Patterns, Probabilities, and People: Making Sense of Quantitative Change in Complex Systems

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Abstract. The learning sciences community has made significant progress in understanding how people think and learn about complex systems. But, less is known about how these understandings relate to the mathematical and quantitative formalisms often used to represent them. In this paper, we make a case for attending to and supporting those connections. We introduce a framework to examine how students connect the behavioral and quantitative aspects of complex systems, and use it to analyze interviews with 11 high school students as they interacted with an agent-based simulation that produces simple exponential-like population growth. While the students were comfortable describing many connections between the simulation's behavior and the quantitative patterns it generated, we found they did not readily describe connections between individual behaviors and patterns of change. Case studies suggest that these missed connections led students to make errors interpreting quantitative patterns in the simulation, but that these difficulties could be resolved by drawing students' attention to the graph of quantitative change featured in the simulation environment, and the underlying rules that generated it. We discuss implications for the design of learning environments, for the study of quantitative reasoning about complex systems, and for the role of mathematical reasoning in complex systems fluency.

Patterns, Probabilities, and People: Making Sense of Quantitative Change in Complex Systems

Complex systems have been a major focus of current learning sciences research (Forrester 1994/2009; NRC, 2012a; Wilensky & Jacobson, in press). This has yielded important insights into how people think and learn about complex systems (Chi, 2005; Jacobson, 2001; Hmelo-Silver & Pfeffer, 2004; Penner, 2000; Wilensky & Resnick, 1999), and how to make complex systems principles more accessible to students (Chi, et al., 2012; Colella, 2000; Hmelo, Holton, & Kolodner, 2000; Wilensky & Stroup, 2000; Wilensky & Reisman, 2006). But most such work focuses on students' understanding of the behavioral aspects of complex systems: their constituent elements and interactions, and how these generate system-level outcomes. Less has focused on whether or how students connect these understandings to quantitative patterns to make sense of the mathematical formalisms that describe such systems.

Exploring how learners make sense of such connections is important both for theoretical and practical reasons. Across the K-12 curriculum, students are expected to use quantitative data and mathematical representations to describe and make predictions about scientific systems and their constituent elements (CCSSI, 2010; NRC, 2012a). Helping learners understand these connections is a major challenge for mathematics and science educators (Ganter, 2001; NRC, 2012b), even more in the context of complex systems where mathematics may be used to measure or represent interconnected, indirect, or counterintuitive phenomena (Goldstone, 2006; Van Dyke Parunak, Savit & Riolo, 1998). Despite these needs and difficulties, many learning environments designed to engage students in thinking about complex systems include mathematical representations such as graphs while taking for granted that students will make sense of them even when their connection to system behavior is not transparent (Chi et al, 2012).

In this study we interviewed 11 high school precalculus students about the connections between the behavioral and quantitative aspects of a NetLogo simulation (Wilensky, 1999). In the simulation, “humans” each had a probabilistic chance to reproduce, approximating an exponential pattern of growth. Drawing from literature in complex systems and math education, we analyze what resources (graphs in the simulation, ideas about reproduction, etc.) students leveraged to make sense of the *individual behaviors*, *group interactions*, *patterns of change,* and *patterns of accumulation* that together described the population system. Our goal was to articulate how students connected these aspects of the simulation, and how to support students in doing so. Our questions were: (1) What resources did students use to describe the quantitative patterns generated by the simulation? (2) What resources did students use to describe connections between the quantitative and behavioral aspects of the simulation? and, (3) What connections did students hesitate or struggle to describe, and how were such struggles resolved?

Our analysis reveals that even when asked exclusively about quantitative patterns, students cited resources that spoke to both quantitative and behavioral aspects of the simulation. Yet although they cited both aspects, many hesitated to describe the specific connections between individual behavior and patterns of change in the simulation. These difficulties led students to make errors similar to those documented in the complex systems thinking literature, but that arose specifically with respect to mathematical representations in the simulation. Case studies further suggest these difficulties could be resolved by drawing students’ attention to the jagged nature of the graph of quantitative change featured in the simulation environment. Our findings suggest that while many studies in the complex systems literature focus on understanding the behavior of complex systems and their qualitative impact on system dynamics, learners may also be able to explore formal mathematical descriptions of those systems with carefully designed supports. Our study has direct implications for the design of learning environments, contributes analytic tools and baseline data for the study of students' quantitative reasoning about complex systems, and illustrates the importance of attending to quantitative and mathematical issues as a key component of complex systems fluency.

Background

Complex systems are dynamic; their behavior, inter-level structure and outcomes of interest unfold over time. While most complex systems education literature focuses on how learners make sense of the behavioral aspects of complex systems and their organization, mathematics also plays an important role in describing such properties (Bar-Yam, 2003; Holland, 2000). Many complex systems are characterized by quantitative patterns of change over time, such as oscillation, escalation, and equilibration (AAAS, 1993; Mitchell, 2009). Because of this, studying how learners make sense of patterns exhibited by complex systems requires attending to learners' thinking about both the behavior of complex systems and the mathematics of change. We situate our work at the intersection of these two literatures. We then describe why computer simulation environments comprise an especially well-suited context for exploring student thinking at this intersection.

Thinking and Learning about Complex Systems

Neither the scientific community nor the educational community has converged on a formal definition of complex systems (Guckenheimer & Ottino, 2008; Holland, 2000; Kolodner, 2006; Wilensky & Jacobson, in press). However, there is general consensus that reasoning about complex systems can be difficult for a number of reasons. They are unpredictable and have a variety of potential outcomes (Chinn & Malhotra, 2002), involve multiple interacting elements which can task working memory (Feltovich, Coulson, & Spiro, 2001; Hmelo-Silver & Azevedo, 2006), and exhibit counterintuitive behavior and relationships since events at one level of a system can have unexpected consequences at another (Casti, 1994; Penner, 2000). Correspondingly, there are a number of perspectives for exploring learners' understanding of complex systems. These include the Structure-Behavior-Function framework which emphasizes the roles and interdependencies of heterogeneous components in a system (Hmelo-Silver & Pfeffer, 2004; Vattam, et al., 2011), mindset theories which focus on learners' attention to the decentralized and stochastic nature of complex systems (Resnick & Wilensky, 1998; Jacobson, 2001), and analytic approaches that explore students' understanding of the different types of causal relationships that exist within a system (Chi, 2005; Perkins & Grotzer, 2005).

Our study adopts an *emergence* based perspective toward complex systems (Bar-Yam, 2003; Chi, 2005; Wilensky & Resnick, 1999). Complex systems exhibit emergence when many entities at a "micro" level interact locally and simultaneously to produce behavior at a global or "macro" level of observation. For example, the movement and collision of molecules in a gas at the micro level collectively create what we observe to be air pressure at the macro level (Holland, 2000; Wilensky, 2003). Emergent phenomena are interesting from a quantitative perspective precisely because of these different levels. Mathematical and quantitative measures can describe the dynamics of these systems at one level of analysis, but those dynamics are indirectly generated by collective elements and interactions that occur at a different level. Moreover, mathematics does not distinguish among these levels. For example, the ideal gas law $PV=nRT$ is used to describe and predict patterns in air pressure (P) by articulating relationships between the number of particles in the gas (n), the volume of their container (V), the temperature of the gas (T). These factors describe macro-level attributes such as the volume of a container, others describe micro-level elements such as the number of particles in the gas; and still others describe emergent effects such as pressure, which results from the frequency of collisions between particles and their container.

Research suggests students' difficulties in making sense of emergence in complex systems stem from a confusion, or "slippage", between these different levels of analysis (Wilensky & Resnick, 1999; Sengupta & Wilensky, 2009). Students may not explicitly consider a system's behavior at more than one level of analysis, or may incorrectly assign behavior at one level to dynamics at a different one. One common example of such "levels confusion" (p. 3) is peoples' understanding of traffic jams. Each car in a traffic jam moves forward, but the jam itself does not move forward; in fact it propagates backward. Even after students understand the different levels of analysis in an emergent system, they may still struggle to understand~~ing~~ the causal relationships that link behaviors at one level to outcomes at another (Penner, 2000). To understand these causal links between levels, students must consider the aggregated and simultaneous effects of *individual behavior*. Research suggests that this might be accomplished by considering how these individual behaviors relate to the net effects of *group interactions* within subsets of entities (Levy & Wilensky, 2008), or an entire collection of entities considered simultaneously (Chi, et al., 2012).

Thinking and Learning About the Mathematics of Change

Mathematics education research suggests that there are two important factors in understanding patterns of quantitative change. First, students need to understand linkages between the particular parameters and relationships highlighted by a mathematical model or set of data, and the corresponding dynamic situation they describe (Keene, 2007; Roth & Bowen, 2003; Thompson, 1994). This helps students interpret mathematical patterns and make inferences or predictions about the underlying system. It also enables them to leverage what they may know about the situation to understand how they should expect quantities or parameters to co-vary (Carlson et al, 2002; Doerr, 2000; White & Mitchelmore, 1996). Envisioning bacteria iteratively splitting, for instance, can help students better understand the structure of exponential growth (Confrey & Smith, 1994; 1995).

Second, students must disentangle which aspects of a system correspond to the *rate of change* of a system, versus which correspond to the *accumulation* or summation of those changes over time. These two ideas are closely intertwined mathematically and conceptually (Piaget, 1946:1971; Schwartz, 1988; Johnson, 2012). For example, a bicyclist that starts from a standstill and steadily increases her speed by 1 mph for every minute she pedals is travelling faster as time goes on, and covering more distance as time goes on. When she decides to slow down, she will cover less distance from moment to moment, but she will still be adding to the total distance she has travelled during her trip. If this bicyclist's total distance travelled were graphed relative to time, the graph's height would always increase relative to time. However, the slope of the graph at different points would reflect relative increases or decreases in speed during the trip. In this case, the *rate of change* is the bicyclist's speed, and the *accumulation* is distance. These quantities and their relationship to one another are the foundation of reasoning about the mathematics of dynamic systems (Kaput, 1994; Nemirovsky, 1994; Stroup, 2002).

Learning with Agent-Based Simulations

One way that educators have successfully engaged learners in making sense of complex systems is through simulation (Clark et al., 2009; Wilensky & Jacobson, in press; Repenning, Ioannidou & Zola, 2000; Hmelo-Silver & Azevedo, 2006; Klopfer & Yoon, 2004). Agent-based simulation environments such as NetLogo (Wilensky, 1999), AgentSheets (Repenning, Ioannidou, & Zola, 2000) and StarLogo TNG (Begel & Klopfer, 2005) are especially well-suited for students to explore how many interacting agents can produce unexpected emergent outcomes.

But while there is evidence that agent-based simulations can help students understand the behavior of complex systems, not much is known about how students understand the mathematical representations that often accompany them. Some studies claim that agent-based simulations are effective because they provide an *alternative* to formal mathematical representations (Goldstone & Janssen, 2005; Sengupta & Wilensky, 2009; Tan & Biswas, 2007; Wilensky & Reisman, 2006), and highlight mechanisms that mathematical formulae may not (Goldstone & Wilensky, 2008). Others indicate that agent-based simulations can help students develop a conceptual grounding for scientific formulae or mathematical patterns, but focus on qualitative trends rather than measurable relationships (Blikstein & Wilensky, 2009; Wilensky, 2003; Wilensky & Stroup, 2000). Still others claim that interacting with agent-based simulations can help students make sense of mathematical ideas such as statistical variation or probability (Wilensky, 1997; Abrahamson & Wilensky, 2005), but do not explore ~~the~~ how these ideas might connect back to a particular mathematical model or scientific context.

