A}$ and sets of indices $\mathcal{I}_1$ and $\mathcal{I}_2$, define $\boldsymbol{A}\{\mathcal{I}_1, \mathcal{I}_2\}$ as the submatrix of $\boldsymbol{A}$ corresponding to the rows whose indices belong to $\mathcal{I}_1$, and the columns whose indices belong to $\mathcal{I}_2$. For a symmetric matrix $\boldsymbol{B}$ of size n, symbol $\boldsymbol{B}(:)$ indicates a vector of size $\binom{n}{2}$ consists of all unique elements of $\boldsymbol{B}$. Symbols $\boldsymbol{I}$, $\boldsymbol{e}_i$, and $\boldsymbol{0}$ denote the identity matrix, standard basis vector, and zero matrix of appropriate dimensions, respectively. For integer n, symbol $\mathcal{I}_n$ shows the set of all integer numbers from 1 to n.

The remainder of this work are organized as follows: Section II states the non-convex BMI optimization problem to be solved and discuss about its challenges. Then, we provide different convex relaxations in Section III to find the solution of problem. Further in Section IV, we propose a penalization technique to recover feasible points for the original BMI problem. Section V offers a sequential scheme to improve the quality of feasible points. In Sections VI and VII, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design problems are cast as BMI programs and we use the sequential method to find controllers of desired structures. Finally, the last section offers a conclusion to the paper.

II. Problem Formulation

This paper is concerned with the class of bilinear matrix inequality (BMI) optimization problems of the form

$$\underset{\boldsymbol{Y}\in\mathbb{R}^{n\times m}}{\text{minimize}} \quad f(\boldsymbol{Y}) \tag{1a}$$

$$\text{subject to} \quad p(\boldsymbol{Y}, \boldsymbol{Y}\boldsymbol{Y}^\top) \preceq 0, \tag{1b}$$

$$\langle \boldsymbol{W}_i, \boldsymbol{Y}\rangle = 0, \qquad i \in \{1,\dots,l\} \tag{1c}$$

where $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ is a piecewise linear objective function:

$$f(\boldsymbol{Y}) \triangleq \langle \boldsymbol{C}_0, \boldsymbol{Y}\rangle + \sum_{i=1}^{o} |\langle \boldsymbol{C}_i, \boldsymbol{Y}\rangle - c_i| \tag{2}$$

where $\{\boldsymbol{C}_i \in \mathbb{R}^{n\times m}\}_i$ and $\{c_i \in \mathbb{R}\}_i$ are given, and $p : \mathbb{R}^{n\times m} \times \mathbb{S}_n \to \mathbb{S}_m$ is a matrix pencil defined as

$$p(\boldsymbol{Y}, \boldsymbol{X}) \triangleq \boldsymbol{P} + \sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij}\boldsymbol{Q}_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}\boldsymbol{R}_{ij}, \tag{3}$$

where $\boldsymbol{P}$, $\{\boldsymbol{Q}_{ij}\}$, and $\{\boldsymbol{R}\}_{ij}$ are all given $q\times q$ real symmetric matrices. With no loss of generality, we assume that $\boldsymbol{R}_{ij} = \boldsymbol{R}_{ji}$ for every $(i,j) \in \{1,\dots,n\}^2$. Let $\mathcal{F} \subseteq \mathbb{R}^{n\times m}$ denote the feasible set of the problem (1a) – (1c). Throughout the paper, we assume that (1a) – (1c) is feasible with an attainable optimal value.

The BMI problem (1a) – (1c) is of particular importance due to its wide range of applications in optimal control design [41], [46], [60]. For instance, consider the problem of finding a sparse controller which is of great interest for computational and practical purposes [61], [62]. For an appropriate choice of the non-smooth function f, this problem can be conveniently cast as a BMI optimization of form (1a) – (1c). Additionally, the matrices $\{\boldsymbol{W}_i \in \mathbb{R}^{n\times m}\}_i$ can be employed to impose any linear structure on the elements of matrix $\boldsymbol{Y}$. In many

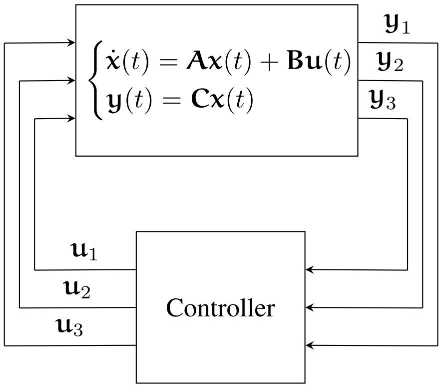


Fig. 1: Static output feedback controller. A small controller plant with three inputs and three outputs. Depending on the desired zero-nonzero pattern of the controller, it uses a subset of observations $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ to generate control commands $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$
.

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \\ \hline K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \quad \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \\ \hline K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix} \quad \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \\ \hline K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix}$$

(a) (b) (c)

Fig. 2: Matrix variables containing the Lyapunov matrix P and the controller gain K. The Lyapunov matrix is symmetric and the controller can have different structures (a) arbitrary (b) partially decentralized, and (c) fully decentralized.

control applications, the matrix variable $\boldsymbol{Y}$ encapsulates submatrices with specific structures (e.g., symmetric, diagonal, etc). Constraint (1c) allows us to take such structures into consideration. Figures 1 and 2 demonstrate a structured control design problem with three different zero/non-zero patterns.

Despite the advantages of the BMI formulation (1a) – (1c), constraint (1b) renders the problem non-convex and computationally challenging. To circumvent this challenge, it is common practice to solve convex surrogates to find approximate solutions. In order to formulate convex relaxations, we first define an auxiliary matrix variable $\boldsymbol{X} \in \mathbb{S}_n$, accounting for $\boldsymbol{Y}\boldsymbol{Y}^\top$, which leads to the so called *lifted reformulation*:

$$\begin{aligned} &\underset{\substack{\boldsymbol{Y}\in\mathbb{R}^{n\times m}\\ \boldsymbol{X}\in\mathbb{S}_n}}{\text{minimize}} && f(\boldsymbol{Y}) && \text{(4a)}\\ &\text{subject to} && p(\boldsymbol{Y},\boldsymbol{X}) \preceq 0, && \text{(4b)}\\ &&& \langle \boldsymbol{W}_i, \boldsymbol{Y}\rangle = 0, \qquad i\in\{1,\dots,l\} && \text{(4c)}\\ &&& \boldsymbol{X} = \boldsymbol{Y}\boldsymbol{Y}^\top && \text{(4d)} \end{aligned}$$

where (4c) is imposed to preserve the equivalence between (1a) – (1c) and (4a) – (4d). Notice that the lifted problem is cast in a higher dimensional space in which the entire non-convexity is captured by the constraint (4d). In what follows, we will replace (4d) with convex alternatives and revise the objective function to find feasible and near-optimal points for the original BMI problem (1a) – (1c).

III. Convex Relaxation

This section aims at introducing a family of convex relaxations whose solutions approximate the solution of (1a) – (1c). To this end, we relax (4d) to transform (4a) – (4d) into the following convex problem

$$\begin{aligned} &\underset{\substack{\boldsymbol{Y}\in\mathbb{R}^{n\times m}\\ \boldsymbol{X}\in\mathbb{S}_n}}{\text{minimize}} && f(\boldsymbol{Y}) && \text{(5a)}\\ &\text{subject to} && p(\boldsymbol{Y},\boldsymbol{X}) \preceq 0, && \text{(5b)}\\ &&& \langle \boldsymbol{W}_i, \boldsymbol{Y}\rangle = 0, \qquad i\in\{1,\dots,l\} && \text{(5c)}\\ &&& (\boldsymbol{Y},\boldsymbol{X}) \in \mathcal{C} && \text{(5d)} \end{aligned}$$

where $\mathcal{C} \subseteq \mathbb{R}^{n\times m}\times\mathbb{S}_n$ is a convex set to be designed. For any choice of $\mathcal{C}$, the optimal cost of (5a) – (5d) can serve as a lower bound for the global cost of the BMI problem (1a) – (1c). In what follows, we first discuss the standard SDP relaxation and then, introduce a novel *parabolic relaxation*, which transforms the constraint (4d) into a set of convex quadratic inequalities.

To formulate an SDP relaxation of (4a) – (4d), we replace $\mathcal{C}$ with $\mathcal{C}^{\text{SDP}}_{n,m}$ which is defined as

$$\mathcal{C}^{\text{SDP}}_{n,m} = \{(\boldsymbol{Y},\boldsymbol{X}) \in \mathbb{R}^{n\times m}\times\mathbb{S}_n \mid \boldsymbol{X} - \boldsymbol{Y}\boldsymbol{Y}^\top \succeq 0\}. \quad (6)$$

Although SDP relaxation is efficiently solvable in polynomial time, its applicability is limited to moderate-sized problems. Motivated by this, we introduce a computationally efficient alternative to SDP relaxation, named parabolic relaxation, which transforms the non-convex constraint (4d) to a set of convex quadratic inequalities. To formulate the parabolic relaxation, one can replace $\mathcal{C}$ with the following set:

$$\begin{aligned} \mathcal{C}^{\text{PRB}}_{n,m} = \big\{&(\boldsymbol{Y},\boldsymbol{X}) \in \mathbb{R}^{n\times m}\times\mathbb{S}_n \mid \\ &X_{ii}+X_{jj}+2X_{ij} \geq \|(\boldsymbol{e}_i+\boldsymbol{e}_j)^\top\boldsymbol{Y}\|^2,\ i,j\in\{1,\dots,n\},\\ &X_{ii}+X_{jj}-2X_{ij} \geq \|(\boldsymbol{e}_i-\boldsymbol{e}_j)^\top\boldsymbol{Y}\|^2,\ i,j\in\{1,\dots,n\}\big\}. \quad (7) \end{aligned}$$

where $\{\boldsymbol{e}_i\}_{i=1}^n$ represents the standard basis for $\mathbb{R}^n$.

Notice that the solution of the aforementioned relaxations are not necessarily feasible for (1a) – (1c). In the following section, we propose to revise the objective function (5a) to direct convex relaxations towards finding feasible points for the original non-convex problem (1a) – (1c).

IV. Penalized Convex Relaxation

In this section, we incorporate a penalty term into the objective function (5a) to formulate the following *penalized convex relaxation*:

$$\begin{aligned} &\underset{\substack{\boldsymbol{Y}\in\mathbb{R}^{n\times m}\\ \boldsymbol{X}\in\mathbb{S}_n}}{\text{minimize}} && f(\boldsymbol{Y}) + \eta\times\text{tr}\{\boldsymbol{X} - 2\check{\boldsymbol{Y}}^\top\boldsymbol{Y} + \check{\boldsymbol{Y}}^\top\check{\boldsymbol{Y}}\} && \text{(8a)}\\ &\text{subject to} && p(\boldsymbol{Y},\boldsymbol{X}) \preceq 0, && \text{(8b)}\\ &&& \langle \boldsymbol{W}_i, \boldsymbol{Y}\rangle = 0, \qquad i\in\{1,\dots,l\} && \text{(8c)}\\ &&& (\boldsymbol{Y},\boldsymbol{X}) \in \mathcal{C} && \text{(8d)} \end{aligned}$$

where $\mathcal{C} \in \{\mathcal{C}^{\text{SDP}}_{n,m}, \mathcal{C}^{\text{PRB}}_{n,m}\}$, the point $\check{\boldsymbol{Y}} \in \mathbb{R}^{n\times m}$ is an initial guess (not necessarily feasible) for the optimal solution of

(1a) – (1c), and $\eta > 0$ is a regularization parameter to control the balance between the original objective function and the penalty term.

