Round Two Math Contest Solutions Spring 2013

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1. If the exam questions were all unique, the total number of exam questions would be 2 x 25 = 50. In this case, if you take 2 x 25, you would count 12 questions twice, so you have to reduce the number by 12, giving 50 - 12 = 38, and the answer is (D).

2. Since a triangle only has three sides, the two given sides of 8.1 and 1.4 have to be adjacent to each other with an angle between them. This angle can vary from almost zero to almost 180 degrees. If the angle between the two sides is almost zero, then the two sides are almost on top of each other, and the third side has a minimum length of about 8.1 - 1.4 = 6.7.

On the other hand, if the angle between the two sides is almost 180 degrees, the third side is almost equal to the sum of the two given sides: 8.1 + 1.4 = 9.5, which would be the maximum possible value. So the third side must lie between 6.7 and 9.5, and the only even integer choice between 6.7 and 9.5 is 8, and the answer is (D).

Notice that if odd integers were allowed, then seven or nine would have been possible answers, so the restriction to even integers was added to make the answer unique.

3. Let us multiply out the terms with parentheses, and the two given equations become

$$\pi x + \pi y + ey = \pi + 2e \qquad (1)$$

and

$$\pi x + 3ex + \pi y + 4ey = \pi + 5e \qquad (2)$$

Notice that in each equation the number π appears always as π or πx or πy, whereas, the number e has different coefficients in front of it. Making use of this symmetry of the two equations, we subtract the equation (1) from equation (2). The resulting equation is:

$$x + y = 1 \qquad (3)$$

where a common factor of 3e was cancelled.

Now substitute the value x = 1 - y for x in equation (1), and obtain the following result:

$$\pi - \pi y + \pi y + ey = \pi + 2e \qquad (4)$$

Equation (4) has the solution y = 2. Then from equation (3), we have x = -1. We assume that the ordered pair (a, b) = (x, y), so then b -a = x – y = 2 – (-1) = 3, and the answer is (E).

4. I did not find a better way to solve this problem then just checking the list of answers given (not very satisfying). A prime number is one which has only itself and one as factors. Starting with 2014, we have $2014 = 2 \cdot 19 \cdot 53$, and reducing each prime factor by one yields 1, 18, and

52. One is a factor of 18, but 18 is not a factor of 52, so 2014 is out. We check $2015 = 5 \cdot 13 \cdot 31$, and reducing each by one gives 4, 12, and 30. We see that 12 is not a factor of 30, so 2015 is out.

The year 2016 has a prime factorization with exponents greater than one, since $2016 = 2^5 \cdot 3^2 \cdot 7$. Reading the problem again carefully, we see that the distinct prime factors of 2016 are 2, 3, and 7, written in increasing order.

Reducing each of these by one, we get: 1, 2, and 6, and we see that 1 is a factor of 2, and 2, is a factor of 6. So it looks like 2016 satisfies the requirements of the problem and the answer is (C).

I think a year that is itself a prime number would satisfy the requirements of the problem since it has only one prime factor, which if reduced by one will still be a factor of itself. So 2017 works by default since it is a prime number, but it is bigger than 2016.

Also, $2018 = 2 \cdot 1009$, and 1009 is prime, and 1 is clearly a factor of 1008, so 2018 works too. There seem to be a lot of numbers with the desired property that reducing each distinct prime factor by one yields a list of numbers in increasing order such that each number is a factor of the next one in the list.

Mathematicians would want to know things about these numbers like: Are there an infinite number of them? Do they continue to occur as often as in the series 2014 - 2018, or do they begin to "thin out" as you go higher? Are there always consecutive numbers with the property? Is there a simple formula, which generates all of these numbers, etc. Answering questions like this is done in the branch of mathematics known as number theory. In this field, questions tend to be simple to state, but difficult to solve! I do not know whether the questions above have answers or not.

Finally, note that one (1) is not considered to be a prime number. Why not? It has only itself and one as factors. Indeed, the number one seems to have been considered a prime among mathematicians through the nineteenth century. But they more recently decided to exclude one from the list of primes. One reason is the fundamental theorem of arithmetic, which states that each number has a unique factorization into prime factors. If we let one be a prime, then each number would have two distinct factorizations. For example, $6 = 1 \cdot 2 \cdot 3 = 2 \cdot 3$ would be distinct factorizations into primes. There are more serious reasons for excluding one from the primes. Some of these involve the fact that 1 and (-1) are unique in the integers in having multiplication inverses. Including one as a prime causes difficulty in at least one branch of more advanced mathematics, known as abstract algebra.