The relationship between agent-based simulations and mathematical representations of complex systems is nuanced. Like mathematical models, simulations highlight the quantities, relationships, and changes over time that define a system. Simulations execute and record these relationships as quantitative data, whereas mathematical models encode them symbolically. Moreover, agent-based simulations define relationships in terms of individual behaviors, rather than in terms of aggregate measures like mathematical models (Holland, 2000). These nuanced relationships, combined with the established potential for agent-based simulations to engage students in constructing scientific knowledge, make agent-based simulation an ideal context in which to explore how students connect quantitative and mathematical knowledge to complex systems behavior.

Conceptual Framework: The Calculus of Complex Systems

To make sense of the quantitative change exhibited by complex systems, we bring together the literature on complex systems thinking and reasoning about quantitative change to argue that learners need to understand and construct connections across four interrelated aspects. First, to make sense of the emergent patterns that are characteristic of complex systems, learners need to identify how micro-level *individual behaviors* within the system generate *group interactions* evident at a macro-level of observation. Second, learners must understand how quantitative patterns generated by the system are substantively connected to these individual and group-level behaviors. To build these connections, learners need to distinguish between the *patterns of change* and *patterns of accumulation* that are reflected in a given quantitative pattern, and what each measure illustrates about the complex system's behavior. We call these four aspects (and the connections between them) the *Calculus of Complex Systems (CCS) Framework*.

The CCS Framework highlights individual behaviors, group interactions, patterns of change, and patterns of accumulation as mutually informative *levels of description*. By 'levels', we do not refer to a developmental trajectory for understanding complex systems. Instead, we refer to descriptions that highlight some aspects of a system of interest, such as individual interactions, that constrain (but may not entirely illuminate) aspects at a different level, such as mathematical patterns (Holland, 2000; Stroup & Wilensky, 2014). For example, describing the micro-level physical relationship between gas particles and their container highlights what sort of behavior might emerge from an air pressure system. However, the mathematical formula

PV=nRT that describes a macro-level aspect of the system highlights how air pressure is affected by changes in particle count, temperature, or volume.

We are interested in whether and how students learn to navigate across these levels of description as they interact with simulation-based learning environments. Therefore, our theoretical orientation focuses on how students make use of and build connections between their existing knowledge and the mediating tools with which they are engaged. We draw from theories of learning that focus on learners' existing *resources* for sense making, and the role those resources play in helping the learner build *connections* across different representations, experiences, and bodies of knowledge (diSessa, 1993; Noss & Hoyles, 1996; diSessa & Sherin, 1998; Wagner, 2010). By *resources*, we draw from Pratt & Noss's (2002) notion of resources as "both external tools and internal knowledge" (p. 456) which learners leverage in activity to coordinate existing understandings and construct new knowledge. By *connections*, we are interested in the links students are constructing between the levels of description in the CCS Framework. This allows us to explore "the complementary roles played in internal (cognitive) and external (physical or virtual) sources of meaning making" (Pratt & Noss, 2002, p. 456) that are the basis for simulation-based educational environments.

An Example: The Case of Population Growth

To illustrate the Calculus of Complex Systems Framework, we consider the example of population growth dynamics. Mathematical models of population growth are part of most high school math and science curricula (AAAS, 1993; CCSSI, 2010; NRC, 2012). But, their study also includes attention to the behaviors that cause populations to fluctuate such as birth, death, immigration, competition for resources, or population density (NRC, 2003; Sandholm, 2011).

Typically, mathematical models of population growth focus on describing dynamics in terms of a *Pattern of Accumulation* of total population. A logistic model represents the growth of a population that is eventually limited by resource or space constraints. It is often represented with a graph of population over time that follows a characteristic logistic "S" shape, or by the Verhulst equation $P(t) = \frac{KP_0e^{rt}}{K+P_0(e^{rt}-1)}$[1]. These two external resources—the graph and the equation—focus on descriptions of *Patterns of Accumulation*. They can also be used to find information about *Patterns of Change*: examining the slope of the graph indicates how many members are added to the population during a given interval of time. Population models also incorporate assumptions about what are the important *Individual Behaviors* and *Group Interactions* that underlie a system and constrain its global behavior. Certain implicit information about these levels might also appear in resources that deal primarily at a different level of description. The Verhulst equation, for instance, assumes that there is some carrying capacity K after which more reproduction cannot occur, because of the competition for space or resources introduced by *Group Interactions*.

Depending on what resources learners leverage to make sense of different levels of description, they might arrive at contradictory or logically consistent understandings of the system as a whole. Recalling our example, a logistic pattern of population growth does not necessarily mean that each member of the population reproduces slowly, then quickly, then slowly again over the duration of the population's existence. Therefore, resources that describe the population's *Pattern of Accumulation* might not be the most appropriate ones to leverage to make sense of *Individual Behavior*. However, learners may also have some implicit knowledge

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[1] Here P(t) is the population at a given time, P_0 is the initial population, r is the rate of reproduction, and K is the highest number of population members the environment can support.

of how *Individual Behaviors* such as reproducing or dying might combine to generate *Patterns of Change* – the total number of births and deaths per year. This in turn might help them build a connection between *Individual Behavior* and *Patterns of Accumulation*.

In this study, we focused on exponential population growth as a first step toward understanding how students identify relevant resources, and use those resources to build connections across different levels of description. However, we anticipate that the insights drawn from the example we explored here illustrate how the CCS Framework can accommodate phenomena that involve more behaviors and interactions, as do many complex systems.

Methods

Our research questions were: (1) What resources did students use to describe the quantitative patterns generated by the simulation? (2) What resources did students use to describe connections between the individual behaviors, group interactions, patterns of change, and patterns of accumulation in the simulation? And, (3) what connections did students hesitate or struggle to describe, and how was this resolved?

We conducted eleven one-on-one semi-structured clinical interviews (Clement, 2000; Ginsberg, 1997) with 6 male and 5 female 11th and 12th grade students enrolled in a summer preparatory Calculus program at a large, urban Midwestern public high school. The students completed a lesson on exponential growth and its rate of change a week earlier in class. We introduced a NetLogo agent-based simulation as an alternative way to explore population growth that was based on individual probabilities, rather than the overall growth rates used for exponential models. Participating students did not have experience using NetLogo or other agent-based modeling environments. Each interview lasted between 30 and 45 minutes.

The simulation focused specifically on simple exponential population growth. The pattern produced in the simulation was generated by probabilistic individual reproduction, and intentionally featured “human” agents, so that students were also likely to have existing commonsense expectations about individual behavior that might inform their interpretation of the simulation. This combination of study design and student educational background allowed us to focus specifically on what *new* challenges and relationships might arise when students work to make sense of quantitative change in the context of complex systems dynamics, even when they had considerable background from which to draw. Our introduction of the NetLogo simulation explicitly prompted students to explore the relationship between a complex systems treatment of population growth (i.e., the explicit linkages between individual and collective behavior), how that pattern is expressed using the simulation, and the familiar mathematical models traditionally used to make predictions about population.

Simulation

During the interviews, we asked students to engage with a NetLogo (Wilensky, 1999) agent-based simulation (Figure 1). We first introduced them to the simulation paused in its initial state, with 100 “people” (computational agents) randomly distributed in the visualization window. We explained that each agent would move within the window a small amount in a random direction[2] and would have a .01 probability of cloning themselves during each unit of time (or “tick”, as we will refer to them in this paper). The simulation interface also featured two graphs, one labeled “population”, which would dynamically plot the total number of simulated agents during each tick while the simulation ran; and another labeled “people born”, which

[2] Movement was not important mathematically. However, if agents did not move the simulation would “stack” them in the visualization, so that the population did not appear to grow. Therefore, we introduced a rule to make them move so that students could use the visualization to observe changes in the number of agents over time.

would plot the number of new agents in the world each tick (Figure 2). These graphs were vertically aligned with the population graph on top, so that each point along the time axes was aligned vertically.

[INSERT FIGURE 1]

Next, we ran the model for approximately 200 ticks, during which the simulated agents moved about and replicated in the visualization window. The graphs of population and people born were plotted over time as the simulation ran (Figure 3). Since each simulated agent was known to have a .01 probability to reproduce during each tick, *approximately* 1% of the total population would reproduce and be added to the original population during each tick, creating an exponential-like pattern of growth. The number of individuals added to the population would be plotted in the "people born" graph, while the total number of individuals (including those recently added) would be plotted on the "total population" graph. The probabilistic rule in the simulation introduced variability in the number of humans added to the population, so that sometimes *fewer* simulated humans would be added later during the simulation, even though one would expect the number to consistently rise. This produced a 'jagged' graph of the number of people born – something not expected from the purely mathematical model.

[INSERT FIGURE 2]

For example, in Figure 2 there is a 'peak' of 10 individuals born at about ¾ of the way through the duration of the simulation's run. The probabilistic element of the simulation caused 10 individuals to be born during that unit of time, which is much higher than the expected 1% of the total population (around only 300 agents at the time). However, during the next unit of time, only 4 agents were born, which is much closer to the expected value. This variation in the number of individuals added per tick also created small perturbations in the exponential-like shape produced by the graph of total population over time. In Figure 2, a small “bump” in total population can be seen at the same time that 10 individuals were born. The simulation was truly probabilistic in that each simulation run generated slightly different specific results. Screenshots of one execution of the simulation at time 0, 66, 132, and 198 are featured in Figure 3. Readers can interact with an online version of the simulation at http://bit.ly/10avPIz.

[INSERT FIGURE 3]

Interview Protocol

After the simulation reached 200 ticks, we paused it and left it in its paused state on the computer screen for reference. Next, we asked participating students a series of questions designed to probe their understanding of different aspects of the system, corresponding to the four levels of description articulated in the CCS Framework. Interviews were semi-structured, such that the interviewer (Michelle) asked the same set of questions to students, but would probe or follow up on students’ ideas differently depending on how they responded to questions. First, to determine what resources students used to describe and justify the quantitative patterns generated by the simulation, we asked them to describe when population was highest and why (*Pattern of Accumulation*) and when it was changing the most and why (*Pattern of Change*). Next, to better understand whether and how students established connections across the behavioral and quantitative aspects of the simulation, we asked them to describe the relationship between an individual agent in the simulation (*Individual Behavior)* and the total population measure; the relationship between the graphs of people born and total population, and to explain why the graph of people born was so ‘jagged’ (something that was ostensibly contradictory to the lesson on exponential growth they had had just a week before; *Group Interactions*).

Each interview was video recorded using two cameras—one positioned to capture activity on and students' gestures toward the computer screen, the other to capture interactions between the interviewer and each participant. The interviews were transcribed, and gestures toward simulation elements on computer screen were noted in the transcript as evidence for the use of specific graphs or other resources within the simulation.

Methods of Analysis

Our analysis corresponds to our three research questions as follows. First, we coded students' responses to each interview question for the presence of each of nine *Resources*, described in the next section. This allowed us to determine what students attended to when describing the various quantitative patterns generated by the simulations (Research Question 1). Next, we identified which *Levels of Description* in the CCS Framework students were reasoning about when they cited each of those different resources. This allowed us to determine which resources students leveraged to make sense of particular levels of description within the simulation (Research Question 2), and to identify when and how different resources highlighted connections across those four levels (Research Question 3). Finally, we selected two interviews to analyze in further depth that exemplify the difficulties that emerge from the missed connections we identified through our coding.

