Vector Spaces

A **vector space** ($\mathbb{V}$, $\mathbb{F}$) is a set of vectors $\mathbb{V}$, a set of scalars $\mathbb{F}$, and two operators that satisfy the following properties:

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ for any $\vec{v},\vec{u},\vec{w} \in \mathbb{V}$.
  - Commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ for any $\vec{v},\vec{u} \in \mathbb{V}$.
  - Additive Identity: There exists an additive identity $\vec{0} \in \mathbb{V}$ such that $\vec{v}+\vec{0} = \vec{v}$ for any $\vec{v} \in \mathbb{V}$.
  - Additive Inverse: For any $\vec{v} \in \mathbb{V}$, there exists $-\vec{v} \in \mathbb{V}$ such that $\vec{v}+(-\vec{v}) = \vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for any $\vec{v} \in \mathbb{V}$, $\alpha,\beta \in \mathbb{F}$.
  - Multiplicative Identity: There exists $1 \in \mathbb{F}$ where $1\cdot\vec{v} = \vec{v}$ for any $\vec{v} \in \mathbb{F}$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$ for any $\alpha \in \mathbb{F}$ and $\vec{u},\vec{v} \in \mathbb{V}$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v}$ for any $\alpha,\beta \in \mathbb{F}$ and $\vec{v} \in \mathbb{V}$.

You have already seen vector spaces before! For example, $(\mathbb{R}^n,\mathbb{R})$ with vector addition and scalar multiplication defined in the previous notes is a vector space — you could show that it satisfies all the properties above.

Bases

We can use a series of vectors to define a vector space. We call this set of vectors a **basis**, which we define formally below:

Definition 7.1 (Basis):

Given a vector space $(\mathbf{V},\mathbf{F})$, a set of vectors $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ is a **basis** of the vector space if it satisfies the following two properties:

- $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n$ are linearly independent vectors
- For any vector $\mathbf{v} \in \mathbb{V}$, there exist scalars $\alpha_1,\alpha_2,\ldots,\alpha_n \in \mathbb{F}$ such that $\mathbf{v} = \alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\ldots+\alpha_n\mathbf{v}_n$.

Intuitively, a basis of a vector space is the minimum set of vectors needed to represent all the vectors in the vector space. If a set of vectors are linearly dependent and "spans" the vector space, it is not a basis because we can remove at least one vector from the set and the resulting set will still span the vector space.

Now the natural question to ask is: given a vector space, is the basis unique? Intuitively, the answer is no because if a set is a basis for the vector space we are considering, we can multiply one of the vectors by a nonzero scalar while keeping the set of vectors linearly independent and its span identical. Alternatively, we can also construct another basis by replacing one of the vectors with the sum of itself and any other vectors in the set.

To illustrate this mathematically, suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for the vector space we are considering. Then

$$\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \tag{1}$$

where $\alpha \neq 0$ is also a basis because just like what we've seen in row operations in Gaussian Elimination, multiplying a row by a nonzero constant does not change the linear independence/dependence of the rows, here multiplying a vector by a nonzero scalars also does not change the linear independence of the set of vectors. In addition, we know that

$$span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}) = span(\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}). \tag{2}$$

We can use a similar argument to show that $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$ is a also basis for the vector space.

Example 7.1 (Vector space $(\mathbb{R}^3, \mathbb{R})$**):** Let's try to find a basis for the vector space $(\mathbb{R}^3, \mathbb{R})$. We want to find a set of vectors that can represent any vector of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $a, b, c \in \mathbb{R}$. One basis could be

$$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}. \tag{3}$$

The set of vectors is linearly independent and we can represent any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in the vector space using the three vectors:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4}$$

Alternatively, we could show that

$$\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\}. \tag{5}$$

is a basis for the vector space.

Now that we have defined bases, we can define the term dimension for vector spaces.

Definition 7.2 (Dimension): The dimension of a vector space is the number of basis vectors.

Example 7.2 (Dimension of $(\mathbb{R}^3, \mathbb{R})$**)**: Previously, we identified a basis

$$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}. \tag{6}$$

for the vector space $(\mathbb{R}^3, \mathbb{R})$. The basis consists of three vectors and hence the dimension of the vector space is equal to three.

Note that a vector space can have many bases, but all its bases must have the same number of vectors in it. We will not prove this rigorously but let's illustrate our arguments. Suppose a basis for the vector space we're considering has n vectors, i.e., the minimum number of vectors we can use to represent all vectors in the vector space is n. Then we can show that any set with less than n vectors cannot be a basis because it does not have enough vectors to span the vector space, i.e. some vectors in the vector space cannot be expressed as a linear combination of the vectors in the set. In addition, we can show that any set with more than n vectors has to be linearly dependent of each other. Combining the two arguments, we have that any other set of vectors that forms a basis for the vector space must have exactly n vectors in it.

We introduced quite a few terminologies in this lecture note, and we'll see how we can connect these with our understanding of matrices in the next lecture note!