Next, we use the well-known Mangasarian-Fromovitz constraint qualification (MFCQ) condition from [63], [64] in order to characterize well-behaved feasible points of the BMI problem (1a) – (1c).

Definition 1. *A point $\boldsymbol{Y} \in \mathbb{R}^{n\times m}$ is said to satisfy the Mangasarian-Fromovitz constraint qualification (MFCQ) condition if there exists $\boldsymbol{Z} \in \mathbb{R}^{n\times m}$ such that*

$$p(\boldsymbol{Y}+\boldsymbol{Z},\ \ \boldsymbol{Y}\boldsymbol{Y}^\top+\boldsymbol{Z}\boldsymbol{Y}^\top+\boldsymbol{Y}\boldsymbol{Z}^\top) \prec 0, \tag{9a}$$
$$\langle \boldsymbol{W}_i, \boldsymbol{Z}\rangle = 0, \qquad i\in\{1,\dots,l\}. \tag{9b}$$

Moreover, define the singularity function $s:\mathbb{R}^{n\times m}\to\mathbb{R}$ as

$$s(\boldsymbol{Y}) \triangleq \max_{\boldsymbol{Z},\delta}\Big\{ \lambda_{\min}\{(1-\delta)\boldsymbol{P}-p(\delta\boldsymbol{Y}+\boldsymbol{Z},\delta\boldsymbol{Y}\boldsymbol{Y}^\top+\boldsymbol{Z}\boldsymbol{Y}^\top+\boldsymbol{Y}\boldsymbol{Z}^\top)\}\ \big|\ \langle\boldsymbol{W}_i,\boldsymbol{Z}\rangle=0 \quad i\in\{1,\dots,l\},\ \delta^2+\|\boldsymbol{Z}\|_{\mathrm{F}}^2\le 1\Big\} \tag{10}$$

where $\lambda_{\min}$ denotes the minimum eigenvalue operator.

Observe that given any $\boldsymbol{Y}\in\mathbb{R}^{n\times m}$ the value of $s(\boldsymbol{Y})$ can be easily calculated by solving a convex optimization problem. Additionally, $s(\boldsymbol{Y})>0$, if and only if $\boldsymbol{Y}$ satisfies the MFCQ condition. The following definition introduces a few constant values that help with the statement of our theoretical results.

Definition 2. *Define $\alpha,\beta>0$ as two arbitrary constants that satisfy:*

$$\|-\boldsymbol{P}+p(\boldsymbol{Y},\boldsymbol{X})\|_{\mathrm{F}} \le 2\alpha\|\boldsymbol{Y}\|_{\mathrm{F}}+\beta\|\boldsymbol{X}\|_{\mathrm{F}} \tag{11}$$

for every $\boldsymbol{Y}\in\mathbb{R}^{n\times m}$ and $\boldsymbol{X}\in\mathbb{S}^{n\times n}$. Additionally, define

$$\kappa \triangleq \big\|\big[\|\boldsymbol{R}_{ij}\|_2\big]_{ij}\big\|_1 \tag{12}$$

where $\big[\|\boldsymbol{R}_{ij}\|_2\big]_{ij}$ denotes the $n\times n$ symmetric matrix whose i,j element is equal to $\|\boldsymbol{R}_{ij}\|_2$.

Given the above definitions, the next theorem investigates conditions under which the penalized convex relaxation problem (8a) – (8d) with a feasible initial point $\check{\boldsymbol{Y}}\in\mathcal{F}$ leads to a feasible point for the original BMI problem (1a) – (1c).

Theorem 1. *Let $\check{\boldsymbol{Y}}\in\mathcal{F}$ be a feasible point for the problem (1a) – (1c), which satisfies the MFCQ condition. If*

$$\eta > \frac{\left(\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_2\right)\left(3\kappa+2(\alpha+\beta+\beta\|\check{\boldsymbol{Y}}\|_{\mathrm{F}})+\sqrt{\beta s(\check{\boldsymbol{Y}})}\right)}{s(\check{\boldsymbol{Y}})} \tag{13}$$

then the penalized convex relaxation (8a) – (8d) has a unique optimal solution $(\overset{*}{\boldsymbol{Y}},\overset{*}{\boldsymbol{X}})$ which satisfies (4d) *and additionally, $f(\overset{*}{\boldsymbol{Y}})\le f(\check{\boldsymbol{Y}})$.*

Proof. See Appendix for the proof. □

According to Theorem 1, the penalized convex relaxation (8a) – (8d) preserves the feasibility of an initial point. In the next theorem, we show that even if the initial point is not feasible for (1a) – (1c), but sufficiently close to its feasible set, the penalized convex relaxation problem is still guaranteed to provide a feasible point. The next definition gives a measure of distance between an arbitrary point in $\mathbb{R}^{n\times m}$ and $\mathcal{F}$.

Algorithm 1 Sequential Penalized Relaxation

Input: $\check{\boldsymbol{Y}}\in\mathbb{R}^{n\times m}$, a fixed parameter $\eta>0$.
1: $k \leftarrow 0$
2: **repeat**
3: $k\leftarrow k+1$
4: $\boldsymbol{Y}^k \leftarrow$ solve the penalized relaxation (8a) – (8d).
5: $\check{\boldsymbol{Y}}\leftarrow\boldsymbol{Y}^k$
6: **until** stopping criteria is met
Output: $\boldsymbol{Y}^k$

Definition 3 (Feasibility Distance)**.** *Define the feasibility distance function $d_{\mathcal{F}}:\mathbb{R}^{n\times m}\to\mathbb{R}$ as:*

$$d_{\mathcal{F}}(\boldsymbol{Y}) \triangleq \min\{\|\boldsymbol{Y}-\bar{\boldsymbol{Y}}\|_{\mathrm{F}}\ \big|\ \bar{\boldsymbol{Y}}\in\mathcal{F}\}, \tag{14}$$

where $\mathcal{F}$ denotes the feasible set of the problem (1a) – (1c).

Theorem 2. *Consider an arbitrary point $\check{\boldsymbol{Y}}\in\mathbb{R}^{n\times m}$, that satisfies*

$$\check{d} \triangleq \frac{s(\check{\boldsymbol{Y}})}{3\kappa+2(\alpha+\beta+\beta\|\check{\boldsymbol{Y}}\|_{\mathrm{F}})+\sqrt{\beta s(\check{\boldsymbol{Y}})}} - d_{\mathcal{F}}(\check{\boldsymbol{Y}}) > 0. \tag{15}$$

If

$$\eta > \check{d}^{-1}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}}, \tag{16}$$

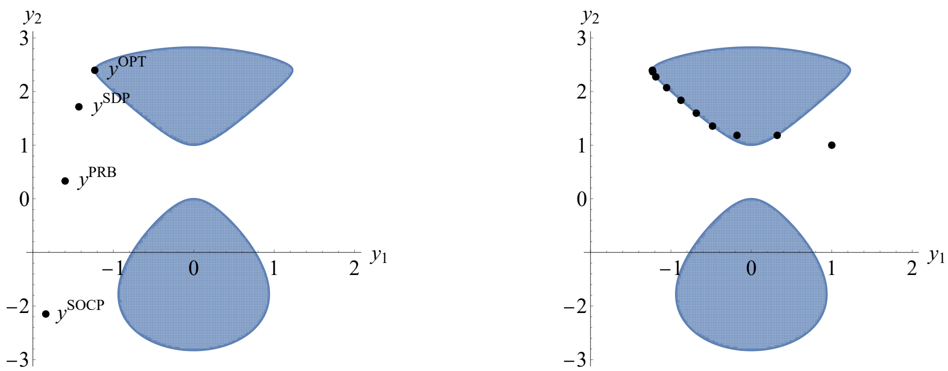
then the penalized convex relaxation (8a) – (8d) has a unique optimal solution $(\overset{*}{\boldsymbol{Y}},\overset{*}{\boldsymbol{X}})$ which satisfies (4d).

Proof. See Appendix for the proof. □

Remark 1. *It should be noted that both* (13) *and* (16) *are of theoretical importance only. They show that a sufficiently large η can lead to a feasible solution. In practice, one can resort to bisection in order to find an appropriate η. For instance, in all of our experiment, in order to obtain a feasible point, we have tested a few values of the form $k_1\times 10^{k_2}$ for η, where $k_1\in\{1,2,5\}$ and $k_2\in\mathbb{Z}$. Additionally, we acknowledged that the calculation of $d_{\mathcal{F}}(\check{\boldsymbol{Y}})$ is computationally hard in general, which further limits the practicality of the bound* (16).

V. Sequential Penalized Relaxation

Motivated by Theorems 1 and 2, this section presents a sequential approach that solves a sequence of penalized relaxations of the form (8a) – (8d) to infer high-quality feasible points for the non-convex problem (1a) – (1c). The proposed scheme starts from an arbitrary initial point $\check{\boldsymbol{Y}}$. Once a feasible point for (1a) – (1c) is obtained, according to Theorem 1, the proposed scheme preserves feasibility and generates a sequence of points whose objective values monotonically improve. The details of this sequential approach are delineated in Algorithm 1. The following theorem guarantees the convergence of Algorithm 1 to at least a locally optimal solution.



(a) (b)

Fig. 3: The feasible set of the BMI problem (18a) – (18b): (a) the globally optimal solution $\boldsymbol{y}^{\mathrm{OPT}}$, as well as the outputs of SDP, parabolic, and SOCP relaxations $\boldsymbol{y}^{\mathrm{SDP}}$, $\boldsymbol{y}^{\mathrm{PRB}}$, and $\boldsymbol{y}^{\mathrm{SOCP}}$. (b) The sequence generated by Algorithm (1), starting from $\check{\boldsymbol{y}} = [1,1]^\top$ with $\eta = 1$.

Theorem 3. *Let $\check{\mathcal{F}} \triangleq \{\boldsymbol{Y} \in \mathcal{F} \,|\, f(\boldsymbol{Y}) < \tilde{f}\}$ denote an epigraph of the problem (1a)–(1c) such that $s(\boldsymbol{Y}) > \tilde{s} > 0$ and $\|\boldsymbol{Y}\|_{\mathrm{F}} < \tilde{d}$ for every $\boldsymbol{Y} \in \check{\mathcal{F}}$. If*

$$\eta > \frac{\left(\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_2\right)\left(3\kappa+2(\alpha+\beta+\beta\tilde{y})+\sqrt{\beta\tilde{s}}\right)}{\tilde{s}} \tag{17}$$

and $\check{\boldsymbol{Y}} \in \check{\mathcal{F}}$, then the sequence generated by Algorithm 1 converges to at least a local minimizer of the problem (1a)–(1c).