Resource Coding

To identify what resources students attended to over the course of the interview, we iteratively developed a set of *resource codes.* These codes describe those resources within and beyond the simulation that at least two students leveraged to describe or justify quantitative aspects of the simulation at any point during the interviews. We started by first identifying a set of resources that we expected students to rely on when reasoning about each of the four CCS levels of description within the simulation[3]. For example, we expected students to reference the visuospatial component of the simulation to find information about agents' behavior and the behavior of the population as a whole. We also anticipated that students would consult the graph of the number of agents born per tick to find information about quantitative change.

We coded each student transcript using this first iteration of codes, paying attention to resources that were not adequately captured by the existing scheme. We eliminated codes for resources that were not cited by at least two students (for example, 'Visualization'), and added or reconceptualized others. For example, while our 'Programmatic' code was intended to capture students' reference to the computational simulation rules, this only manifested as attention to the *random* command as representing the probabilistic nature of the program and was hence redefined as 'Chance'. Similarly, our 'Graphical' code proved too broad, since students used the graphs of total population and people born differently. So, we divided this initial code into two codes to indicate the appropriate graph. Our finalized resource codes are identified in Table 1.

During our analysis, we will at times refer to particular resources as "mathematical" or "behavioral". We do not argue that this is how the *students* perceived their use of those resources; rather, we make these distinctions as designers to better understand whether and when students are beginning to construct~~ion~~ connections across what we identify as "quantitative/mathematical" and "behavioral" aspects of the simulation environment per our research question and design goal. We have marked those resources deemed "mathematical" with an M in Table 1, and those that we refer to as "behavioral" with a B.

[INSERT TABLE 1]

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[3] The initial set of codes included eight categories: Visuospatial, Quantity, Graphical, Behavioral-Agent Level, Behavioral-Aggregate Level, Functional, Programmatic, and Systemic.

Once our categories were established, we analyzed student responses to each interview question for the presence of each code. For example, the excerpt below is Irene’s response to the question “Why is population growing fastest at the end of the simulation?”:

> [Irene] If you have a single number and that’s raised, if you have a constant rate of change but your initial value is greater then your end amount will be greater.

Irene’s response was coded as featuring *exponential* as a resource (“a … number … raised”) to justify why the *Pattern of Accumulation* was highest at the end. Appendices A and B feature a more complete set of example responses and resource codes for each interview question. An independent second rater used the same coding scheme and Appendices A and B as a training set to analyze four additional randomly selected answers to each interview question (over 30% of total data). Inter-rater agreement on resource codes was 86% raw agreement and 75% agreement on presence (Smith, 1992)[4]. We met to discuss conflicts and revised our codes correspondingly, after which agreement rose to 95% and 93%, respectively. Two codes, *quantity* and *systemic,* were each referenced by only two participants over the course of the entire interview, and so these codes are not reported in our analysis.

Levels of Description

Next, we wanted to find out which of these resources students leveraged to make sense of different *levels of description* in the CCS Framework. Our first set of interview questions made the level of description explicit by asking students to describe *Patterns of Change* (when the most people were being born, and why) and *Patterns of Accumulation* (when the population was

[4] ‘Raw’ agreement calculates the total percent agreement between coders on the presence and absence of each code for each response: $\frac{\#\, agreed\ present + \#\, agreed\ absent}{total\ opportunities\ for\ present\ or\ absent}$. ‘Agreement on presence’ calculates agreement only on presence, and adjusts for inflation when only a few codes might be applied to each response: $\frac{2 * \#\, agreed\ present}{\#\, present\ per\ coder\ A + \#\, present\ per\ coder\ B}$.

highest, and why). Our next set of questions prompted students to describe connections across different levels of description. For each resource coded, we also coded what levels of description the student was using to make sense of that resource. In the following excerpt, Kevin is answering the question “What is the relationship between the people born graph and the total population graph?”:

> [Kevin] To, um, I mean you could simply like we already related how this graph's irregular [indicates people born graph] and so is this [indicates population graph] but um, simply putting um, this [population graph] isn't going to model dips in population nearly as well as this [people born graph], so you could simply I guess say that this graph is simply almost like a best fit line of this graph so it takes like the top points are the most important pertinent points of the bottom graph and it simply shows up on the top.

This response is coded as involving the people born graph (“bottom graph”), the population graph (“top graph”), and the actions or properties of the population as an entity (“dips in population”) as resources. In terms of levels of description, Kevin describes the population graph as a best fit line that reproduces important points of the people born graph, implying that he is using both graphs to speak to the *Pattern of Accumulation* of the population. Kevin is also referring to actions/properties of the population to describe both the *Pattern of Accumulation* and the *Group Interactions* those patterns reflect (“dips in population”).

Appendix B features a more complete set of example responses and level of description codes for each interview question. Inter-rater agreement on level of description codes was 89% agreement on presence or absence of each level, and 82% agreement on presence (Smith, 1992). We met to discuss conflicts and revise our codes correspondingly, after which agreement rose to 93% and 90%, respectively.

Results

As we will describe in further detail below, we found that while students leveraged both behavioral and mathematical resources to make sense of quantitative change in complex systems, they made some connections across levels of description more readily than others. Most notably, most students did not articulate connections between Individual Behaviors and Patterns of Change, even though they were comfortable describing the relationships between a single agent in the simulation (Individual Behavior) and overall population growth (Pattern of Accumulation), and between the graph of population (Pattern of Change) and graph of people born in the simulation (Pattern of Accumulation). In fact, most students only connected Individual Behaviors and Patterns of Change when they were explicitly asked to explain why the graph of people born was jagged.

To better understand the implications of this missed connection, we present two more detailed case studies (complemented with data from other interviews). These case studies reveal that missed connections between Individual Behaviors and Patterns of Change led participating students to experience difficulty when describing the mathematical relationships that underlie the simulation. In both cases, these difficulties were resolved once students' attention was drawn to the jagged nature of the people born graph.

Part 1: Resources Cited By Participants ~~(1.5 pages)~~

Our first objective is to describe what resources in the simulation students cited when answering questions specifically about quantitative change. We do this by exploring students' responses to our first set of interview questions: "When is population highest in this simulation," and "Why is population highest then?" (*Pattern of Accumulation*), and "When is population growing the most in this simulation," and "Why is it growing the most then?" (*Pattern of Change*).

Resources For Describing the Pattern of Accumulation. Table 2 shows which resources each student (indicated by the first letter of his or her pseudonym) cited to describe when population was highest in the simulation, and why. To identify when population was highest, all participating students referred to mathematical representations and ideas, citing the population graph or the monotonic pattern of growth, rather than behavioral ones. But when asked to describe why population was highest at the end, more than half of the students cited behavioral resources, and almost half also cited more than one resource as part of their explanation.

[INSERT TABLE 2]

Resources For Describing the Pattern of Change. Table 3 shows which resources each student cited to describe when population was changing the most in the simulation, and why. As in questions related to accumulation, all but one participant cited mathematical resources in their responses. Notably, many of the students that cited the total population graph by attending to the slope of the graph (see Alex in Appendix A for an example), in contrast to attending to its height during the question about total population. When asked to explain why population was changing most at the end, all but two students cited at least one behavioral resource as part of their explanation, and all students who cited at least one behavioral resource cited multiple resources as part of their explanation.

[INSERT TABLE 3]

Table 4 summarizes the resources cited by each student across all four interview questions focused on quantitative patterns in the simulation. All participating students leveraged both behavioral and mathematical resources at some point during this portion of the interview.

And, all but one used more than one resource to respond to at least one question. This suggests that these participants already attended to and recognized the utility of both mathematical and behavioral resources for answering questions about the quantitative patterns generated by the simulation, and leveraged those resources and the connections between them to make sense of those quantitative patterns of change – a key component of making sense of complex systems.

[INSERT TABLE 4]

Across both Tables 2 and 3, participating students more frequently used behavioral resources, and more frequently cited both mathematical and behavioral resources together, to answer questions about quantitative patterns when explaining why those patterns emerged, rather than when describing the patterns themselves. This was particularly true when students explained why pattern of change emerged the way it did. Together, this suggests that asking learners to explore the *causes of patterns of change* can help build on learners' existing strengths to explore and articulate connections across different types of resources, and levels of description, for a given complex system.

While all students recognized the utility of a diverse collection of both mathematical and behavioral resources to describe the quantitative patterns in the simulation, there were notable differences in what types of resources each preferred. For example, Gary (G) relied heavily on resources that are most appropriate for describing individual behaviors in the system, such as person actions and probabilistic rules. This suggests that Gary was especially attuned to agent-level descriptions of system behavior, even when answering questions about the aggregate-level quantitative patterns that emerged in the simulation. Sarah (S), on the other hand, cited mathematical resources most appropriate for describing patterns of accumulation such as the population graph and the idea of exponential growth, and mentioned the actions or properties of the population only once. This suggests Sarah found mathematical descriptions of the system behavior to be more useful than behavioral or agent-level information. Thus even while students articulated connections across a diversity of types of resources and levels or description, these connections did not necessarily extend across all four levels of connection we identified above or reflect fluidity across those levels.

Finally, we found that participating students rarely attended to some resources during this portion of the interview. Only two participating students ever referenced the “1% chance to reproduce” rule that was the programmatic basis for the simulation, and only one cited the graph of people born during the first set of interview questions. This was especially surprising because the people born graph directly represented the pattern of change (number of individuals added to the population per tick), something we explicitly asked about. That participants did not attend to these two resources will become more important later in our analysis.

Part II: Identifying Connections across Levels ~~(1.5 pages)~~

Our second objective is to understand which resources students used to describe connections across different levels of description as outlined in the Calculus of Complex Systems Framework. In this section, we report patterns in student responses to three questions designed to probe their understandings of various connections across levels of description. For each question, we feature a table that shows what resources were leveraged to describe the relationship between an individual ‘person’ agent in the simulation and overall population growth, and at what level(s) of description those resources were used. Each cell in the matrix includes the first initial of each student who used a particular combination of resource and level. Initials are aligned across cells to make it easier to track the responses of each student.

Describing Connections Between A 'Person' Agent and Total Population Growth. Table 5 shows the resources and levels of description participating students leveraged to describe how the behavior of a 'person' agent in the simulation (Individual Behavior) would contribute to the total pattern of population growth (Pattern of Accumulation). Most students leveraged the chance and/or 'person' agent behavior resources to describe connections between Individual Behavior and Group Interactions, and leveraged the actions of the population as a whole to describe connections between Group Interactions and the overall Pattern of Accumulation.

[INSERT TABLE 5]

Our participants stitched these descriptions together to yield coherent multi-level explanations for how each 'person' agent, even with a low and consistent probability to reproduce, contributed to a total population growth pattern that increased at an increasing rate. Most participants described qualitative connections across levels, rather than more precise mathematical connections. Indeed, the only mathematical resource that was heavily leveraged by participants during this portion of the interview was chance, and was not usually employed quantitatively but rather as evidence that individual behavior was probabilistic in general. This excerpt from Irene's interview is an example:

> [Irene] Because there's so many people like the, it, you eventually have, let's say 100 people there and so someone's bound to have another person or have another reproduce and so that why, that's why you have a population growth because, as, as the population grows there's more chances of people being born.