5. Intersecting the x-axis occurs when y = 0, so we put y = 0 in both of the given equations, yielding: 0 = 2x + b, and 0 = mx - 6. We wish to eliminate x between these equations, so solve the first one for x, which gives x = - b/2. Substitute this expression for x into the second equation, which gives: 0 = - mb/2 - 6. This can be put into the form:

$$mb + 12 = 0 \qquad (1)$$

and the answer is (B), since equation (1) is required to be true given the conditions of the problem. I investigated a bit further to see if any of the other answers could be true, even if not necessarily required.

The answer cannot be (A), since the substitution mb = 12 into equation (1) gives 24 = 0, a contradiction.

For answer (C), suppose we eliminate m from equation (1) using m = 3b. The resulting equation is

$$3b^2 + 12 = 0 \qquad (2)$$

which does not have a real solution for b. So answer (C) is not possible with m and b real numbers.

For answer (D), substitute m = -3b into equation (1) and get

$$3b^2 = 12 \qquad (3)$$

which has solutions $b = \pm 2$. So answer (D) could also be true, but it is not required to be true by the conditions of the problem.

Finally, for answer (E), substitute $m = \frac{b}{3}$ into equation (1) and get

$$\frac{b^2}{3} + 12 = 0 \qquad (4)$$

and once again, equation (4) requires b to be complex.

So answer (B) is required to be true, but answer (D) could also be true, but is not required. The other answers cannot be true.

6. Combine the terms on the left side of the given equation with the result:

$$\frac{a+b}{ab} = \frac{1}{n} \qquad (1)$$

We set the reciprocal of each side of equation (1) equal, with the result

$$\frac{ab}{a+b} = n \qquad (2)$$

Note that equation (2) is symmetric under interchange of a and b, so if (a, b) is a solution, then so is (b, a). Since we are requiring $b \geq a$, then only one allowed solution is generated for any values of a and b.

Let us put a = x and b = x + m into equation (2), where m is an integer.

$$\frac{x(x+m)}{2x+m} = n \qquad (3)$$

Re-write equation (3) as a quadratic equation in x:

$$x^2 + (m - 2n)x - mn = 0 \qquad (4)$$

Solve equation (4) with the quadratic formula:

$$x = \frac{(2n - m) \pm \sqrt{(m - 2n)^2 + 4mn}}{2} \qquad (5)$$

Since we must have x > 0, we use the plus sign in equation (5) if m > 0, and the minus sign if m < 0. In order for x to be an integer, the expression under the square root must be a perfect square, so we write

$$(m - 2n)^2 + 4mn = j^2 \qquad (6)$$

where j is an integer. Equation (6) simplifies to

$$4n^2 = j^2 - m^2 = (j - m)(j + m) \qquad (7)$$

Recall that $\sqrt{j^2} = |j|$, so we can write equation (5) as

$$x = \frac{(2n - m) \pm |j|}{2} \qquad (8)$$

Now let us look for solutions of equation (7).

If m = 0, then equation (7) gives $|j| = 2n$, and equation (8) gives x = 2n = a = b for m = 0 (recall $b = a + m$). The minus sign in equation (8) gives x = 0, which is not allowed, so the solution (a, b) = (2n, 2n) is available for any value of n.

Let us proceed by assuming that $4n^2 = fg$, where f and g are positive factors. Then equation (7) becomes

$$(j - m)(j + m) = fg \qquad (9)$$

We look at several cases:

Case 1. (j – m) = f, and (j + m) = g. Then we solve two equations for the two unknowns j and m with the result $j = \frac{f+g}{2}$, and $m = \frac{g-f}{2}$. Then equation (8) with the plus sign gives $x = a = \frac{2n+f}{2}$, and $b = a + m = \frac{2n+g}{2}$.