In what follows, we give an example to provide insights into the performance of Algorithm 1.

Example 1: Consider the following small-scale BMI in terms of two scalar variables y_1 and y_2:

$$\underset{\boldsymbol{y}\in\mathbb{R}^2}{\text{minimize}} \quad y_1 \tag{18a}$$

$$\text{subject to} \quad \begin{bmatrix} 2y_1^2 - y_2^2 + y_2 & -y_1y_2 + 2y_1 \\ -y_1y_2 + 2y_1 & y_1^2 + y_2^2 - 8 \end{bmatrix} \preceq 0. \tag{18b}$$

Observe that (18a) – (18b) can be cast as an optimization problem of form (1a) – (1c) since it has a linear objective function and a BMI constraint. The point $\overset{*}{\boldsymbol{y}} = [-1.2302, 2.3975]^\top$ is the optimal solution of (18a) – (18b) with the corresponding objective value -1.2302. In what follows, we leverage convex relaxation techniques to recover feasible and near-optimal points of (18a) – (18b). To this end, define auxiliary variable $\boldsymbol{X} \in \mathbb{S}_2$ to formulate the SDP relaxation as

$$\underset{\boldsymbol{y}\in\mathbb{R}^2, \boldsymbol{X}\in\mathbb{S}_2}{\text{minimize}} \quad y_1 \tag{19a}$$

$$\text{subject to} \quad \begin{bmatrix} 2X_{11} - X_{22} + y_2 & -X_{12} + 2y_1 \\ -X_{12} + 2y_1 & X_{11} + X_{22} - 8 \end{bmatrix} \preceq 0, \tag{19b}$$

$$\boldsymbol{X} - \boldsymbol{y}\boldsymbol{y}^\top \succeq 0, \tag{19c}$$

which has the optimal solution $\overset{*}{\boldsymbol{y}} = [-1.4280, 1.7156]^\top$ corresponding to the objective value -1.4280. Additionally, we can derive the parabolic relaxation of (18a) – (18b) by replacing (19c) with a set of quadratic constraints as

$$\underset{\boldsymbol{y}\in\mathbb{R}^2, \boldsymbol{X}\in\mathbb{S}_2}{\text{minimize}} \quad y_1 \tag{20a}$$

$$\text{subject to} \quad \begin{bmatrix} 2X_{11} - X_{22} + y_2 & -X_{12} + 2y_1 \\ -X_{12} + 2y_1 & X_{11} + X_{22} - 8 \end{bmatrix} \preceq 0, \tag{20b}$$

$$X_{11} + X_{22} - 2X_{12} \geq (y_1 - y_2)^2, \tag{20c}$$

$$X_{11} + X_{22} + 2X_{12} \geq (y_1 + y_2)^2, \tag{20d}$$

$$X_{11} \geq y_1^2, \qquad X_{22} \geq y_2^2 \tag{20e}$$

which has the optimal solution $\overset{*}{\boldsymbol{y}} = [-1.5988, 0.3319]^\top$ corresponding to the objective value -1.5988. As illustrated in Figure 3a, neither of these points belong to the feasible set of the original problem (18a) – (18b). To direct the relaxations towards feasible points, we adopt the initial guess $\check{\boldsymbol{y}} = [1.0000, 1.0000]^\top$ and revise the objective functions (19a) and (20a) as

$$y_1 + \eta(\mathrm{tr}\{\boldsymbol{X}\} - 2\boldsymbol{y}^\top\check{\boldsymbol{y}} + \check{\boldsymbol{y}}^\top\check{\boldsymbol{y}}). \tag{21}$$

By doing so, we formulate the penalized SDP and penalized parabolic relaxations which provide feasible points for appropriate choices of η. Moreover, for sufficiently large η, both penalized relaxations provide feasible point for the original problem. For $\eta = 1$, both penalized relaxations give point $\overset{*}{\boldsymbol{x}} = [0.3214, 1.1835]^\top$ which is feasible for (18a) – (18b) as well. Given that, we can employ Algorithm 1 to solve a sequence of penalized relaxations and improve the quality of points. Figure 3b illustrates the sequence of points generated by Algorithm 1 for both penalized relaxations. Notice that since a feasible point is recovered in the first round and η is sufficiently large, the algorithm preserves feasibility and monotonically improves the objective value until convergence is achieved.

The next section seeks to formulate $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal structured control synthesis problems in the form the optimization problem (1a) – (1c).

VI. Applications to Optimal Control

This section investigates the application of Algorithm 1 in solving structured controller design problems. Many studies have been extensively explored the design of structured controllers for several systems, including spatially distributed systems [14], [65]–[68], localizable systems [69], power systems [61], [70], [71], optimal static distributed systems [72], [73], strongly connected systems [74], heterogeneous systems [75] etc.

The problems of designing distributed state-feedback and output-feedback controllers for linear time-invariant systems have been of great interest in the literature [76]–[83]. Papers [84]–[88] have considered special cases which make controller design problems computationally tractable. Paper [89] introduces a condition regarded as quadratic invariance, which enables the transformation of optimal distributed controller design problems to convex optimization. This condition is further explored in other studies, including [90]–[97].

In what follows, we consider $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design problems for a linear time-invariant plant G of the following form,

$$\mathrm{G}:\begin{cases}\dot{\mathbf{x}}=\mathbf{A}\ \mathbf{x}+\mathbf{B}_1\ \boldsymbol{w}+\mathbf{B}\quad\mathbf{u}\\ \boldsymbol{z}=\mathbf{C}_1\mathbf{x}+\mathbf{D}_{11}\boldsymbol{w}+\mathbf{D}_{12}\mathbf{u}\\ \mathbf{y}=\mathbf{C}\ \mathbf{x}+\mathbf{D}_{21}\boldsymbol{w}\end{cases}\tag{22}$$

where $\mathbf{x}\in\mathbb{R}^{n_x}$ is the state vector, $\boldsymbol{w}\in\mathbb{R}^{n_w}$ is the system input, $\mathbf{u}\in\mathbb{R}^{n_u}$ is the control command vector, $\boldsymbol{z}\in\mathbb{R}^{n_z}$ is the output, and $\mathbf{y}\in\mathbb{R}^{n_y}$ is the sensor measurement vector. Matrices $\mathbf{A}$, $\mathbf{B}_1$, $\mathbf{B}$, $\mathbf{C}_1$, $\mathbf{C}$, $\mathbf{D}_{11}$, $\mathbf{D}_{12}$, $\mathbf{C}$, $\mathbf{D}_{21}$ are all fixed and of appropriate dimensions. We show that finding structured $\mathcal{H}_2$ and $\mathcal{H}_\infty$ static output-feedback controllers for G can be cast as BMI problems of form (1a) – (1c). Therefore, Algorithm 1 can be used to solve the resulting BMI problems and obtain the desired controllers. Our work is related to the body of literature on convex relaxation of optimal controller design based on semidefinite programming in [20], [71], [98]–[101], as well as sequential methods in [102]–[105].

We define the matrix function $\mathbf{K}:\mathbb{R}^l\to\mathbb{R}^{n_u\times n_y}$ as follows to characterize structured controllers

$$\mathbf{K}(\mathbf{h})\triangleq\sum_{i=1}^{l}\mathrm{h}_i\boldsymbol{E}_i,\tag{23}$$

where $\{\boldsymbol{E}_i\}_{i=1}^l\in\{0,1\}^{n_u\times n_y}$ are binary matrices used to represent pre-defined structures and $\mathbf{h}\in\mathbb{R}^l$ indicates the non-zero elements of the controller. Given that, an observation vector $\mathbf{y}$ is applied to the controller $\mathbf{K}(\mathbf{h})$ as input, through which the control command $\mathbf{u}=\mathbf{K}(\mathbf{h})\mathbf{y}$ is generated. To formulate such control design problem, we first derive the dynamic equations describing the closed-loop plant as:

$$\mathrm{G}_{\mathrm{cl}}:\begin{cases}\dot{\mathbf{x}}=(\mathbf{A}+\mathbf{B}\quad\mathbf{K}(\mathbf{h})\mathbf{C})\mathbf{x}+(\mathbf{B}_1\ +\mathbf{B}\quad\mathbf{K}(\mathbf{h})\mathbf{D}_{21})\boldsymbol{w}\\ \boldsymbol{z}=(\mathbf{C}_1+\mathbf{D}_{12}\mathbf{K}(\mathbf{h})\mathbf{C})\mathbf{x}+(\mathbf{D}_{11}+\mathbf{D}_{12}\mathbf{K}(\mathbf{h})\mathbf{D}_{21})\boldsymbol{w}\end{cases}\tag{24}$$

In what follows, we formulate $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control design problems of $\mathcal{G}$ and show how they can be cast as BMI optimizations of form (1a) – (1c).

A. $\mathcal{H}_2$ Optimal Control

The primary goal of this problem is to find a controller gain $\mathbf{K}(\mathbf{h})$ for the linear system G such that $\mathbf{A}+\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}$ becomes a Hurwitz matrix (i.e. all of its eigenvalues have negative real part) and the $\mathcal{H}_2$ norm of the closed-loop system G_{cl} is minimized. With no loss of generality, we assume $\mathbf{D}_{11}=\mathbf{0}$, $\mathbf{D}_{21}=\mathbf{0}$ and there exists a controller gain matrix $\mathbf{K}(\mathbf{h})$ that stabilizes system G (or equivalently $\mathbf{A}+\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}$ is Hurwitz). Given that, the $\mathcal{H}_2$ norm of the closed-loop plant is given by,

$$\|\mathrm{G}_{\mathrm{cl}}\|_{\mathcal{H}_2}=\mathrm{tr}\big\{\big(\mathbf{C}_1+\mathbf{D}_{12}\mathbf{K}(\mathbf{h})\mathbf{C}\big)\mathbf{P}\big(\mathbf{C}_1+\mathbf{D}_{12}\mathbf{K}(\mathbf{h})\mathbf{C}\big)^\top\big\},\tag{25}$$

where $\mathbf{P}\succeq 0$ is the solution of the following Lyapunov equation,

$$\big(\mathbf{A}+\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}\big)\mathbf{P}+\mathbf{P}\big(\mathbf{A}+\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}\big)^\top+\mathbf{B}_1\mathbf{B}_1^\top=\mathbf{0}.\tag{26}$$

It is well-known [106] that, the solution of the above equation is not affected if it is relaxed to,

$$\big(\mathbf{A}+\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}\big)\mathbf{P}+\mathbf{P}\big(\mathbf{A}+\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}\big)^\top+\mathbf{B}_1\mathbf{B}_1^\top\preceq 0.\tag{27}$$