While nearly all participants made productive qualitative connections across three of the four main levels of description identified in our framework—Individual Behaviors, Group

Interactions, and Patterns of Accumulation—few explicitly talked about Patterns of Change during this portion of the interview.

Describing Connections between the People Born Graph and Total Population Graph. Table 6 shows the resources and levels of description participating students leveraged to describe how graph of people born (Pattern of Change) was related to to the graph of total population growth (Pattern of Accumulation). Most participating students leveraged resources that speak to aggregate levels of description—the graph of people born, the graph of total population, and the actions of the population as a whole as resources—to describe these connections.

[INSERT TABLE 6]

Whereas the connections participants described between Individual Behaviors and Total Population Growth were typically qualitative, many participants connected the mathematical notions of change and accumulation to how specific group interactions emerged within the simulation – specifically, the number of people born that is added to the population and hence change it. This makes sense given that participants leveraged more mathematical than behavioral resources in general to describe these connections. An excerpt from Zoe's interview below provides one example of how these students identified the actions or properties of the population as the underlying cause that underpinned quantitative connections between Group Interactions and the Patterns of Change and Accumulation that manifest in the simulation:

> [Zoe] Um, well, you see the, an increase of people born [indicates people born graph] and that's because as the population's growing, you have more population [indicates population graph] to have more kids you're gonna have more kids born, (okay) so that's like the correlation.

Though many students made productive quantitative across Group Interactions, Patterns of Change, and Patterns of Accumulation, few students included any description of Individual Behaviors as part of their responses. This is especially notable since the graphs were generated by the behaviors of these individual 'person' agents. Similarly, few students leveraged resources that speak directly to individual level of description, such as chance or person actions.

Across Tables 5 and 6, results reveal that while participating students noted connections across most levels of description including levels that were not explicitly part of the question asked, they rarely leveraged connections between Individual Behaviors and Patterns of Change. And, while participants' connections between Individual Behavior and Patterns of Accumulation were qualitative and focused on connections between individual-level and population-level behavior, their connections between Patterns of Change and Patterns of Accumulation were quantitative, but did not include the agent level.

<u>Describing Why the People Born Graph is Jagged: Linking Individual Behavior and Patterns of Change.</u> It was not until we asked students to describe why the graph of people born in the simulation was so jagged that most participating students connected resources that spoke to both Individual Behavior and Patterns of Change. This question was also the first time that many students quantitatively connected aspects of the simulation that describe Individual Behavior to aggregate-level Patterns of Change and Patterns of Accumulation within the simulation.

[INSERT TABLE 7]

The excerpts below from Caroline and Kevin provide examples of how students leveraged agent behavior generally (in Caroline's response), and the probabilistic reproduction rule specifically (in Kevin's response), to describe the jagged nature of the graph:

[Caroline] Um, well, there are, there could be years where no one was born because they were wandering around or whatever so they couldn't reproduce.

[Kevin] Like you said that um, they have a 1% chance of reproducing and for instance, at the spikes maybe here and here, the, just at that moment of that 1% chance actually occurred, so, they had a higher of reproducing, while at the dips over here it just didn't come through and then, um, the other chances of not reproducing kicked in and just right there they didn't reproduce as much.

Before asking about the jagged nature of the people born graph, participating students were adept at identifying quantitative connections between patterns of change and patterns of accumulation. They were also adept at identifying qualitative connections between the behavior of the system at both the agent and aggregate levels and the overall patterns of accumulation that emerged from those behaviors. However, even though they leveraged both mathematical and behavioral resources to make sense of patterns of change and patterns of accumulation in the simulation, most participating students did not articulate a quantitative connection between individual behavior and these patterns before this question. Figure 4 provides a diagrammatic summary of these findings, using the Qualitative Calculus of Complex Systems Framework as an organizing device. In the next section, we argue that strengthening connections between the individual behaviors in a system and the patterns of change that result from those behaviors is critical for learners to develop fluency with the mathematics of complex systems.

[INSERT FIGURE 4 HERE]

Part III: Missed Connections, Complications, and Resolutions

Part II reveals that the students in our study were adept with both mathematical and behavioral resources for making sense of the population growth simulation. They readily identified many connections across these resources to describe the system across multiple levels of description. However, most students did not identify connections between Individual Behavior and Patterns of Change until they were explicitly asked why the graph of people born was so jagged. In this section, we present two case studies (along with supplemental data from other interviews) that suggest without this particular connection between Individual Behavior and Patterns of Change students' reasoning about the mathematics that underlie complex systems dynamics can become problematic. We suggest that explicitly drawing students' attention to the jagged nature of the people born graph is one effective way to encourage students to make sense of this particular connection, which in turn can help them make progress toward understanding the system's mathematical connections across levels of description.

The two cases we have selected to focus on are those of Gary and Sarah. These interviews were chosen for three reasons. First, they exemplify two different complications we saw in our data more generally: describing an inappropriate mathematical connection from Individual Behavior to a Pattern of Accumulation (exemplified by Gary), and struggling to interpret quantitative patterns generated in the simulation environment (exemplified by Sarah). Second, in both cases the resolution of these complications unfolded in a way that yielded rich opportunities for analysis. Third, each student began the interview with a different pattern of response: Gary relied on resources related to the probabilistic rules and behaviors in the simulation to describe the mathematical trends, while Sarah relied primarily on mathematical ideas and representations ~~to do so~~ (Table 4). Despite these differences, both Gary and Sarah resolved those complications in the same way—by attending to jagged nature of the graph of people born.

Gary: Learning to Connect Individual Behavior to Mathematical Representations.

Like most participants in our study, Gary readily described the qualitative connections between individual reproduction and exponential-like accumulation in the simulation. Unlike most of his peers, he also worked to articulate these connections mathematically. But even though Gary drew upon many appropriate resources and worked hard to make sense of the connections between them – both productive practices that should be encouraged in mathematics and science education – he still struggled to develop a coherent explanation of the mathematical connections between individual probabilistic behavior and overall exponential patterns of accumulation in the simulation. After Michelle drew Gary’s attention to the jagged nature of the graph of people born in the simulation, however, he began to draw more clear and coherent connections between quantitative descriptions of individual behaviors, their influence on patterns of change, and resulting patterns of accumulation he would expect to see in the simulation.

In the following excerpt, Michelle had just asked Gary to describe the connections between how ‘people’ agents behaved, and the resulting patterns of accumulation generated in the simulation. When Gary mentioned that ‘people’ agents had more of a chance of reproducing, Michelle asked him to clarify what he meant:

M: So when you say they have more chance of reproducing if we're
talking about that blue guy right there does he have more chance of
reproducing?
G: He starts at 1%, right? And it's 1% percent every tick isn't, then isn't
it that af- there's 1% every tick then for every tick that goes his
chance like increases? or does it, like, I think, yeah.
M: Can you talk more about that?

G: If it's, can I write on?

M: Yeah, oh yeah, that's why there's paper here.

G: So there's, hold on, 1% chance for every tick right? So for, if it's 1%, wait, let me think in my head real quick.

M: Yeah that's fine, if you can say what you're thinking, too, you know [laughs]

G: So I'm trying to remember how I do this, if it's 1% probability per tick, over the span of five ticks, I think the probability increases, you multiply this, oh wait no, it decreases, I think. Cause it's .01 to the fifth power cause it's for every tick you multiply again by .01.

M: I see, so you're saying for like the blue guy, since, since each tick it's a .01 chance that for five ticks altogether it's-

G: .01 times .01 five times. Which is actually smaller then, yeah I think it's smaller.

M: Does that make sense?

G: Yeah

M: Okay, why is that? Like if you just think about a person in the world, you know?

G: Because as they get older, their uh reproduction system it like, it's not as healthy because it peaks at a certain point and then like, as you age, it becomes harder to produce like you know, like reproduce.

In many ways, what Gary did here was evidence of productive reasoning. He leveraged a number of relevant resources: his understanding of ‘person’ behaviors such as reproduction and aging, his understanding of the 1% chance simulation rule, and the mathematical idea of exponential – likely because this was the shape made by the total population graph, and a population growth model that had been recently discussed in class. He also worked to connect these resources in a way that was coherent and connected to his understanding of real phenomena.

During the exchange, Gary first noted the individual behavior embodied by the simulation – that the “blue guy”, an agent within the visualization featured in the simulation environment, starts at and maintains a 1% chance to reproduce (lines 1-3). But in the same turn of talk, Gary also suggested that “every tick that goes his chance like increases?” (line 4). One possible explanation for this is that Gary was attributing behavior at one level of observation (exponential growth at the population level, $P(t)=P_0e^{rt}$) to behavior at another level (individual agents) – that is, exhibiting “slippage between levels” (Wilensky & Resnick, 1999).

Next, when Michelle asked Gary to elaborate, he wrote the expression $(.01)^5$ to describe what he understood to be an individual agent’s probability to reproduce over 5 units of time (lines 14-17). This is not the exponential growth formula applied to an individual, but rather a formula that includes an exponential term and calculates the probability of repeated independent events[5]. As he wrote and worked through this new mathematical description, however, Gary realized that this solution implies an individual’s likelihood to reproduce would decrease as time

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[5] The formula for calculating the probability of repeated independent events is ($P(e_1$ and $e_2 \ldots e_n) = P(e_1) \times P(e_2) \ldots P(e_n)$), where e_x represents an event. In this case the probability for each event, individual reproduction during each successive tick in the simulation, is the same at 1%. This reduces to .01 x .01 x .01… n times, or $(.01)^n$—$(.01)^5$ if one were to seek the probability of 5 births happening in a row.

increases, in contrast to his initial prediction. Michelle asked him to explain his calculation again, and he confirmed that an individual's probability to reproduce will decrease over time (lines 20-21) and seems to accept this description of individual behavior. Michelle asked Gary to make sure that his claim makes sense, since it conflicted with what he proposed in lines 4-6 of the excerpt, that an agent's chance to reproduce should stay the same or increase. Gary responded that a decreasing probability to reproduce makes sense, since it can represent decreased fertility with age.

One interpretation for the inconsistencies in Gary's explanation is that in describing the connection between an individual agent's behavior and the overall pattern of population growth, Gary actively worked to reconcile the 1% probabilistic behavior of individual agents with the exponentially growing behavior of the overall population. To do this, he leveraged a formula that included elements of both: a formula used to calculate probabilities over multiple event trials (or in Gary's case, over multiple ticks), and that included an exponential term. However, neither the exponential growth formula nor the formula for calculating the probability of repeated independent events can be applied to individual behaviors in a way that illuminates how those behaviors connect to overall population growth patterns. We argue that this exchange provides clear evidence that even though he leveraged appropriate resources and pieced those resources together in creative and locally coherent ways, Gary experienced difficulty creating broadly coherent mathematical connections across Individual Behavior to Patterns of Accumulation.

Later during the same interview, Gary continued to assign this exponential representation to agent behavior in this way. When at one point he suggested that the exponential trend in population growth resulted from multiplying 1% by smaller versus larger numbers (rather than individual behavior), Michelle asked him how this corresponded to his earlier claim that individuals had less of a chance to reproduce. Gary said “Um I think this [pointing to written $.01^5$] is uh only counting one person, because it's just one person's probability, (mhm) but then you also have to take into account that there's multiple people that have that .1, .01 percent chance.” It seems that although here Gary described mathematical patterns in the simulation as they related to Group Interactions rather than Individual Behaviors, he still applied the exponential idea directly to Individual Behavior as well.