Equation (8) with the minus sign gives: $a = \frac{2n-g}{2}$, and b=$\frac{2n-f}{2}$. However, since $4n^2 = fg$, we have $2n = \sqrt{fg}$. With this substitution into the values of a and b here, we obtain

$$a = \frac{\sqrt{g}\left(\sqrt{f} - \sqrt{g}\right)}{2} \quad and \quad b = \frac{\sqrt{f}\left(\sqrt{g} - \sqrt{f}\right)}{2} \qquad (10)$$

If f = g. then equation (10) gives a = b = 0. If $f \neq g$, suppose f > g. Then equation (10) gives b < 0, which is not allowed. If g > f, then a < 0.

It follows that we only need to consider the plus sign in equations (8) and (5).

Case 2. Interchange the factor assignments and take: (j – m) = g, and (j + m) = f. This yields $j = \frac{g+f}{2}$ and $m = \frac{f-g}{2}$, the opposite of case 1. Then equation (8) gives $a = \frac{2n+g}{2}$, and $b = \frac{2n+f}{2}$. This interchanges a and b, compared to case 1 and as discussed above, this is not a new solution.

Case 3. Use negative factors, replacing f with (-f) and g with (-g) in case 1. This changes the sign of both j and m, but since equation (8) has absolute value of j, changing the sign of j has no effect on a and b. As seen in case 2 above, changing the sign of m just interchanges the values of a and b.

To summarize, we have found that interchanging the factor assignments in equation (9) interchanges the values of a and b. Using negative factors also interchanges the values of a and b, and using the minus sign in equations (8) and (5) makes either a or b negative, or zero if the factors are equal. So from now on, we will assume that j and m are positive integers, and we will not miss any solutions.

Let us now suppose for the moment that n is a prime number, having only itself and one as factors. Equation (7) gives a factorization of $4n^2$on the right side. If n is prime, then the possible factorizations are $4n^2 = 1 \cdot 4n^2,\ \ 2 \cdot 2n^2,\ \ 4 \cdot n^2,\ \ or\ \ 4n \cdot n.$

1. Let us start with $4n^2 = 1 \cdot 4n^2$. In equation (7) we take $j - m = 1$ and $j + m = 4n^2$. Solve these two equations for the two unknowns, j and m with the result

$$j = 2n^2 + \frac{1}{2}, \quad m = 2n^2 - \frac{1}{2} \qquad (11)$$

However, in equation (11), m cannot be an integer, and b = a + m would not be an integer if a is one. Or if b was an integer, then a could not be one also. So this factorization yields no integer solutions for (a, b).

2. Now consider $4n^2 = 2 \cdot 2n^2$: In equation (7) we take $j - m = 2$, and $j + m = 2n^2$. This gives the values $j = 1 + n^2$ and $m = n^2 - 1$. With these values for j and m, equation (8) gives

$$x = a = n + 1,\ \ and\ \ \ b = n + n^2 \qquad (12)$$

Since the factorization used here did not depend on n being a prime number, we see that there are two solutions available for any value of n. Either (a, b) = (2n, 2n) for m = 0, obtained above, or the solution in equation (12). You can verify by substitution that the solution of equation (12) does indeed satisfy the given equation or equation (1).

3. Now consider $4n^2 = 4 \cdot n^2$: In this case, equation (7) gives $j - m = 4$ and $j + m = n^2$. The solution is $j = 2 + \frac{n^2}{2}$ and $m = \frac{n^2}{2} - 2$. If n = 2, we have m = 0, which gives (a, b) = (2n, 2n) as

above. If n is a prime number greater than two, it must be odd, and j and m will not be integers. As discussed above, this forces either a or b to be non-integer, which is not allowed.

4. Now consider $4n^2 = 4n \cdot n$: Then we have from equation (7), $j - m = n$ and $j + m = 4n$. The solution is $j = \frac{5}{2}n$ and $m = \frac{3}{2}n$. But in equation (8), this gives $x = a = \frac{3n}{2}$, which is not an integer for any prime greater than two, since such a prime has to be odd.

The values of a and b are integers if n = 2, and equation (8) then gives x = a = 3, and b = 6. However, this solution is the same as the one provided by equation (12) with n = 2, so we still only have two solutions for n = 2.