Therefore, we can formulate the $\mathcal{H}_2$ controller design problem for G as follows,

$$\underset{\substack{\mathbf{P}\in\mathbb{S}_{n_x},\boldsymbol{W}\in\mathbb{S}_{n_z}\\ \mathbf{h}\in\mathbb{R}^l}}{\text{minimize}}\qquad\langle\boldsymbol{W},\boldsymbol{I}\rangle\tag{28a}$$

$$\text{subject to}\qquad f_{\text{LMI}}(\mathbf{P},\boldsymbol{W})+f_{\text{BMI}}(\mathbf{P},\mathbf{h})\preceq 0,\tag{28b}$$

where matrix functions $f_{\text{LMI}}:\mathbb{S}_{n_x}\times\mathbb{S}_{n_z}\to\mathbb{S}_{2n_x+n_z}$, $f_{\text{BMI}}:\mathbb{S}_{n_x}\times\mathbb{R}^l\to\mathbb{S}_{2n_x+n_z}$ are defined as,

$$f_{\text{LMI}}(\mathbf{P},\boldsymbol{w})\triangleq\begin{bmatrix}\mathbf{A}\mathbf{P}+\mathbf{P}\mathbf{A}^\top+\mathbf{B}_1\mathbf{B}_1^\top & \mathbf{0} & \mathbf{0}\\ * & -\boldsymbol{W} & \mathbf{C}_1\mathbf{P}\\ * & * & -\mathbf{P}\end{bmatrix},\tag{29a}$$

$$f_{\text{BMI}}(\mathbf{P},\mathbf{h})\triangleq\begin{bmatrix}\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}\mathbf{P}+(\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}\mathbf{P})^\top & \mathbf{0} & \mathbf{0}\\ * & \mathbf{0} & \mathbf{D}_{12}\mathbf{K}(\mathbf{h})\mathbf{C}\mathbf{P}\\ * & * & \mathbf{0}\end{bmatrix},\tag{29b}$$

and $*$ accounts for the symmetric elements of matrices.

Proposition 1. *Assume that* $\mathbf{B}_1\mathbf{B}_1^\top\succ 0$, $\mathbf{D}_{11}=\mathbf{0}$, *and* $\mathbf{D}_{21}=\mathbf{0}$. *Matrix* $\mathbf{K}(\overset{*}{\mathbf{h}})$ *is the optimal* $\mathcal{H}_2$ *static output-feedback controller gain for plant* G, *if* $(\overset{*}{\mathbf{P}},\overset{*}{\boldsymbol{W}},\overset{*}{\mathbf{h}})$ *is an optimal solution of problem (28a) – (28b).*

Proof. Observe that due to the BMI constraint (28b), matrix $\overset{*}{\mathbf{P}}$ is positive-definite and satisfies the Lyapunov inequality (27), which certifies that $\mathbf{K}(\overset{*}{\mathbf{h}})$ is stabilizer. On the other hand, we have $\overset{*}{\boldsymbol{W}}=\big(\mathbf{C}_1+\mathbf{D}_{12}\mathbf{K}(\overset{*}{\mathbf{h}})\mathbf{C}\big)\overset{*}{\mathbf{P}}\big(\mathbf{C}_1+\mathbf{D}_{12}\mathbf{K}(\overset{*}{\mathbf{h}})\mathbf{C}\big)^\top$ which implies that the closed-loop norm $\langle\overset{*}{\boldsymbol{W}},\boldsymbol{I}\rangle=\|\mathrm{G}_{\mathrm{cl}}\|_{\mathcal{H}_2}$ is minimized. □

Notice that (29b) is a BMI constraint due to the presence of the matrix product $\mathbf{K}(\mathbf{h})\mathbf{C}\mathbf{P}$. Hence, we can cast (28a) – (28b) as a BMI problem of form (1a) – (1c). To this end, it suffices to stack all variables into a large vector $\boldsymbol{y}$ defined as,

$$\boldsymbol{y}\triangleq[\boldsymbol{W}(:)^\top,\mathbf{P}(:)^\top,\mathbf{h}^\top]^\top\in\mathbb{R}^{\tilde{n}},\tag{30}$$

where $\tilde{n}=\binom{n_x}{2}+\binom{n_z}{2}+l$.

B. $\mathcal{H}_\infty$ Optimal Control

The $\mathcal{H}_\infty$ control design problem for plant G seeks to find a vector $\mathbf{h}$ such that the controller gain $\mathbf{K}(\mathbf{h})$ stabilizes the system and the $\mathcal{H}_\infty$ norm of the closed-loop system G_{cl} is minimized. With no loss of generality, we assume $\mathbf{D}_{21} = \mathbf{0}$, $\gamma > 0$, and there is a stablizing controller $\mathbf{K}(\mathbf{h})$ for system G. Then, $\|\mathsf{G}_{\mathrm{cl}}\|_{\mathcal{H}_\infty} < \gamma$ is satisfied if there exist a unique matrix $\mathbf{Y} \succeq 0$ and controller $\mathbf{K}(\mathbf{h})$ satisfying the following algebraic Riccati equation [59]:

$$\begin{aligned}&\mathbf{A}_{\mathrm{cl}}(\mathbf{h})\mathbf{Y} + \mathbf{Y}\mathbf{A}_{\mathrm{cl}}(\mathbf{h})^\top + \gamma^{-1}\mathbf{B}_{\mathrm{cl}}(\mathbf{h})\mathbf{B}_{\mathrm{cl}}(\mathbf{h})^\top + \\ &\gamma^{-1}\mathbf{M}(\mathbf{Y},\mathbf{h},\gamma)^\top\mathbf{R}(\mathbf{h},\gamma)^{-1}\mathbf{M}(\mathbf{Y},\mathbf{h},\gamma) = 0,\end{aligned} \tag{31}$$

where the matrix functions $\mathbf{M} : \mathbb{S}_{n_x} \times \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}^{n_z \times n_x}$ and $\mathbf{R} : \mathbb{R}^l \times \mathbb{R} \to \mathbb{S}_{n_z}$ are defined as,

$$\mathbf{R}(\mathbf{h},\gamma) \triangleq \boldsymbol{I} - \gamma^{-2}\mathbf{D}_{\mathrm{cl}}(\mathbf{h})\mathbf{D}_{\mathrm{cl}}(\mathbf{h})^\top, \tag{32a}$$

$$\mathbf{M}(\mathbf{Y},\mathbf{h},\gamma) \triangleq \mathbf{C}_{\mathrm{cl}}(\mathbf{h})\mathbf{Y} + \gamma^{-1}\mathbf{D}_{\mathrm{cl}}(\mathbf{h})\mathbf{B}_{\mathrm{cl}}(\mathbf{h})^\top. \tag{32b}$$

The existence of such solution is guaranteed if there is $\mathbf{Q} \succ \mathbf{Y} \succeq 0$ and $\mathbf{K}(\mathbf{h})$ such that

$$\begin{aligned}&\mathbf{A}_{\mathrm{cl}}(\mathbf{h})\mathbf{Q} + \mathbf{Q}\mathbf{A}_{\mathrm{cl}}(\mathbf{h})^\top + \gamma^{-1}\mathbf{B}_{\mathrm{cl}}(\mathbf{h})\mathbf{B}_{\mathrm{cl}}(\mathbf{h})^\top + \\ &\gamma^{-1}\mathbf{M}(\mathbf{Q},\mathbf{h},\gamma)^\top\mathbf{R}(\mathbf{h},\gamma)^{-1}\mathbf{M}(\mathbf{Q},\mathbf{h},\gamma) \prec 0.\end{aligned} \tag{33}$$

Therefore, we can use Schur complement to form the $\mathcal{H}_\infty$ control design problem as,

$$\underset{\substack{\mathbf{Q}\in\mathbb{S}_{n_x},\gamma\in\mathbb{R}\\ \mathbf{h}\in\mathbb{R}^l}}{\text{minimize}} \quad \gamma \tag{34a}$$

$$\text{subject to} \quad g_{\mathrm{LMI}}(\mathbf{Q},\gamma) + g_{\mathrm{BMI}}(\mathbf{Q},\mathbf{h}) \preceq 0, \tag{34b}$$

where the matrix functions $g_{\mathrm{LMI}} : \mathbb{S}_{n_x} \times \mathbb{R} \to \mathbb{S}_{2n_x+n_w+n_z}$ and $g_{\mathrm{BMI}} : \mathbb{S}_{n_x} \times \mathbb{R}^l \to \mathbb{S}_{2n_x+n_w+n_z}$ are defined as,

$$g_{\mathrm{LMI}}(\mathbf{Q},\gamma) \triangleq \begin{bmatrix} -\mathbf{Q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{A}\mathbf{Q}+\mathbf{Q}\mathbf{A}^\top & (\mathbf{C}_1\mathbf{Q})^\top & \mathbf{B}_1 \\ * & * & -\gamma\boldsymbol{I} & \mathbf{D}_{11} \\ * & * & * & -\gamma\boldsymbol{I} \end{bmatrix}, \tag{35a}$$

$$g_{\mathrm{BMI}}(\mathbf{Q},\mathbf{h}) \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}\mathbf{Q}+(\mathbf{B}\mathbf{K}(\mathbf{h})\mathbf{C}\mathbf{Q})^\top & (\mathbf{D}_{12}\mathbf{K}(\mathbf{h})\mathbf{C}\mathbf{Q})^\top & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} \\ * & * & * & \mathbf{0} \end{bmatrix}. \tag{35b}$$

Proposition 2. *Assume that* $\mathbf{D}_{11} \neq \mathbf{0}$ *and* $\mathbf{B}_1\mathbf{B}_1^\top \succ 0$. *If* $(\overset{*}{\mathbf{Q}}, \overset{*}{\mathbf{h}}, \overset{*}{\gamma})$ *is an optimal solution for problem (34a)–(34b), then* $\mathbf{K}(\overset{*}{\mathbf{h}})$ *is the optimal* $\mathcal{H}_\infty$ *static output-feedback controller gain for plant* G.

Proof. From the BMI constraint (34b), it can be easily verified that the assumption $\mathbf{D}_{11} \neq \mathbf{0}$ concludes $\overset{*}{\gamma} > 0$. Moreover, matrix $\overset{*}{\mathbf{Q}}$ is positive-definite and satisfies the inequality (33), which certifies that $\mathbf{K}(\overset{*}{\mathbf{h}})$ is stabilizer. On the other hand, $\overset{*}{\gamma}$ ($\|\mathsf{G}_{\mathrm{cl}}\|_{\mathcal{H}_\infty} < \overset{*}{\gamma}$) is minimized through (34a)–(34b). □

Observe that (35a) is a BMI constraints, due to the presence of the matrix product $\mathbf{Q}\mathbf{B}\mathbf{K}(\mathbf{h})$. Therefore, we can cast (28a)–(28b) as a BMI optimization problem of form (1a)–(1c), in terms of the vector

$$\bar{\boldsymbol{x}} \triangleq [\mathbf{Q}(:)^\top, \mathbf{h}^\top, \gamma]^\top \in \mathbb{R}^{\bar{n}} \tag{36}$$

where $\bar{n} = \binom{n_x}{2} + l + 1$.