Directly following this interaction, Michelle asked Gary to talk more about the “people born” graph and its relationship to the “total population” graph:

> M: So, okay, now we got this second graph down here, and it's the number of people born at each tick. What does this tell you about the model, er, does it look the way you expect?
>
> G: Yes, because as the ticks increase, and as you see from here (points to upper graph) as it relates to this as the population increases, the number of people born also increases.
>
> M: Okay, can you talk more about they're related? I mean are there more specific ways that they're related?
>
> G: I think it's be.. uh, this is.. this kinda looks like the area under the population, kinda (hmm) like the shape, (okay) it's not like the same amount of space, but it looks as shape because as more people are born, the population will increase more, so as this gets higher, this will also get higher.

This time, Gary related both graphs to the system’s behavior at the level of Group Interactions - that the number of people born was dependent on the number that were already present in the population [lines 33-34]. As he continued to explain, Gary also articulated the relationship between the people born graph and the total population graph in terms of the mathematical relationship between Patterns of Change and Patterns of Accumulation, stating that "… if we start off at fifty and four people are born, then it's gonna be fifty four". However, Gary still did not relate the graphs specifically to Individual Behaviors in the simulation.

Next, Michelle drew Gary's attention to the jagged nature of the people born graph:

M: Does it makes sense that this is as jagged as it is?

G: Um, kinda because, I'm kinda confused by why peaks to nothing, like is this over like, is this like just squished amount the ticks, like is this one year at the bottom or is this, like within one-is this one tick exactly or is it in between two?

To address Gary's question, Michelle re-ran the simulation so that he could see it draw on the screen. The interview continued:

M: If I kept running this model we already predicted what this [points to total population graph] would look like, but what would this one [points to people born graph] look like you think?

G: I think it would um, it would either increase or stabilize because, does, does this program, uh, uh factor in how old each, like person is?

M: That's a good question. It doesn't now but you can add that later if you want.

G: Cause um if it doesn't, then it should start to increase because it won't factor in how each person's probability of 1% will decrease over time, but if you do include that then uh the pop--the people born should either close to level off because the older people will not produce as much while the younger people will produce the same amount and since they produce more younger people, it starts, it starts to balance out for the people, the older people who are not producing.

M: Okay I gotcha, okay. Would there, do you think it'd still be as jagged or would it smooth out?

G: I still think it would be jagged because this is just, is this random is it? How is it determined, it just does the math based on like the 1% chance for people born?

In the first excerpt of this case, Gary attempted to directly connect individual behavior to mathematical formalisms in ways that did not maintain the coherence of agent-level and aggregate-level behavior in the simulation. However, after attending specifically to the jagged nature of the people born graph, Gary began to connect his understanding of Individual Behavior (such as the probability of reproduction, line 21) to the people born graph as a description of the Pattern of Change in the simulation, instead of directly to resources that describe Patterns of Accumulation. He then considered how those interactions would produce the resultant quantitative patterns he would expect in mathematical representations in the simulation [lines 22-24]. We will return to Gary's case at the end of this section to explore why attending to the jagged nature of the graph might have prompted this shift.

Sarah: Learning to Connect Patterns of Change to Simulation Behavior. During early portions of Sarah's interview, Sarah was hesitant to connect the mathematical behavior of the simulation – she rarely cited behavioral resources when responding to questions, and never cited resources that spoke to the individual level of behavior. Instead, Sarah relied on mathematical ideas, procedures and manipulations of quantitative data to talk about rate of change and accumulation. Later during the interview, Sarah noticed and started questioning why the graph of people born in the simulation included unexpected “dips”. With prompting from Michelle, she began to explain the cause of those dips as the probabilistic individual behavior in the simulation – a level of description she had not attended to before describing and working to understand the “dips” in the people born graph. After making this connection between individual behavior and patterns of change in the simulation, Sarah was able to interpret the mathematical ideas she relied on early in the interview in terms of what they implied for individual and group-level behavior in the simulation.

In the excerpt below Sarah uses the graph of total population and its exponential nature to describe how she might measure change for one “tick” in the simulation. Even when Michelle prompted Sarah to consider other possible resources available within the simulation, Sarah suggested she was “stuck” in a mathematical way of thinking:

M: Okay, and then one tick, to find the rate of change, what would you measure?

S: You could use derivatives[6]. [laughs].

M: Mhm, is there any other way you could do it?

S: Uh, I dunno cause the problem with exponential functions using uh, solving for the slope in general, is that you come out with a straight line if you were to use it like you would solve for linear? Which isn't

---

[6] A mathematical derivative measures how much a function *f(x)* will change as *x* changes. Here, Sarah is proposing to find the derivative of a function that describes total population growth over time, which would reveal specifically how much the population is changing at a particular time.

8 realistic for a population. I mean I guess it would be if you were only
9 using one tick.
10 M: So, okay, can you think of any other way, with all the information
11 you're given here [gestures toward simulation environment], that you
12 could do it?
13 S: To do what? To just...
14 M: To like, for a given tick, to say what the rate of change is.
15 S: Um, I dunno, I'd have to think about that. Kind of like derivatives all
16 stuck in my mind [laughs].

To describe how she might find the measure of change for a given “tick” during the simulation run, Sarah cited mathematical procedures she could perform using the total population graph as a resource, such as calculating a derivative (line 3) or solving for the slope of the graph for the time of interest (line 6). Michelle attempted to redirect Sarah’s attention to other resources within the simulation that also provided information about how the population is changing over time (lines 10-12), to probe whether and how Sarah might navigate the connections between the idea of rate of change and simulated behavior. However, Sarah did not take up Michelle’s proposal.

In terms of resources and levels of description, Sarah is leveraging the *exponential* as a mathematical idea, the *population graph* from which she identified that exponential pattern, and her understanding of the *actions/properties of the population* as resources in this excerpt. She used all three of those resources to describe the Pattern of Accumulation in the simulation, and considered how she can use the exponential function or the population graph to find the corresponding Pattern of Change. However, Sarah never directly links her ideas about the Pattern of Change to Group Interactions or Individual Behavior.

Like Gary, many aspects of Sarah's case are interesting and productive. She volunteers more than one relevant solution to Michelle's question, drawing from what she has learned in class. And, she considers how reasonable or "realistic" those approaches might be given the nature of population dynamics as the phenomenon under study (Lines 7, 8). Given that the interview occurred during a mathematics class and dealt with mathematical patterns that were similar to those studied in class, Sarah might have believed she was expected to provide only mathematical answers to the interviewer's questions and that leveraging non-mathematical ideas was not appropriate in this context. Or, she might have been struggling to understand how the mathematical notions of rate of change that she was considering were connected to the specific behavior of the simulation (a problem not uncommon in mathematics education). We argue that regardless of whether it was because of her expectations from the interview or her understanding of the simulation, Sarah was clearly not leveraging all of the resources she had available to connect Patterns of Change to the Individual Behavior or Group Interactions within the simulation during this exchange, and this interfered with the degree to which she could describe the system and constituent parts across all levels of description.

Later, Michelle asked Sarah to explain whether the "people born" graph was what she expected in terms of its appearance and trajectory. Sarah responded that she previously thought that the graph was "another version of population" that would "take the highest and lowest points and find the average between them and that would somehow equal this [population graph]". Once this became clear, Sarah spontaneously noticed the "dips" in the graph:

17 S: Okay, um, then it's saying as incre--, as time increases more and more

are being born throughout the population. It's kind of easy to look at it
with these numbers, um, cause you can watch as time increases uh
people increases cause sometimes like you can see these kind of dips
um, yeah.
M: Why do you think those dips are there?
S: Uh, cause those are the dips that are just like I don't know how to really
word it but the, the, I don't know how to say it, like I can't say the
minimum people born [laughs]
M: Oh yeah I know, I think I know what you're talking about.
S: Uh yeah, but, as like opposed to here where the highest peak would be,
you know what is it, one person? But then most people would have
zero?

During this exchange, Sarah is beginning to connect her understanding of the actions/properties of individual people (people being born) and of the population as an entity (more and more people born over time) as resources to the people born graph. This allows her to describe connections between the levels of Individual Behavior and Group Interactions. However, it is unclear from this excerpt whether at this point Sarah understands the graph of people born relates to the Pattern of Accumulation generated by the simulation, or even whether people born represents a Pattern of Change in the simulation.

Given that Sarah understood the graph of people born in a new way, Michelle decided to run the simulation again for her. Sarah responded:

S: So um, as time kind of like increases some people won't be having, like
people born at certain times, more people won't be born at certain

times.

M: And does that make sense knowing what you know about how this works?

S: Yeah

M: And why is that?

S: Because if they only have a .01 chance of reproducing, it doesn't mean they're gonna be doing it every second.

During this exchange, Sarah maintains the connections she had articulated before between Individual Behavior and Group Interactions, but this time also cites the specific probabilistic rules of the simulation as a new resource to describe these levels. Soon afterward, Michelle decided to ask Sarah to articulate the relationship between the graphs of “total population” and “people born” again, now that she had a better understanding of the latter:

S: Um [points at lower graph, then top graph] okay our original population is taking this [points to lower graph] added to uh people that there were, that there were beforehand, before they were the people were born. Um so it’s taking in account to adding to the uh population beforehand, which is kind of the deal of exponents which is multiplying and multiplying and multiplying from the original

M: And so does that help you talk about rate of change at all?

S: In terms of population or in terms of...?

M: In terms of population.

S: People weren’t, um, then I guess, oh, I guess this could be the rate of change.

M: And why's that?
S: Uh, because, well this divide, is it, yeah, because their rate of change is saying like oh, well this is how many people were added to the original population over a period of time
M: Mhm and how does that relate to kind of like the ideas you learned in class?
S: Um, about derivatives and stuff?
M: Yeah.
S: Um, that derivatives is basically taking like an exact point divided by another exact point finding the exact um, like change, but this gives us the exact change over the exact time. It gives us the exact number of people born at a certain time which is what derivatives is, is solving for.

In the first excerpt of this case, Sarah was unable or unwilling to articulate the connection between her understanding of the Pattern of Change in the simulation (which she thought of as derived mathematically from an exponential) and the Individual Behavior or Group Interactions in the simulation. After attending to and making sense of the jagged nature of the people born graph, however, Sarah was able to use that graph to describe Individual Behaviors and Group Interactions within the simulation (lines 17-38). She then related those behaviors to the total population graph by noting that the number of people who are born are added to the total population (lines 39-42). Importantly, this means that Sarah is implicitly using a measure of the population as an entity – the total born during a tick of time – as a resource to describe a Pattern of Change. Since the number of people born depends on the size of the existing population, Sarah is also able to coordinate her understanding of the actions/properties of the population as a

Pattern of Change with her understanding of how exponential growth describes a Pattern of Change as "multiplying and multiplying from the original" (line 44).

It appears that it is not until lines 48-49 that Sarah refers to the graph of people born itself as an expression of the Pattern of Change exhibited by the simulation. However, by this point, she has added behavioral descriptions of Patterns of Change, based on the collective sum of agent births over time, to her previous mathematically-based understandings of Patterns of Change that were based on her ideas about derivative. This time, Sarah is able to articulate this notion of derivative in terms of the specific behaviors and interactions in the simulation that generate population growth.

Synthesis of the Case Studies: Connecting Individual Behavior and Patterns of Change. We claim that although the two complications exhibited by Gary and Sarah seem different, both emerged because these students did not attend to the connections between Individual Behavior and Patterns of Change in the simulation. For this reason, both were resolved by drawing Gary and Sarah's attention explicitly to the jagged graph of people born. This prompted each student to attend to how the probabilistic and agent-based resources that describe Individual Behavior (probabilistic reproduction) also emerged within mathematically-based descriptions of Patterns of Change (making the graph of people born jagged even though it is mathematically unexpected). Once this happened, the connection between Individual Behavior and Pattern of Change enabled learners to construct coherent quantitative understandings across all levels of description illustrated within the simulation.

In Gary's case, the lack of a connection between Individual Behavior and Patterns of Change led him to connect descriptions of Individual Behavior in the simulation directly to Patterns of Accumulation, without considering that multiple individuals should be aggregated first. Even when asked to describe the relationship between the graph of people born and the graph of total population, Gary did not describe either graph as representing only a collective measure of many individuals. However, once he was asked to attend to why the graph of people born was jagged, Gary linked this graph to the probabilistic behavior of multiple individual agents in the simulation. This provided Gary a way to consider the role of multiple individuals collectively contributing to quantitative patterns, rather than those patterns being assigned to a particular agent.

In Sarah's case, the lack of a connection between Individual Behavior and Patterns of change led her to ignore connections between mathematical descriptions of Patterns of Change (such as the idea of a derivative, or a rate of change) and what the interpretation of those ideas were within the simulation itself. Like Gary, Sarah began to connect the idea of probabilistic group to the graph of people born once she was asked to explain why that graph was so jagged. Once this connection was made, Sarah recognized the graph of people born both as a measure of change in the mathematical descriptions of population, as well as a measure of the results of the probabilistic reproduction behavior the simulation was based on.

While Gary and Sarah were the clearest examples of the complications that can emerge from a missed connection between Individual Behavior and Patterns of Change, there was evidence that other students experienced similar difficulties. In addition to Gary, four other students (including Sarah) explicitly attempted to connect individual behavior directly to the graph of population or the idea of an exponential. These attempts led to statements like "even though it stays at .01 it seems it's becoming exponential [Eddie]," or "I don't know, but it is increasing exponentially, so uh, each person you said has a 1 % chance of reproducing, so like, I dun, I don't know how to explain it [Caroline]." In addition to Sarah, three other students referenced the people born graph as though it were a measure of total population or a representation of raw population data from which the total population graph was determined using a line of best fit, and struggled to connect it to other aspects of the simulation.

Interestingly, the two complications we explored here emerge from two different sides of the Individual Behavior to Pattern of Change connection: Gary struggled to connect Individual Behavior to other resources in the simulation, while Sarah struggled to connect Patterns of Change to other resources in the simulation. In both cases, it is unclear whether Sarah or Gary made these errors based on their actual understandings of the system, or because they thought their task was to focus on mathematical relationships of the sort they had studied in class. In other words, Gary and Sarah might not have known the connections between Individual Behavior and Patterns of Change, or might not have thought that those connections were relevant for this particular class. Either way attending to the jagged nature of the people born graph helped them privilege those connections so that they could better understand the relationships between the behavioral and quantitative aspects of the population growth simulation.

Discussion

Interacting with agent-based simulations is a promising way to engage students in thinking about complex scientific systems. However, little is known about whether or how students make sense of the mathematical representations that accompany those simulations. In this paper, we take a first step toward understanding how students make sense of the mathematical aspects of agent-based simulations, what difficulties they may experience doing so, and how to support students in building those connections. Drawing from research on how students think and learn about complex systems and the mathematics of change and variation, we developed the Calculus of Complex Systems Framework as an analytic tool. The CCS

Framework highlights the importance of understanding and connecting the (1) Individual Behaviors that comprise a complex system, (2) Group Interactions that emerge within that system, (3) Patterns of Change the system exhibits over time, and (4) Patterns of Accumulation that are often used to describe and track the system's dynamics in order to fluently make sense of a complex system and its quantitative aspects. We used this framework to analyze how students who had recently studied the mathematics of exponential growth made sense of an agent-based simulation that generated simple exponential-like population growth from probabilistic rules. This allowed us to focus on what new ways of reasoning and challenges arose specifically when learners work to make sense of complex systems, in which mathematical patterns are often generated by multiple, simultaneous, probabilistic underlying events.

Our analysis suggests that students were adept at describing many connections between simulation behavior and the mathematical patterns it produced across most levels of description identified in the CCS Framework. For example, when asked to explain why particular quantitative patterns emerged within the simulation (Patterns of Accumulation), almost all of the students we interviewed cited the behavior of the simulated population (Group Interactions) in the model at least once. When asked to explain why the quantitative pattern in the simulation *changed* the way it did (Patterns of Change), nearly all students even cited agent behaviors or the probabilistic nature of the simulation (Individual Behavior). Similarly, when asked to describe the connections between different resources within the simulation (such as the available graphs, visuospatial rendering of the simulation, simulation rules, and mathematical ideas), most students generated explanations that incorporated multiple levels of description, including predominantly behavioral aspects such as Individual Behaviors or Group Interactions as well as predominantly mathematical ones such as Patterns of Change or Patterns of Accumulation.

However, while our participants clearly understood the mutual relevance of behavioral and mathematical resources for making sense of the simulation as a whole, there were come missing connections in their descriptions. Although they could provide mathematically sophisticated descriptions and specific quantitative examples of the connections between Group Behavior, Patterns of Change, and Patterns of Accumulation, Individual Behaviors were not part of those descriptions. Instead, connections between Individual Behaviors, Patterns of Change and Patterns of Accumulation were described qualitatively, with focus on general trends rather than measurable mathematical aspects. They did not engage with the *mathematical* contributions of Individual Behaviors in the simulation, even though the quantitative patterns within the simulation were generated exclusively by those behaviors: The 1% probability for reproduction that translated to an approximate, though ‘jagged’, 1% rate of population growth. When they did, they often experienced difficulties or inconsistencies.

Our analysis suggested one possible source for this problem in connecting mathematical aspects of individual behaviors to the quantitative patterns generated by the simulation. Looking across students’ responses to different interview questions, we found that most students did not attend to connections specifically between Individual Behavior and Patterns of Change. When they talked about one of these levels of description, they did not talk about the other. Participating students only began to articulate these connections when we asked them specifically why the graph of people born (which represents the Pattern of Change) was so jagged (a result of Individual Behavior – the probabilistic reproduction rule in the simulation – that was much more exaggerated than it was in the graph of total population growth). We conjecture that attending to this graph’s ‘jagged’ nature can help engage students in constructing connections across Individual Behaviors and Patterns of Change that can then address the errors and difficulties they experienced when describing the mathematical influence of individual behavior.

We explored this conjecture further through two case studies that reflected broader patterns in our data. One student, Gary, attempted to directly attach an exponential formula to individual behavior, which led him to a contradictory interpretation of individual behavior in the simulation. After attending to the ‘jagged’ people born graph, he more clearly articulated how he expected individual patterns of reproduction to quantitatively contribute to change in the population, ultimately changing overall patterns of growth. Another, Sarah, relied mainly on mathematical procedures to describing patterns of change in the simulation, but was hesitant to interpret what those procedures meant in the context of the simulation. After noticing and working to make sense of unexpected “dips” in the people born graph, Sarah connected the probabilistic reproduction of individual agents in the simulation to this quantitative representation of change. She later meaningfully interpreted taking the derivative of population growth as finding the change in number of people in the simulation. We argue that in both of these cases, Gary and Sarah already possessed productive resources for making sense of simulation, and productive dispositions – however, the jagged nature of the people born graph provided an organizing device that helped them connect those resources into a meaningful, coherent understanding of the simulation, its quantitative output, and related mathematical ideas.

These findings have implications for research on complex systems thinking and the design of learning environments. In terms of complex system thinking, our findings highlight nuances in what it means to understand the relationship between levels of observation (“levels thinking”, as described in Wikensky & Resnick, 1999; Levy & Wilensky, 2008) and their relationship to measurable patterns and mathematical formalisms. Often, simulation-based learning environments for complex systems include quantitative patterns such as the emergence of dynamic equilibrium in ecological systems (Wilensky & Reisman, 2006) or canonical mathematical formulae such as the Maxwell-Boltzmann distribution law (Wilensky, 2003) as part of their learning objectives. However, these quantitative patterns are calculated through aggregation mechanisms that are not readily apparent to students, and little work has been done to explore what reasoning is needed for learners to build those connections (Chi, et al., 2012).

Our findings suggest that even if students are quite comfortable describing relationships between levels of the *behavior* of the system, and even between some quantitative and mathematical aspects of those same systems, they still may not attend to the relationship between the agent-level behaviors that comprise a system and their measurable contributions to those quantitative patterns. This, in turn, can lead to specifically *quantitative* forms of complex systems difficulties, such as slippage between (mathematical) levels. And, it limits the degree to which learners might be able to interpret what are the implications of those quantitative patterns, and their corresponding mathematical representations and predictions, for constituent agents in the system and hence, the complex system as a whole.

For participating students in our study, building these connections was not particularly difficult or inaccessible. By drawing learners' attention to mathematical resources that emphasize the ongoing probabilistic contributions of individual agents, learners began to identify new connections across resources they were already using that helped them navigate the quantitative and measurable contributions of individual agent behaviors. In our case, that graph of people born helped learners recognize this connection, because it featured dramatic and unexpected 'jags' or 'dips' that could only be reasonable explained by connecting it to the probabilistic aspect of the simulation rules.

Together, these findings have important implications for the design of simulation-based environments for learning about complex systems. First, designers cannot assume that simply including quantitative descriptions of systems such as graphs alongside visuospatial and programmatic representations of those systems implies students will automatically work to make sense of how the system, its constituent members, and its behavior are reflected in those graphs. However, encouraging students to attend to and make sense of resources at the intersection of Individual Behavior and Patterns of Change – that illustrate clearly how nuances of agent-level behavior contribute to and are recorded within larger quantitative measures – can help foster such connections. We have also been exploring the potential of allowing students themselves to construct agent-based simulations and linked quantitative representations using high-level descriptions of agent behaviors, with the goal of foregrounding the connections between individual behaviors, their quantitative manifestations, and how the result of those behaviors can be measured over time (Wilkerson-Jerde & Wilensky, 2010; Wilkerson-Jerde, 2012).

This study also illustrates the utility of the Calculus of Complex Systems Framework we introduce as an analytic tool and guide for environment design. The framework allowed us to identify what resources students relied upon for different patterns of quantitative change. It also brought students' inattention to the connections between Individual Change and Patterns of Change to the surface as a potential reason that some students struggled to make mathematical connections in the simulation. While our current study dealt with relatively simple emergent behavior in the particular context of agent-based simulations, we believe that the framework holds promise to inform the study of student reasoning about quantitative change in complex systems more generally.

Finally, this study illustrates the importance of explicitly attending to quantitative and mathematical issues in the context of complex systems reasoning. There are nuanced and powerful connections between the individual agents that comprise a complex system and the quantitative and mathematical methods that those agents generate together over time. Exploring these connections is not currently emphasized in curricula, has not been explored much in literature, and was not spontaneously undertaken by students in our study. However, these connections were certainly *accessible* to learners, and led Gary and Sarah to engage in meaningful and sophisticated reasoning that allowed them to clarify and elaborate their understandings of the system more generally. Given the prevalence of mathematical representations in the study of science, complex systems, and even educational simulations, we argue that mathematical and quantitative reasoning are key components of complex systems fluency.