In the next section, we use Algorithm 1 to solve the problem of optimal control design as well as some small-scale BMI instances.

VII. Numerical Results

This section tests the effectiveness of Algorithm 1 through extensive experiments on benchmark control plants from COMPl$_{\mathrm{e}}$ib [59]. The test cases cover a variety of applications, such as aircraft models (AC), academic test problems (NN), and decentralized interconnected systems (DIS), etc. We investigate the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controller design problems for plants that are inherently static output-feedback stabilizable.

We use the HIFOO [107], [108] and the PENBMI [31] packages as competing solvers. The HIFOO is a publicly available MATLAB package which is based on a two-stage method for solving fixed order $\mathcal{H}_2$ and $\mathcal{H}_\infty$ output-feedback controller design problems. The first stage relies on the standard Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm, and the second stage is based on random gradient sampling. The PENBMI package is a commercial local optimization solver, that is able to handle general BMI constrained problems with quadratic objectives. In our experiments, all the solvers are initialized with zero input. Other parameters used in HIFOO and PENBMI are set to default values. All the experiments are performed in MATLAB 2020a on a desktop computer with a 4-core 3.6GHz CPU and 32GB RAM. MOSEK v8.1 [109] is used through CVX to solve the resulting convex programs.

The reminder of this section offers detailed discussion of our experiments on centralized and fully decentralized controller design problems.

A. Case Study I: Centralized Controller

In this scenario, we use Algorithm 1 to find unstructured static output-feedback controllers that stabilize the control plant G and minimizes the norm of the closed-loop system. This controller is allowed to use the entire measurements to generate the control decisions. Numerical results for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers are reported in Table I and Table II, respectively. In the tables, the first and the second columns contain the model names and their corresponding open-loop norms. These norms are computed based on the following system:

$$\mathsf{G}_{\mathrm{ol}} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_{11} \end{bmatrix}.$$

The subsequent sections in Tables I and II include the results of our sequential approach equipped with SDP and parabolic as well as the results of HIFOO and PENBMI. In the following, we give a brief explanation of the numbers reported in the tables:

- η denotes the choice of penalty parameter in (8a). This parameter is chosen from the set $\{1\times10^i, 2\times10^i, 5\times10^i\}_{i=-2}^{4}$ in all experiments.

TABLE I: Results of centralized $\mathcal{H}_2$ controller design for COMPl$_e$ib models.

Name	$\|\|G_{ol}\|\|_2$	SDP						Parabolic						Competitors	
		η	t	k_f	obj$_f$	k_p	obj$_p$	η	t	k_f	obj$_f$	k_p	obj$_p$	HIFOO	PENBMI
AC1	Inf	1e0	0.19	76	0.038	131	0.034	1e0	0.15	102	0.037	152	0.034	-	0.360
AC2	Inf	1e0	0.17	76	0.038	131	0.034	1e0	0.17	102	0.037	152	0.034	0.050	0.300
AC4	Inf	1e4	0.21	1	11.026	1	11.026	1e4	0.17	1	11.026	1	11.026	-	11.014
AC6	24.606	5e1	0.19	29	2.895	34	2.883	1e2	0.18	74	2.869	75	2.868	3.798	-
AC7	Inf	1e2	0.23	39	0.051	81	0.048	1e2	0.19	61	0.053	250	0.052	0.052	1.184
AC15	176.455	1e0	0.18	27	2.554	37	1.908	2e0	0.17	25	2.669	76	1.776	12.612	353.728
AC17	10.265	1e-1	0.17	23	1.592	32	1.560	1e4	0.18	6	2.246	6	2.246	12.298	1.534
NN2	Inf	1e0	0.16	3	1.206	10	1.189	1e0	0.16	11	1.189	18	1.189	1.565	1.189
NN4	5.563	1e1	0.17	8	2.062	33	1.928	5e1	0.16	10	2.159	47	1.964	1.875	1.832
NN8	5.922	-	-	-	-	-	-	1e1	0.16	5	1.772	36	1.596	2.279	1.510
NN11	0.142	2e0	0.96	1	0.153	8	0.142	2e0	0.48	2	0.142	3	0.142	0.118	0.149
NN15	Inf	5e1	0.20	66	0.283	180	0.069	5e1	0.18	66	0.286	195	0.073	0.049	0.000
NN16	Inf	2e2	1.06	5	0.134	85	0.125	1e0	0.35	15	0.119	30	0.119	0.291	-
DIS1	5.149	5e0	0.23	3	1.761	156	0.996	1e0	0.18	214	0.820	214	0.820	2.660	-
DIS3	11.653	5e0	0.17	21	1.633	89	1.102	5e0	0.16	31	1.553	92	1.097	1.839	303.850
AGS	7.041	1e3	0.38	217	7.057	226	7.057	-	-	-	-	-	-	6.995	6.973
PSM	3.847	2e-1	0.18	23	0.242	250	0.072	5e-1	0.15	10	0.496	250	0.091	1.503	0.004
BDT1	0.039	5e-1	0.27	74	0.016	250	0.005	1e0	0.20	206	0.013	250	0.009	0.010	0.006

TABLE II: Results of centralized $\mathcal{H}_\infty$ controller design for COMPl$_e$ib models.

Name	$\|\|G_{ol}\|\|_\infty$	SDP						Parabolic						Competitors	
		η	t	k_f	obj$_f$	k_p	obj$_p$	η	t	k_f	obj$_f$	k_p	obj$_p$	HIFOO	PENBMI
AC1	2.167	5e-1	0.19	14	0.000	250	0.000	5e-1	0.17	14	0.000	110	0.000	0.000	0.008
AC2	2.167	1e3	0.18	83	0.602	250	0.296	5e2	0.16	250	0.237	250	0.237	0.111	0.118
AC4	69.990	1e0	0.18	2	70.078	8	69.990	1e0	0.18	2	70.078	11	69.990	0.935	-
AC6	391.782	5e2	0.21	174	6.584	250	4.410	-	-	-	-	-	-	4.113	4.113
AC7	0.042	1e0	0.22	20	0.000	110	0.000	1e0	0.19	21	0.000	173	0.000	0.064	-
AC15	2.4e3	5e2	0.20	165	39.945	250	20.250	-	-	-	-	-	-	16.865	15.168
AC17	30.823	1e4	0.19	26	79.491	250	7.748	1e4	0.19	34	71.867	250	7.640	16.639	14.855
NN2	Inf	1e0	0.17	17	2.224	47	2.221	5e0	0.15	5	2.769	16	2.222	2.220	2.221
NN4	31.043	5e0	0.17	18	1.792	67	1.416	5e0	0.16	32	1.551	75	1.411	1.369	1.358
NN8	46.508	5e3	0.18	6	59.946	250	3.585	5e3	0.17	12	43.637	250	3.587	3.387	-
NN11	0.170	5e0	2.26	48	0.460	180	0.169	5e0	1.37	179	0.193	207	0.169	0.107	0.124
NN15	Inf	1e-2	0.19	109	0.278	250	0.130	1e-2	0.17	152	0.232	250	0.134	0.098	0.098
NN16	6.4e14	1e3	0.88	56	0.575	81	0.559	-	-	-	-	-	-	1.012	-
DIS1	17.320	1e1	0.45	54	4.574	91	4.286	2e1	0.29	99	4.626	104	4.564	4.182	-
DIS3	32.069	1e1	0.30	73	2.613	116	1.302	2e2	0.19	37	5.876	215	1.350	1.341	1.275
AGS	8.182	2e3	0.35	144	8.872	188	8.192	1e4	0.27	209	10.765	250	9.486	8.173	8.173
PSM	4.232	5e0	0.17	20	1.185	41	0.921	1e1	0.16	25	1.231	52	0.921	0.920	0.920
BDT1	5.142	1e-2	0.84	77	0.565	250	0.311	5e-2	0.53	202	0.523	250	0.431	0.266	0.266

- t denotes the average running time to solve each round of penalized convex relaxation in Algorithm 1.
- k_f denotes the number of rounds required to obtain a feasible point for the original BMI (i.e., the first round whose resulting point satisfies $\boldsymbol{C} = \boldsymbol{Y}\boldsymbol{Y}^\top$) and obj$_f$ represents the corresponding objective value (without the penalty term) at round k_f.
- k_p and obj$_p$, respectively, denote the round number at which the stopping criteria is met and the corresponding objective value.

In all of the experiments, we terminate Algorithm 1 if the percentage of objective value improvement between two consecutive rounds is less than 0.1 for $\mathcal{H}_2$ and 0.05 for $\mathcal{H}_\infty$, or if the number of rounds exceeds 250. In cases where $\mathbf{B}_1\mathbf{B}_1^\top$ is not a positive definite matrix, we use matrix $\mathbf{B}_1\mathbf{B}_1^\top + 10^{-5} \times \boldsymbol{I}$ as the alternative.

Given $\overset{*}{\mathrm{h}}$ as the optimal solution of either (28a) – (28b) or (34a) – (34b), we call $\mathrm{K}(\overset{*}{\mathrm{h}})$ a stabilizing controller for the plant G if the real part of all eigenvalues of $\boldsymbol{A}+\mathbf{B}\mathrm{K}(\overset{*}{\mathrm{h}})\boldsymbol{C}$ are smaller than 10^{-5}.

B. Case Study II: Decentralized Controller

This case study is concerned with the design of decentralized controllers. In this scenario, the controller only have access to a subset of measurements to generate control commands. In this experiment, we only consider the models in

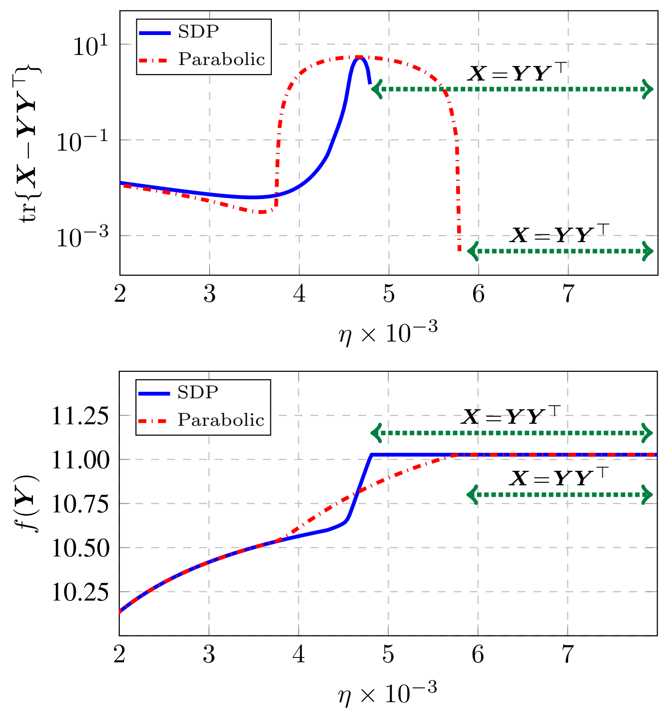


Fig. 4: Sensitivity of the penalized convex relaxations to the choice of penalty parameter η ($\mathcal{H}_2$ centralized controller design for "AC4" model).

which control commands $\mathbf{u}$ and sensor measurements $\mathbf{y}$ are of the same dimensions.