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Table 1. Summary of resource codes.

Resource Code	Description
~~Total~~ Population Graph (M)	Explicit use of the graph of total population, including reference to or gestures toward the shape of the graph, slope, height of the graph at specific points, and so forth.
Exponential (M)	Explicit mention of the exponential function as an algebraic, graphical, or qualitative pattern or operation.
Monotonic (M)	References to the monotonic (always increasing) quantitative behavior.
People Born Graph (M)	Explicit use of the graph of people born, including reference to or gestures toward the shape of the graph, slope, height of the graph at specific points, and so forth.
Population Actions / Properties (B)	Explicit reference to the population as a whole. Examples include referencing actions performed by the population as a whole such as growing or “reproducing more and more” as a collective unit; or referencing properties that exist at the collective level such as a population’s size.
Chance/Probability (M)	Explicit reference to mathematical chance, probability, or randomness in the simulation.
Person Actions / Properties (B)	Explicit reference to an individual member of the population as an autonomous actor. Examples include citing actions such as giving birth, seeking partners, choosing to have kids; or referencing properties at the individual level such as indivisibility or describing “each” person.
Quantities (M)	Explicit reference to a specific quantity revealed in the simulation interface (typically when the mouse is hovered over the graph).
Systemic (B)	Explicit reference to previous knowledge of "general" or known patterns for population growth systems.

Table 2. Resources cited by participants when responding to questions about Patterns of Accumulation.

	When is Population Highest?												Why is Population Highest?											
	T	S	M	K	I	G	E	Z	C	B	A		T	S	M	K	I	G	E	Z	C	B	A	
Population Graph	■		■	■		■	■	■	■	■		8									■			1
Exponential												-		■	■	■						■		4
Monotonic		■			■						■	3	■						■	■				3
People Born Graph												-												-
Population Action/Prop												-	■			■	■	■				■	■	6
Chance												-						■					■	2
Person Action/Prop												-											■	1

Table 3. Resources cited by participants when responding to questions about Patterns of Change.

	When is Population Changing the Most?												Why is Population Changing the Most?											
	T	S	M	K	I	G	E	Z	C	B	A		T	S	M	K	I	G	E	Z	C	B	A	
Population Graph	■	■		■	■		■	■	■		■	8		■		■				■		■		4
Exponential			■							■		2			■		■							2
Monotonic			■									1												-
People Born Graph	■											1												-
Population Action/Prop						■						1	■	■	■	■		■	■	■	■		■	9
Chance												-						■					■	2
Person Action/Prop												-	■						■	■	■		■	5

Table 4. Summary of resources cited by participants across first four interview questions.

	Across First Four Questions											
	T	S	M	K	I	G	E	Z	C	B	A	
Population Graph	2	2	1	3	1	1	2	3	3	2	1	11
Exponential		1	3	1	1					2		5
Monotonic	1	1			1		1	1			1	7
People Born Graph	1											1
Population Action/Prop	2	1	1	2	1	3	1	1	1	1	2	11
Chance						2					2	2
Person Action/Prop	1						1	1	1		2	5
Total Resources Used	7	5	5	6	4	5	5	6	5	5	8	

Table 5. Resources cited by level for each participant describing connections between a 'person' agent and overall pattern of population growth.

	Individual Behaviors	Group Interactions	Patterns of Change	Patterns of Accumulation	Total
Population Graph		M A		M E A	3
Exponential	G			S GE	2
Monotonic					
People Born Graph					0
Population Action/Prop	C	T MKI E CBA	T	TSMKI EZCB	10
Chance	TS KIGE BA	T KI E B	T	S	8
Person Action/Prop	KIGEZCB	K EZ B			7
Total	10	9	2	11	

Table 6. Resources cited by level for each participant describing connections between the people born graph and the total population graph.

	Individual Behaviors	Group Interactions	Patterns of Change	Patterns of Accumulation	Total
Population Graph			T M BA	TSMKIGEZCBA	11
Exponential				S	0
Monotonic					0
People Born Graph		Z A	T IGEZCBA	SMK G	11
Population Action/Prop		S GEZCBA	M GE CBA	SMKIGEZCB	10
Chance	S	S		S	1
Person Action/Prop	S I	I	I	S	2
Total	2	8	9	11	

Table 7. Resources cited by level for each participant describing why the people born graph is jagged.

	Individual Behaviors	Group Interactions	Patterns of Change	Patterns of Accumulation	Total
Population Graph	B	B	G B	B	2
Exponential					0
Monotonic					0
People Born Graph	TS Z	TS KIGEZCB	KIG BA	T A	10
Population Action/Prop	A	MKIGEZCBA	MK G A	T	10
Chance	S K BA	S K B			4
Person Action/Prop	TSMK EZCBA	TSMK EZCB	M		9
Total	9	11	6	4	

Figure 1. NetLogo simulation interface used during student interviews.

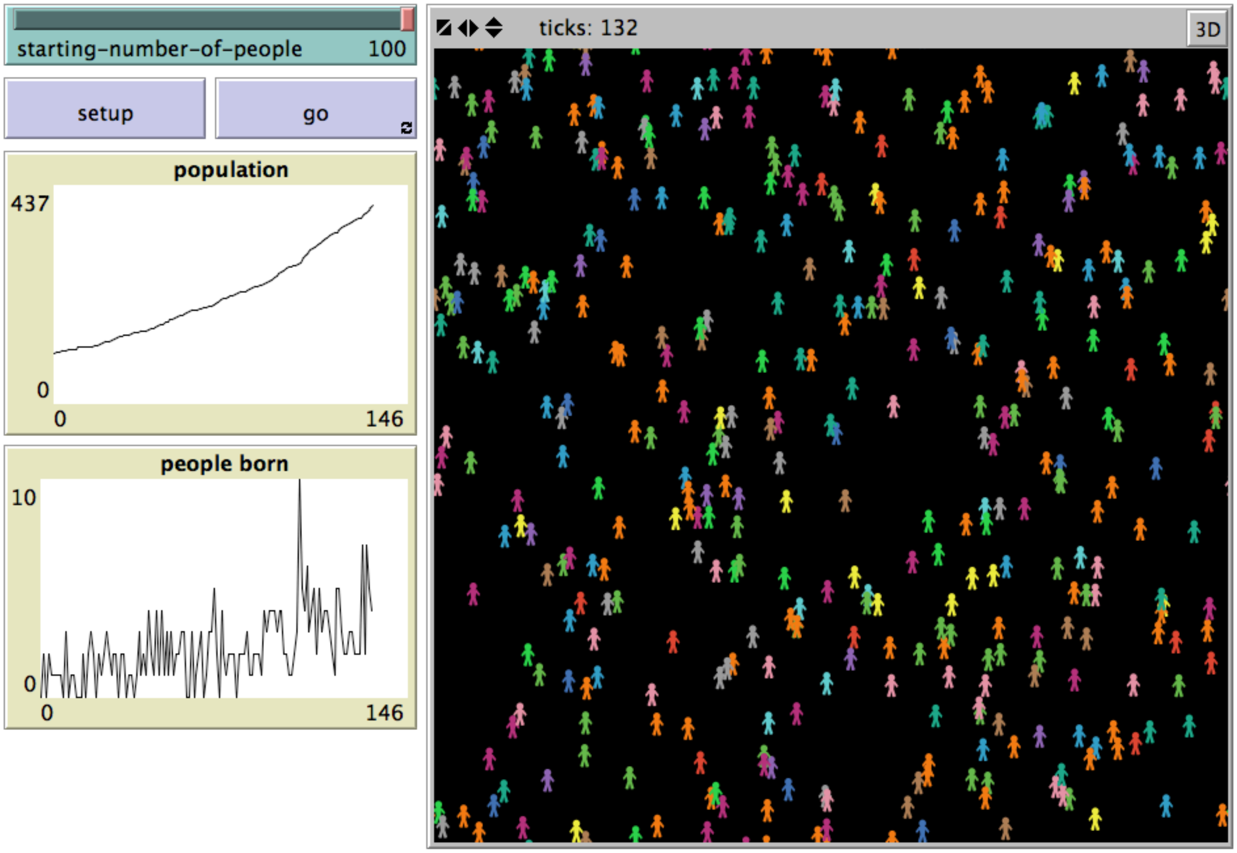


Figure 2. Detail of corresponding population and people born graphs produced by one execution of the NetLogo simulation.

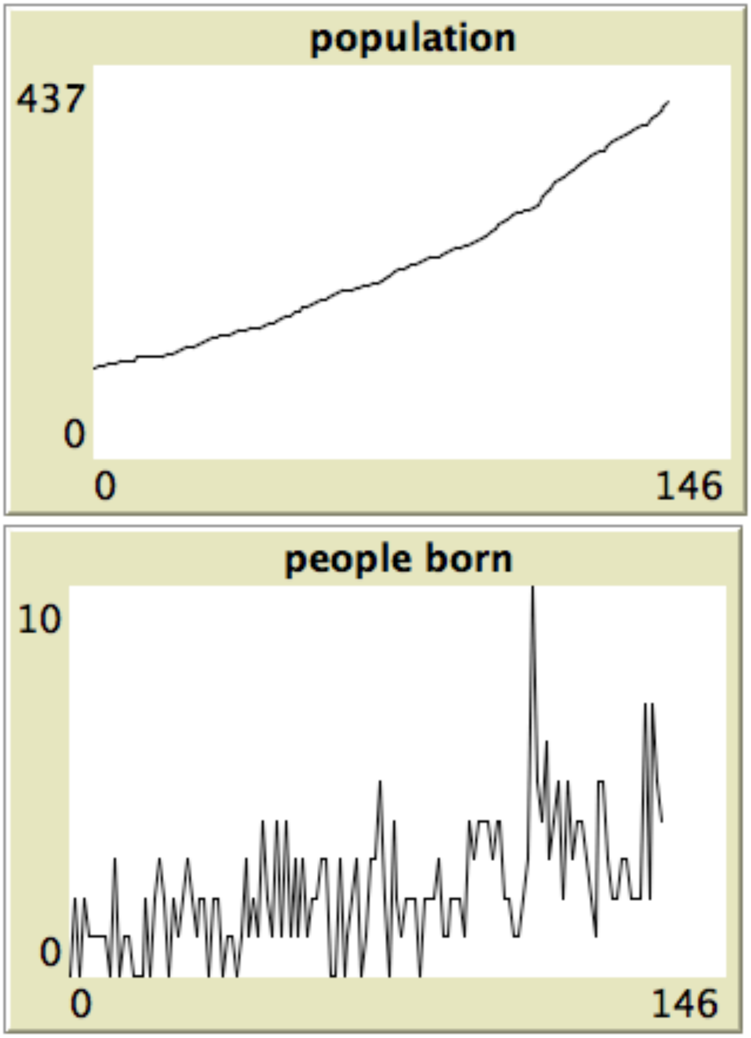


Figure 3. Time lapse of one execution of the NetLogo simulation at 0, 66, 132, and 198 ticks.