We use Algorithm 1 to find $\mathcal{H}_2$ and $\mathcal{H}_\infty$ static output-feedback controllers with diagonal patterns. The results are reported in Tables III and IV.

As the tables indicate, Algorithm 1 equipped with SDP and parabolic relaxations provide promising performance in both centralized and decentralized cases compared to both PENBMI and HIFOO packages (smaller norm means better performance).

C. Case Study III: Choice of Penalty Parameter η

This case investigates the sensitivity of different penalized relaxations to the choice of regularization parameter η. To this end, we solve $\mathcal{H}_2$-norm static output-feedback controller problem for model "AC4" for a wide range of η. Similar to the Example 1, none of the penalized relaxations provide feasible points for relatively small η (shown in Figure 4). As η increases, the feasibility violation $\mathrm{tr}\{\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^\top\}$ abruptly vanishes once η exceeds a certain threshold. As Figure 4 depicted, all penalized relaxations produce feasible points for a wide range of η values, and within that range, the objective cost is not very sensitive to the choice of η.

VIII. Conclusions

This work introduced convex relaxation techniques for solving a class of optimization problems with piecewise linear objective function and bilinear matrix inequality (BMI) constraints. We proposed a novel computationally efficient convex relaxation, called parabolic relaxation, which relies on convex quadratic constraints to transform BMI problems into polynomial time solvable surrogates. To recover feasible points for the BMI problems, we propose a penalization technique which is applicable to common practice convex relaxations as well. We theoretically proved that the penalized relaxations are guaranteed to provide feasible points for the original BMI problem. Moreover, to improve the quality of feasible points for the BMI problems, we proposed a sequential scheme which is guaranteed to converge under certain assumptions. The performance of the sequential scheme is emprically tested on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design problems for benchmark plants from COMPl$_\mathrm{e}$ib [59]. The numerical results verified that the method achieves promising performance in comparison with the HIFOO and the PENBMI packages.

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TABLE III: Results of fully decentralized $\mathcal{H}_2$ controller design for COMPl$_e$ib models.

Name	$\|G_{ol}\|_2$	SDP						Parabolic						Competitors	
		η	t	k_f	obj_f	k_p	obj_p	η	t	k_f	obj_f	k_p	obj_p	HIFOO	PENBMI
AC1	Inf	-	-	-	-	-	-	1e1	0.14	59	0.084	196	0.047	0.054	0.039
AC2	Inf	-	-	-	-	-	-	1e1	0.15	59	0.084	196	0.047	0.090	0.039
NN2	Inf	2e0	0.13	2	1.220	8	1.189	1e0	0.13	11	1.191	18	1.189	1.565	1.189
NN8	5.9220	5e0	0.14	3	1.864	10	1.839	5e0	0.14	4	1.864	10	1.840	2.365	1.838
NN15	Inf	1e0	0.16	238	0.122	250	0.082	5e1	0.15	67	0.283	183	0.070	0.049	0.000
NN16	Inf	1e0	0.19	5	0.139	54	0.121	5e-1	0.16	13	0.132	75	0.119	0.488	0.119
DIS1	5.1491	1e2	0.19	9	2.111	54	1.783	1e3	0.18	21	2.351	132	1.874	2.991	6982.151
DIS2	Inf	5e0	0.15	24	1.501	119	0.512	5e0	0.14	47	1.368	120	0.487	2.047	0.377
DIS3	11.6538	1e1	0.16	20	2.039	107	1.385	1e1	0.15	43	1.976	95	1.370	2.286	-
AGS	7.0412	2e3	0.29	187	7.149	187	7.149	-	-	-	-	-	-	7.029	7.032
BDT1	0.0397	1e0	0.24	34	0.026	250	0.007	2e0	0.19	119	0.020	250	0.008	0.010	0.000

TABLE IV: Results of fully decentralized $\mathcal{H}_\infty$ controller design for COMPl$_e$ib models.

Name	$\|G_{ol}\|_\infty$	SDP						Parabolic						Competitors	
		η	t	k_f	obj_f	k_p	obj_p	η	t	k_f	obj_f	k_p	obj_p	HIFOO	PENBMI
AC1	2.167	5e0	0.14	37	0.164	250	0.056	1e1	0.15	55	0.162	250	0.064	0.067	0.014
AC2	2.167	5e2	0.15	160	0.422	250	0.319	5e3	0.14	149	0.921	250	0.661	0.661	0.167
NN2	Inf	5e0	0.13	5	2.758	16	2.222	5e0	0.13	5	2.769	16	2.222	2.220	2.221
NN8	46.508	1e3	0.14	6	30.859	194	3.405	2e3	0.14	11	30.752	236	3.461	3.272	3.746
NN15	Inf	1e-2	0.14	107	0.288	250	0.130	1e-2	0.14	152	0.231	250	0.135	0.100	0.100
NN16	6.4e14	2e3	0.20	35	1.143	59	0.959	5e3	0.16	78	1.118	111	0.960	0.956	0.957
DIS1	17.320	1e2	0.18	30	11.591	64	7.175	1e2	0.14	52	10.028	84	7.223	7.165	6.843
DIS3	32.069	5e1	0.16	63	5.831	115	1.689	5e1	0.14	37	4.510	91	1.690	1.655	1.656
AGS	8.182	5e3	0.26	106	13.143	231	8.209	1e3	0.20	202	11.041	250	9.420	8.173	8.173
BDT1	5.142	2e-1	0.28	30	2.704	250	0.575	1e-1	0.23	168	0.737	250	0.551	0.266	0.266

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Appendix

In this section, we provide detailed proofs for Theorems 1, 2, and 3. To this end, we first formulate the following auxiliary non-convex optimization problem:

$$\begin{aligned}
\underset{\substack{\boldsymbol{Y}\in\mathbb{R}^{n\times m}\\ \boldsymbol{t}\in\mathbb{R}^{o}}}{\text{minimize}}\quad & \langle \boldsymbol{C}_0,\boldsymbol{Y}\rangle + \mathbf{1}^\top\boldsymbol{t} + \eta\|\boldsymbol{Y}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 && \text{(37a)}\\
\text{subject to}\quad & \bar{\gamma}_i : +\langle \boldsymbol{C}_i,\boldsymbol{Y}\rangle + c_i \le t_i, \quad i\in\{1,\ldots,o\} && \text{(37b)}\\
& \underline{\gamma}_i : -\langle \boldsymbol{C}_i,\boldsymbol{Y}\rangle - c_i \le t_i, \quad i\in\{1,\ldots,o\} && \text{(37c)}\\
& \boldsymbol{\Lambda} : p(\boldsymbol{Y},\boldsymbol{Y}\boldsymbol{Y}^\top)\preceq 0, && \text{(37d)}\\
& \nu_i : \langle \boldsymbol{W}_i,\boldsymbol{Y}\rangle = 0, \qquad i\in\{1,\ldots,l\}. && \text{(37e)}
\end{aligned}$$

where the quadruplet $(\bar{\boldsymbol{\gamma}},\underline{\boldsymbol{\gamma}},\boldsymbol{\Lambda},\boldsymbol{\nu})\in\mathbb{R}^o\times\mathbb{R}^o\times\mathbb{S}_q\times\mathbb{R}^l$ contains the dual variables associated with the constraints (37b), (37c), (37d), and (37e), respectively. Observe that problem (37a) – (37e) is equivalent to (1a) – (1c) if $\eta=0$. This implies that both problems have the same feasible set. In what follows, we state a few lemmas that will be used later to prove the theorems.

Lemma 1. *Every optimal solution* $(\overset{*}{\boldsymbol{Y}},\overset{*}{\boldsymbol{t}})$ *of the problem* (37a) – (37e) *satisfies:*

$$0\le \|\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}} - d_{\mathcal{F}}(\check{\boldsymbol{Y}}) \le \eta^{-1}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}} \tag{38}$$

where $d_{\mathcal{F}}$ *is defined in* (14)*.*

Proof. According to Definition 3, the distance between an arbitrary point $\check{\boldsymbol{Y}}$ and any point in $\mathcal{F}$ is greater than or equal to $d_{\mathcal{F}}(\check{\boldsymbol{Y}})$. This concludes the left side inequality in (38). To prove the right side inequality, let $\boldsymbol{Y}$ be an arbitrary member of $\{\boldsymbol{Y}\in\mathcal{F}\,|\,\|\boldsymbol{Y}-\check{\boldsymbol{Y}}|_{\mathrm{F}} = d_{\mathcal{F}}(\check{\boldsymbol{Y}})\}$. Since $\overset{*}{\boldsymbol{Y}}$ is the minimizer of the optimization problem (37a) – (37e) and due to feasibility of $\boldsymbol{Y}$, one can write:

$$f(\overset{*}{\boldsymbol{Y}}) + \eta\|\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 \le f(\bar{\boldsymbol{Y}}) + \eta\|\bar{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2. \tag{39}$$

Hence,

$$\begin{aligned}
&\eta\|\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 - \|\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}}\\
&\qquad\le \eta\|\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 - \sum_{i=0}^{o}|\langle \boldsymbol{C}_i,\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\rangle| && \text{(40a)}\\
&\qquad\le -f(\check{\boldsymbol{Y}}) + f(\overset{*}{\boldsymbol{Y}}) + \eta\|\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 && \text{(40b)}\\
&\qquad\le -f(\check{\boldsymbol{Y}}) + f(\bar{\boldsymbol{Y}}) + \eta\|\bar{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 && \text{(40c)}\\
&\qquad\le \eta\|\bar{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 + \sum_{i=0}^{o}|\langle \boldsymbol{C}_i,\bar{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\rangle| && \text{(40d)}\\
&\qquad\le \eta\|\bar{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 + \|\bar{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}} && \text{(40e)}
\end{aligned}$$

which concludes the right side of (38). □

The next lemma guarantees that if the initial point $\check{\boldsymbol{Y}}$ satisfies the MFCQ regularity condition, then under some assumptions, the MFCQ condition is satisfied by every optimal point of the problem (37a) – (37e) as well.