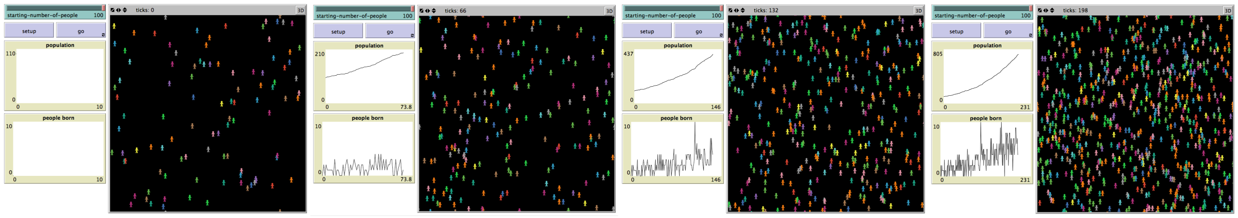
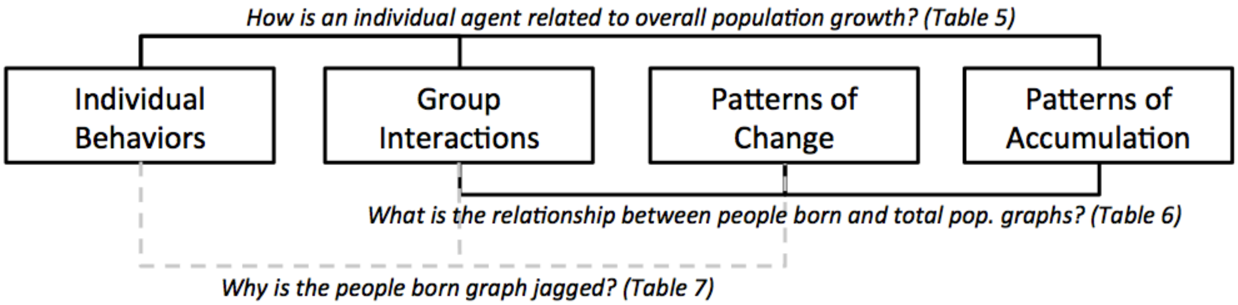


Figure 4. Summary analysis of Part II, organized using the QCS Framework.



Appendix A

Code Examples: Identifying Resources

When is population highest?

[Betsy] Where is it the, uh, [points at population graph] when x is 184.

This statement is coded as referencing the *population graph* to reason about accumulation, since Betsy gestures to and reads from it “x is 184”.

[Gary] The later the tick, which is at like uh, 153ish.

This statement is coded as referencing *monotonic/programmatic behavior* to reason about accumulation, since Gary noted that population would always be highest “the later the tick” or at the end of the simulation’s execution. It is also coded as the *population graph*, which he gestures to and reads the value “153” from.

Why is population the highest at the end?

[Eddie] It doesn’t, it doesn’t decrease it’s just increasing so the latest tick is the highest.

This statement is coded as referencing *monotonic/programmatic behavior* to reason about accumulation, since Eddie noted that population in this particular simulation can only increase, and referred to simulation “ticks”.

[Kevin] It's um, because um, in an exponential growth as time increases the amount of people increases so it's generally given that as more time elapses there will be more people.

This statement is coded as referencing the general properties of *exponential* growth (as the independent variable time increases, so does the dependent variable) to reason about accumulation, as well as connecting this growth to the *collective* behavior of a population (“the amount of people increases”).

When is population changing the fastest?

[Alex] Uh, the rate of, well the rate of change if gonna be tor—it’s gonna be greater when the, if you were to make a tangent line to the curve [on the population graph], if the tangent line was steeper that technically means the rate of change is getting greater.

This statement is coded as referencing the *population graph* to reason about rate of change, since Alex uses it to determine where rate is highest by visually evaluating the slope of the graph.

[Mani] When is it changing the fastest? Um, probably by the next time you go, it’s gonna be fastest, because there’s um a bigger, a big population right now, so by next, 1%, let’s say how many people there are right now, 1% of 287, it’s gonna be 2.87 people next time, so.

This statement is coded as referencing *quantities, collective properties*, and *monotonic/programmatic* aspects to reason about rate of change, because Mani refers to the size of the collective population, notes that this will be largest at the end or "the next time you go", and notes with a specific quantitative example that 1% of a large number represents more differential change than 1% of a smaller number. Note that although Mani cites 1% in her explanation, this is not cited as *probability/chance* since she does not apply it this way (in fact, she contradicts such an application by applying it to the population as a whole).

Why is population changing the fastest?

[Irene] If you have a single number and that's raised, if you have a constant rate of change but your initial value is greater then your end amount will be greater.

This statement is coded as referencing the general properties of *exponential* growth ("a … number … raised") - in this case, that the constant 'rate' characteristic of exponential growth can produce differentially larger values.

[Todd] Um because there's more people to create that change. There's a higher probability I guess of a person being born when there's more people that can have that child.

This statement is coded as referencing *collective behavior* ("more people"), *chance/probability*, and *individual behavior* ("probability … of a person being born") to explain rate of change in the simulation.

Appendix B

Code Examples: Identifying Resources and Levels of Description

How is an individual agent related to overall population growth?

[Betsy] No, him, as like more people, it's not him alone but in a population there are more chances that more people will reproduce.

M: Okay I see. Okay

B: Because there is a greater number, a greater population.

In terms of **resources**, this response is coded as connecting *individual behavior/properties* because she references an individual alone, *probability/chance*, and *collective behavior/properties* because she notes that when there are a greater number of people there are more individual chances to reproduce.

In terms **levels of description**, this response is coded as using both agent behavior and probability/chance to describe the system at both the *Individual Behavior* and *Group Interaction* levels of description. This is because by noting that individuals' role in population growth is "not alone", their individual behavior ("him") can also be thought of as happening "in a population" to create multiple "chances". We code Betsy's reference to collective behavior in terms of "more people reproducing" a *Group Interaction*, and her reference to a "greater number" in the population to speak to population as a *Pattern of Accumulation*. It is unclear from this exchange alone whether Betsy recognized quantitative, or only qualitative connections between individual behavior and population growth.

[Zoe] So at first you start with a smaller amount of people and then they each have like a kid or something then the next time around you have twice as much people so you have a bigger base so therefore, that change, the next change is gonna be greater because you're gonna have two times the amount of people doing the same thing.

In terms of **resources**, this response is coded as involving *individual behavior/properties* because she describes that "each" agent "has a kid", *collective behavior/properties* to describe properties of the population, and a specific *quantity*.

In terms of **levels of description**, this response is coded as using agent behavior/properties to describe both *Individual Behavior* and *Group Interactions* ("each have like a kid"). Collective behavior ("start with a smaller amount of people") is coded as describing *Group Interactions*. Finally, the quantity is used to describe *Group Interactions* ("each have a kid… twice as much people") and *Patterns of Change* ("the change is gonna be greater because you're gonna have two times the amount of people"). Note, however, that while Zoe is specifying specific quantitative connections across these levels, she is not considering the actual 1% chance rule that drives the simulation she is interacting with.

What is the relationship between people born graph and total population graph?

[Kevin] To, um, I mean you could simply like we already related how this graph's irregular [gestures toward born graph] and so is this [gestures toward population graph] but um, simply putting um, this [population graph] isn't going to model dips in population nearly as well as this, so you could simply I guess say that this graph is simply almost like a best fit line of this graph so it takes like the top points are the most important pertinent points of the bottom graph and it simply shows up on the top.

In terms of resources, this response is coded as involving the *people born graph* ("bottom graph") and the *population graph* ("top graph"). Kevin also explicitly relates these graphs to the *collective behavior/properties* of population as an entity.

In terms of **levels of description**, Kevin describes the population graph as a best fit line that reproduces important points of the people born graph, which suggests that he is using both graphs to describe the *Pattern of Accumulation* of the population. Kevin is also referring to collective behavior/properties of the population to describe its *Pattern of Accumulation* and the *Group Interactions* it describes ("dips in population").

[Irene] Well, yes, I, there is a relation because the population depends on the number of people being born (mhm), but um, the people being, the, it's not vice versa, so, like the people being born is almost independent and the population depends on that, so.
M: Okay, almost independent how, can you talk a little more about that?
I: Um, because, you're um, people being born, is, is going to affect the population but, the number of people there isn't going to really affect the number of people being born. Yeah, I don't know if that makes sense.

In terms of resources, this response is coded as involving the *population graph* and the *people born graph*. She also talks about the population's *collective behavior* and dependent and independent relationships within that behavior. Irene's insistence that people being born is "almost independent" from the population provides some evidence that Irene is aware of behavior at something other than the collective level of population. This excerpt represents an example where looking at other portions of Irene's transcript is useful. Returning to her interview, Irene's responses to the previous questions indicated that she was aware the individuals have a "one percent chance" and that when there are more people "someone's bound to have another person". Therefore, we interpret the claim for independence in this excerpt as also implying *individual behavior/properties*.

In terms of **levels of description**, Irene is using the population graph to describe Patterns of Accumulation, and using the people born graph to describe *Patterns of Change*. While she knows the two are interdependent, she does not articulate how these resources embody information at different levels of description. Irene uses agent behaviors to describe *Individual Behaviors* and *Group Interactions* ("people being born"). She also connects agent behavior to *Patterns of Change* and population behavior to *Patterns of Accumulation*.

Why is the people born graph jagged?

[Mani] I think so, cause you can't have half a person or .8 person.
M: Mmkay. So sometimes it's, it gets kind of lower and then it goes back up, does that make sense?
Well uh, [pause] I guess so, uh.
M: It's okay to say no.

Um, cause, I don't know if this simulation is like, perfect like, cause people may not choose to have kids. So some years there might be less and some years there might be more. But it evens out since it's an average 1, .01%, er 1%.

In terms of **resources**, this exchange is coded as citing *individual behavior/properties* and *quantities* to justify a feature of the *people born graph*, since people cannot be reasonably treated as divisible entities. After further probing, Mani cites more individual behavior ("people may not choose to have kids") and asserts that individuals born comprise a *collective behavior/property*. However, there is no more evidence in the interview that Mani identifies mathematically random or probabilistic behavior as an element of the simulation – instead, she identifies the 1% as resultant average.

In terms of **levels of description**, Mani is using chance to describe *Group Interactions*. She considers agent behavior/properties that "you can't have half a person", specific quantities, and the people born graph as resources for describing the *Pattern of Change* exhibited by the simulation. She also considers differnt agent behaviors, that "people may not choose to have kids", to describe the *Individual Behaviors* in the simulation.

[Kevin] Like you said that um, they have a 1% chance of reproducing and for instance, at the spikes maybe here and here, the, just at that moment of that 1% chance actually occured, so, they had a higher uh reproducing, while at the dips over here it just didn't come through and then, um, the other chances of not reproducing kicked in and just right there they didn't reproduce as much.

In terms of **resources**, this statement is coded as involving *individual behavior/properties*, since he is referencing the chance individuals have to reproduce, as well as *collective behavior/properties*, since he suggests the collective "they" reproduce more or less at different points in time, to describe the *people born graph*. Although this excerpt could be interpreted as Kevin claiming that the population as a whole has a chance to increase by 1% at any point in time (rather than each individual), we code the except as also involving *probability/chance* because there are other moments in the interview where Kevin refers to the 1% chance as belonging to an individual, and he specifically mentions 1% chance "like you said" (when the interviewer described the agent-based rules of the simulation).

In terms of **levels of description**, Kevin is relating *agent behavior/properties*, "reproducing", to the 1% reproduction chance rule and is using both of these resources to describe Individual Behavior and Group Interactions. He also connects this agent behavior and chance to *Patterns of Change* by describing how they directly affect the graph of people born, which also measures *Patterns of Change*.