Lemma 2. *Consider an arbitrary point* $\check{\boldsymbol{Y}}\in\mathbb{R}^{n\times m}$. *Every optimal solution* $(\overset{*}{\boldsymbol{Y}},\overset{*}{\boldsymbol{t}})$ *of the problem* (37a) – (37e) *satisfies*

$$s(\overset{*}{\boldsymbol{Y}})\ge s(\check{\boldsymbol{Y}}) - 2(\alpha+\beta+\beta\|\check{\boldsymbol{Y}}\|)\|\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}} - \beta\|\overset{*}{\boldsymbol{Y}}-\check{\boldsymbol{Y}}\|_{\mathrm{F}}^2. \tag{41}$$

Proof. Based on the definition of s, there exist $\check{\boldsymbol{Z}} \in \mathbb{R}^{n\times m}$ such that $\check{\delta}^2 + \|\check{\boldsymbol{Z}}\|_{\mathrm{F}}^2 \leq 1$, $\langle \boldsymbol{W}_i, \check{\boldsymbol{Z}}\rangle = 0$ for every $i \in \{1,\dots,l\}$, and $s(\check{\boldsymbol{Y}})$ is equal to

$$\lambda_{\min}\{(1-\check{\delta})\boldsymbol{P} - p(\check{\delta}\check{\boldsymbol{Y}} + \check{\boldsymbol{Z}}, \check{\delta}\check{\boldsymbol{Y}}\check{\boldsymbol{Y}}^\top + \check{\boldsymbol{Z}}\check{\boldsymbol{Y}}^\top + \check{\boldsymbol{Y}}\check{\boldsymbol{Z}}^\top)\}. \quad (42)$$

On the other hand,

$$\begin{aligned}
& s(\overset{*}{\boldsymbol{Y}}) \geq \\
& \lambda_{\min}\{(1-\check{\delta})\boldsymbol{P} - p(\check{\delta}\overset{*}{\boldsymbol{Y}} + \check{\boldsymbol{Z}},\ \check{\delta}\overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Y}}{}^\top + \check{\boldsymbol{Z}}\overset{*}{\boldsymbol{Y}}{}^\top + \check{\boldsymbol{Y}}\overset{*}{\boldsymbol{Z}}{}^\top)\} \geq \\
& \lambda_{\min}\{(1-\check{\delta})\boldsymbol{P} - p(\check{\delta}\check{\boldsymbol{Y}} + \check{\boldsymbol{Z}},\ \check{\delta}\check{\boldsymbol{Y}}\check{\boldsymbol{Y}}^\top + \check{\boldsymbol{Z}}\check{\boldsymbol{Y}}^\top + \check{\boldsymbol{Y}}\check{\boldsymbol{Z}}^\top)\} - \\
& \lambda_{\max}\{-\boldsymbol{P} + p(\check{\delta}(\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}),\ \ \check{\delta}(\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}})(\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}})^\top + \\
& \qquad (\check{\delta}\check{\boldsymbol{Y}} + \check{\boldsymbol{Z}})(\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}})^\top + (\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}})(\check{\delta}\check{\boldsymbol{Y}} + \check{\boldsymbol{Z}})^\top)\} \geq \\
& s(\check{\boldsymbol{Y}}) - 2(\alpha|\check{\delta}| + \beta\|\check{\delta}\check{\boldsymbol{Y}} + \check{\boldsymbol{Z}}\|_{\mathrm{F}})\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}} - \beta|\check{\delta}|\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 \geq \\
& s(\check{\boldsymbol{Y}}) - 2(\alpha + \beta + \beta\|\check{\boldsymbol{Y}}\|_{\mathrm{F}})\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}} - \beta\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}}^2 \qquad (43)
\end{aligned}$$

which concludes (41). □

The next lemma guarantees the existence of Lagrange multipliers corresponding to optimal solutions of the problem (37a) – (37e).

Lemma 3. *Let $\check{\boldsymbol{Y}} \in \mathbb{R}^{n\times m}$ be an initial point that satisfies*

$$\check{d} \triangleq \frac{s(\check{\boldsymbol{Y}})}{3\kappa + 2(\alpha + \beta + \beta\|\check{\boldsymbol{Y}}\|_{\mathrm{F}}) + \sqrt{\beta s(\check{\boldsymbol{Y}})}} - d_{\mathcal{F}}(\check{\boldsymbol{Y}}) > 0. \quad (44)$$

If

$$\eta > \check{d}^{-1}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}}, \quad (45)$$

then for every primal optimal pair $(\overset{*}{\boldsymbol{Y}}, \overset{*}{\boldsymbol{t}})$ of (37a) – (37e), there exists Lagrange multipliers $(\overset{*}{\underline{\boldsymbol{\gamma}}}, \overset{*}{\bar{\boldsymbol{\gamma}}}, \overset{*}{\boldsymbol{\Lambda}}, \overset{*}{\boldsymbol{\nu}}) \in \mathbb{R}^o \times \mathbb{R}^o \times \mathbb{S}_q \times \mathbb{R}^l$ that satisfy the following Karush–Kuhn–Tucker (KKT) conditions

$$\begin{aligned}
& \boldsymbol{C}_0 + 2\eta(\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}) + \sum_{i=1}^{o}(\overset{*}{\bar{\gamma}}_i - \overset{*}{\underline{\gamma}}_i)\boldsymbol{C}_i \\
& \quad + [\langle \boldsymbol{Q}_{ij}, \overset{*}{\boldsymbol{\Lambda}}\rangle]_{ij} + 2[\langle \boldsymbol{R}_{ij}, \overset{*}{\boldsymbol{\Lambda}}\rangle]_{ij}\overset{*}{\boldsymbol{Y}} + \sum_{i=1}^{l}\nu_i\boldsymbol{W}_i = 0 && (46a) \\
& 1 - \overset{*}{\bar{\gamma}}_i - \overset{*}{\underline{\gamma}}_i = 0, \qquad l \in \{1,\dots,o\} && (46b) \\
& \overset{*}{\bar{\gamma}}_i(+\langle \boldsymbol{C}_i, \boldsymbol{Y}\rangle + c_i - t_i) = 0, \qquad l \in \{1,\dots,o\} && (46c) \\
& \overset{*}{\underline{\gamma}}_i(-\langle \boldsymbol{C}_i, \boldsymbol{Y}\rangle - c_i - t_i) = 0, \qquad l \in \{1,\dots,o\} && (46d) \\
& \overset{*}{\boldsymbol{\Lambda}} p(\overset{*}{\boldsymbol{Y}}, \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Y}}{}^\top) = 0 && (46e) \\
& \overset{*}{\underline{\gamma}}_i \geq 0, \qquad l \in \{1,\dots,o\} && (46f) \\
& \overset{*}{\bar{\gamma}}_i \geq 0, \qquad l \in \{1,\dots,o\} && (46g) \\
& \overset{*}{\boldsymbol{\Lambda}} \succeq 0, && (46h)
\end{aligned}$$

and

$$\frac{\mathrm{tr}\{\overset{*}{\boldsymbol{\Lambda}}\}}{\eta} \leq \kappa^{-1}. \quad (47)$$

Proof. Consider an arbitrary optimal point $(\overset{*}{\boldsymbol{Y}}, \overset{*}{\boldsymbol{t}})$. According to Lemma 1 and the assumptions (44) and (45), we have

$$\begin{aligned}
\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}} &\leq d_{\mathcal{F}}(\check{\boldsymbol{Y}}) + \eta^{-1}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}} && (48a) \\
&< d_{\mathcal{F}}(\check{\boldsymbol{Y}}) + \check{d} && (48b) \\
&< \frac{s(\check{\boldsymbol{Y}})}{3\kappa + 2(\alpha + \beta + \beta\|\check{\boldsymbol{Y}}\|_{\mathrm{F}}) + \sqrt{\beta s(\check{\boldsymbol{Y}})}}. && (48c)
\end{aligned}$$

Hence, according to Lemma 2, $s(\overset{*}{\boldsymbol{Y}}) > 0$ and $\overset{*}{\boldsymbol{Y}}$ satisfies the MFCQ condition and there exists $(\overset{*}{\underline{\boldsymbol{\gamma}}}, \overset{*}{\bar{\boldsymbol{\gamma}}}, \overset{*}{\boldsymbol{\Lambda}}, \overset{*}{\boldsymbol{\nu}})$ that satisfies the KKT conditions (46a) – (46h). Now, in order to prove (47), consider the inner product (46a) and $\overset{*}{\boldsymbol{Z}}$:

$$\begin{aligned}
& \langle \boldsymbol{C}_0, \overset{*}{\boldsymbol{Z}}\rangle + 2\eta\langle \overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}, \overset{*}{\boldsymbol{Z}}\rangle + \sum_{i=1}^{o}(\overset{*}{\bar{\gamma}}_i - \overset{*}{\underline{\gamma}}_i)\langle \boldsymbol{C}_i, \overset{*}{\boldsymbol{Z}}\rangle + \\
& \langle [\langle \boldsymbol{Q}_{ij}, \overset{*}{\boldsymbol{\Lambda}}\rangle]_{ij} + 2[\langle \boldsymbol{R}_{ij}, \overset{*}{\boldsymbol{\Lambda}}\rangle]_{ij}\overset{*}{\boldsymbol{Y}}, \overset{*}{\boldsymbol{Z}}\rangle + \sum_{i=1}^{l}\nu_i\langle \boldsymbol{W}_i, \overset{*}{\boldsymbol{Z}}\rangle = 0
\end{aligned}$$

On the other hand, based on the definition of s, we have:

$$\begin{aligned}
& s(\overset{*}{\boldsymbol{Y}})\mathrm{tr}\{\overset{*}{\boldsymbol{\Lambda}}\} \\
& = \lambda_{\min}\{-p(\overset{*}{\boldsymbol{Y}} + \overset{*}{\boldsymbol{Z}}, \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Y}}{}^\top + \overset{*}{\boldsymbol{Z}}\overset{*}{\boldsymbol{Y}}{}^\top + \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Z}}{}^\top)\}\mathrm{tr}\{\overset{*}{\boldsymbol{\Lambda}}\} && (49a) \\
& \leq \langle -p(\overset{*}{\boldsymbol{Y}} + \overset{*}{\boldsymbol{Z}}, \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Y}}{}^\top + \overset{*}{\boldsymbol{Z}}\overset{*}{\boldsymbol{Y}}{}^\top + \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Z}}{}^\top), \overset{*}{\boldsymbol{\Lambda}}\rangle && (49b) \\
& = \langle \boldsymbol{P} - p(\overset{*}{\boldsymbol{Y}}, \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Y}}{}^\top) - p(\overset{*}{\boldsymbol{Z}}, \overset{*}{\boldsymbol{Z}}\overset{*}{\boldsymbol{Y}}{}^\top + \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Z}}{}^\top), \overset{*}{\boldsymbol{\Lambda}}\rangle && (49c) \\
& = \langle \boldsymbol{P} - p(\overset{*}{\boldsymbol{Z}}, \overset{*}{\boldsymbol{Z}}\overset{*}{\boldsymbol{Y}}{}^\top + \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Z}}{}^\top), \overset{*}{\boldsymbol{\Lambda}}\rangle && (49d) \\
& = -\big\langle [\langle \boldsymbol{Q}_{ij}, \overset{*}{\boldsymbol{\Lambda}}\rangle]_{ij} + 2[\langle \boldsymbol{R}_{ij}, \overset{*}{\boldsymbol{\Lambda}}\rangle]_{ij}\overset{*}{\boldsymbol{Y}}, \overset{*}{\boldsymbol{Z}}\big\rangle && (49e) \\
& = \langle \boldsymbol{C}_0, \overset{*}{\boldsymbol{Z}}\rangle + 2\eta\langle \overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}, \overset{*}{\boldsymbol{Z}}\rangle + \sum_{i=1}^{o}(\overset{*}{\bar{\gamma}}_i - \overset{*}{\underline{\gamma}}_i)\langle \boldsymbol{C}_i, \overset{*}{\boldsymbol{Z}}\rangle && (49f) \\
& \leq 2\eta\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}} + \sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}}. && (49g)
\end{aligned}$$

Hence,

$$\frac{\mathrm{tr}\{\overset{*}{\boldsymbol{\Lambda}}\}}{\eta} \leq \frac{2\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}} + \eta^{-1}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}}}{s(\overset{*}{\boldsymbol{Y}})}. \quad (50)$$

which according to Lemma 2, concludes:

$$\frac{\mathrm{tr}\{\overset{*}{\boldsymbol{\Lambda}}\}}{\eta} \leq \frac{2\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}} + \eta^{-1}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}}}{s(\check{\boldsymbol{Y}}) - 2(\alpha + \beta + \beta\|\check{\boldsymbol{Y}}\|_{\mathrm{F}})\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}} - \beta\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}}^2}. \quad (51)$$

Now, substituting $\|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|_{\mathrm{F}}$ and $\eta^{-1}\sum_{i=0}^{o}\|\boldsymbol{C}_i\|_{\mathrm{F}}$ with the upper bound from (48c), proves (47). □

Next, the upper bound (47) will be used to show that the problem (37a) – (37e) can be relaxed to (8a) – (8d) with no effect on the solution.

Proof of Theorem 2. In order to prove the theorem, it suffices to construct a strictly diagonally dominant Lagrange multiplier associated with the constraint (8d) to certify that $\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{Y}^\top$. Let $(\overset{*}{\boldsymbol{Y}}, \overset{*}{\boldsymbol{t}})$ and $(\overset{*}{\underline{\boldsymbol{\gamma}}}, \overset{*}{\bar{\boldsymbol{\gamma}}}, \overset{*}{\boldsymbol{\Lambda}}, \overset{*}{\boldsymbol{\nu}})$ represent a pair of primal and dual solutions for the problem (37a) – (37e) whose existence is guaranteed by Lemma 3. We claim that the matrix

$$\overset{*}{\boldsymbol{\Omega}} \triangleq \boldsymbol{I} + \eta^{-1}[\langle \boldsymbol{R}_{ij}, \overset{*}{\boldsymbol{\Lambda}}\rangle]_{ij} \quad (52)$$

is strictly diagonally-dominant and therefore serves as a Lagrange multiplier for (8d) which is an immediate consequence of (47).

To prove the uniqueness of solution, assume by contradiction that the penalized relaxation (8a) – (8d) has two distinct solutions $(\overset{*}{\boldsymbol{Y}}_1, \overset{*}{\boldsymbol{X}}_1)$ and $(\overset{*}{\boldsymbol{Y}}_2, \overset{*}{\boldsymbol{X}}_2)$. Due to convexity of (8a) – (8d), the point $((\overset{*}{\boldsymbol{Y}}_1 + \overset{*}{\boldsymbol{Y}}_2)/2, (\overset{*}{\boldsymbol{X}}_1 + \overset{*}{\boldsymbol{X}}_2)/2)$ is a solution as well. Hence,

$$\frac{\overset{*}{\boldsymbol{X}}_1 + \overset{*}{\boldsymbol{X}}_2}{2} = \frac{(\overset{*}{\boldsymbol{Y}}_1 + \overset{*}{\boldsymbol{Y}}_2)(\overset{*}{\boldsymbol{Y}}_1 + \overset{*}{\boldsymbol{Y}}_2)^\top}{4} \tag{53a}$$

$$\Rightarrow \quad \frac{\overset{*}{\boldsymbol{Y}}_1 \overset{*}{\boldsymbol{Y}}_1^\top + \overset{*}{\boldsymbol{Y}}_2 \overset{*}{\boldsymbol{Y}}_2^\top}{2} = \frac{(\overset{*}{\boldsymbol{Y}}_1 + \overset{*}{\boldsymbol{Y}}_2)(\overset{*}{\boldsymbol{Y}}_1 + \overset{*}{\boldsymbol{Y}}_2)^\top}{4} \tag{53b}$$

$$\Rightarrow \quad (\overset{*}{\boldsymbol{Y}}_1 - \overset{*}{\boldsymbol{Y}}_2)(\overset{*}{\boldsymbol{Y}}_1 - \overset{*}{\boldsymbol{Y}}_2)^\top = 0 \tag{53c}$$

which contradicts the original assumption that $(\overset{*}{\boldsymbol{Y}}_1, \overset{*}{\boldsymbol{X}}_1)$ and $(\overset{*}{\boldsymbol{Y}}_2, \overset{*}{\boldsymbol{X}}_2)$ are distinct. □

Proof of Theorem 1. The exactness of relaxation is an immediate consequence of Theorem 2 for $d_{\mathcal{F}}(\check{\boldsymbol{Y}}) = 0$. Due to optimality of $\overset{*}{\boldsymbol{Y}}$ and feasibility of $\check{\boldsymbol{Y}}$, we have:

$$f(\overset{*}{\boldsymbol{Y}}) + \eta \times \|\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}}\|^2 \leq f(\check{\boldsymbol{Y}}) \tag{54}$$

which proves $f(\overset{*}{\boldsymbol{Y}}) \leq f(\check{\boldsymbol{Y}})$. □

Lemma 4. *Consider a set $\mathcal{Q} \in \mathcal{F}$ for which there exists $\tilde{s}, \tilde{y} > 0$ such that $s(\boldsymbol{Y}) > \tilde{s}$ and $\|\boldsymbol{Y}\|_{\mathrm{F}} < \tilde{y}$ for every $\boldsymbol{Y} \in \mathcal{Q}$. For every η that satisfies* (17)*, define $h_{\mathcal{Q},\eta} : \mathcal{Q} \to \mathcal{F}$ as the function mapping any initial point $\check{\boldsymbol{Y}} \in \mathcal{Q}$ in the problem* (37a) – (37e) *to its unique solution $\overset{*}{\boldsymbol{Y}}$ (whose existence and uniqueness is guaranteed by Theorem 1). Then $h_{\mathcal{Q},\eta}$ is continuous throughout $\mathcal{Q}$.*

Proof. According to Berge's maximum theorem, $h_{\mathcal{Q},\eta}$ is upper hemicontinuous. However, according to Theorem 1 it is a function and therefore, it is continuous. □

Proof of Theorem 3. Let $\{\check{\boldsymbol{Y}}^k\}_{k=0}^{\infty}$ represent the sequence generated by Algorithm 1. Assume by induction that $\boldsymbol{Y}^k \in \check{\mathcal{F}}$ and let $(\overset{*}{\boldsymbol{Y}}, \overset{*}{\boldsymbol{Y}}\overset{*}{\boldsymbol{Y}}^\top)$ be the solution of the problem (8a) – (8d) with $\check{\boldsymbol{Y}} = \boldsymbol{Y}^k$. According to Theorem 1, we have $f(\overset{*}{\boldsymbol{Y}}) \leq f(\boldsymbol{Y}^k)$ and $\boldsymbol{Y}^{k+1} = \overset{*}{\boldsymbol{Y}} \in \check{\mathcal{F}}$. As a consequence, the sequence $\{f(\boldsymbol{Y}^k)\}_{k=0}^{\infty}$ is monotonically non-increasing and convergent. On the other hand, due to optimality of $\boldsymbol{Y}^{k+1}$ and feasibility of $\boldsymbol{Y}^k$, we have:

$$\|\boldsymbol{Y}^{k+1} - \boldsymbol{Y}^k\|_{\mathrm{F}}^2 \leq \eta^{-1}(f(\boldsymbol{Y}^k) - f(\boldsymbol{Y}^{k+1})) \tag{55}$$

which implies that the sequence $\{\boldsymbol{Y}^k\}_{k=0}^{\infty}$ is convergent to a point $\boldsymbol{Y}^\infty \in \check{\mathcal{F}}$.

Define $h_{\check{\mathcal{F}},\eta} : \check{\mathcal{F}} \to \check{\mathcal{F}}$ as the function mapping any initial point $\check{\boldsymbol{Y}}$ of the problem (8a) – (8d) to its unique solution. According to Lemma 4, $h_{\check{\mathcal{F}},\eta}$ is continuous and therefore:

$$h_{\check{\mathcal{F}},\eta}(\boldsymbol{Y}^\infty) = \boldsymbol{Y}^\infty. \tag{56}$$

Now according to Lemma 3, there exists Lagrange multipliers that along with $\boldsymbol{Y}^\infty$ satisfy (46a) – (46h). With the term $2\eta(\overset{*}{\boldsymbol{Y}} - \check{\boldsymbol{Y}})$ vanishing from (46a), the equations (46a) – (46h) boil down to the KKT conditions for the original BMI problem (1a) – (1c). This implies that $\boldsymbol{Y}^\infty$ is a locally optimal solution for the BMI problem (1a) – (1c). □

Mohsen Kheirandishfard is a machine learning scientist at Cognitiv Corp. He received his Ph.D. in computer science from the University of Texas at Arlington in 2019. Prior to that, he obtained his B.Sc. and M.S. degrees, both in Electrical Engineering, from Shiraz University in 2010 and 2013, respectively. His research interests include Deep Learning, Machine Learning, and Optimization.

Fariba Zohrizadeh received her B.Sc. and M.S. degrees in electrical engineering from Shiraz University in 2012 and 2014, and her Ph.D. degree in Computer Science from the University of Texas at Arlington in 2019 respectively. She has worked on different interdisciplinary problems in Machine Learning, Nonlinear Optimization, and Deep Learning. Currently, she is working as an applied research scientist in Expedia Group where she develops machine algorithms for complex and large-scale problems in the marketing domain.

Muhammad Adil is currently working towards the Ph.D. degree in Electrical Engineering at University of Texas at Arlington. He received the B.S and M.S degrees in Electrical Engineering from University of Engineering and Technology and Pakistan Institute of Engineering and Applied Sciences in 2012 and 2014. His research interests include developing numerical algorithms for solving large scale optimization problems.

Ramtin Madani received the Ph.D. degree in electrical engineering from Columbia University, New York, NY, USA, in 2015. He was a Postdoctoral Scholar with the Department of Industrial Engineering and Operations Research at the University of California, Berkeley in 2016. He is an Assistant Professor with the Department of Electrical Engineering Department, University of Texas at Arlington, Arlington, TX, USA. His research interests include developing algorithms for optimization and control with applications in energy.