We have shown that for n equal to any prime number, there are only two solutions available: $(a, b) = (2n, 2n)\ \ or\ \ (1 + n, n + n^2)$. In order to get a third solution, n must be a composite number so that there are additional factorizations beyond the four considered above.

Let us suppose that n is even, so we have n = 2k, where k is an integer. Then we can write $4n^2 = 16k^2$, and we can try the factorization: $4n^2 = 16 \cdot k^2$. This factorization was not available when n was assumed to be a prime number. In this case, equation (7) gives $j - m = k^2$ and $j + m = 16$. Solving for j and m gives $j = \frac{k^2}{2} + 8$ and $m = 8 - \frac{k^2}{2}$. Substitute n = 2k, and the values of j and m into equation (8) with the result

$$x = a = 2k + \frac{k^2}{2} \quad and \quad b = a + m = 2k + 8 \qquad (13)$$

We see that b is always an integer for any value of k, while a needs k to be even in order to be an integer. (Recall that the square of an odd integer is odd.) Re-writing equation (13) in terms of n gives (recall n = 2k):

$$a = n + \frac{n^2}{8}, \quad and \quad b = n + 8 \qquad (14)$$

This is a third solution distinct from equation (12) and the solution (a, b) = (2n, 2n) when m = 0. If the integer k is even, then n is a multiple of four. The smallest multiple of four is n = 4. We write down the three solutions: (a, b) = (8, 8) from (2n, 2n), (a, b) = (5, 20) from equation (12), and (a, b) = (6, 12) from equation (14). There could be more solutions, we have not checked all possible factorizations when n is a multiple of 4. Since n = 4 is one of the given answers, the desired answer is (B).

7. The given equation is

$$a^3 + b^2 + c^2 = 2013 \qquad (1)$$

First note the since $\sqrt[3]{2013} \approx 12.63$, the positive integer $a \leq 12$. Let us use the concept of division with remainder. If you use 3 as a divisor, then any integer can be written as $m = 3q + r$, where 3 is the divisor, q is the quotient, and r is the remainder. Note that the remainder r can only be 0, 1, or 2 because if, for example we had r = 4, you could subtract off three from it, increase the quotient q by one, and the actual remainder would be 1.

Mathematicians use the notation $4 \equiv 1(mod\ 3)$, which is read “4 is congruent to 1 mod 3”. This means that 4 leaves remainder 1 when divided by three, which is the divisor (modulus). Let us write $a = 3q + r_a$ and work out the cube of this expression for a with the result:

$$a^3 = (3q)^3 + 3(3q)^2 r_a + 3(3q) r_a^2 + r_a^3 \qquad (2)$$

Note that every term on the right side of equation (2) is a multiple of three, except the last one, r_a^3. So we can write equation (2) as

$$a^3 = (multiple\ of\ 3) + r_a^3 \qquad (3)$$

Similarly, if we write $b^2 = (3q + r_b)^2$and multiply it out, every term except r_b^2 will be a multiple of three. It follows that

$$b^2 = (multiple\ of\ 3) + r_b^2 \qquad (4)$$

A similar equation holds for c^2. Note that 2013 is a multiple of three: $2013 = 3 \cdot 671$. Now substitute equation (4) and (3) into equation (1) with the result:

$$(multiple\ of\ 3) + r_a^3 + r_b^2 + r_c^2 = (multiple\ of\ 3)$$

We group all the multiples of three together on the right side and get:

$$r_a^3 + r_b^2 + r_c^2 = (multiple\ of\ 3) \qquad (5)$$

Recall that the values of the three remainders can only be 0, 1, or 2, so what are the possible values of the remainder when a number is squared or cubed? Consider a cube first, and note that $0^3 = 0$, $1^3 = 1$, and $2^3=8 = 6 + 2$. Notice that for 2^3, we subtract off the highest possible multiple of three, leaving an actual remainder of 2. Any multiple of three occurring on the left side of equation (5) can be grouped with the multiples on the right side. So we conclude that the possible values of r_a^3 are 0, 1, and 2. We did not reduce the possibilities by cubing the number.

Now consider the square of a remainder. We see that $0^2 = 0$, $1^2 = 1$, and $2^2 = 4 = 3 + 1$. In the case where $r_b^2 = 4$, we can subtract off 3 and the actual remainder is 1. So when a number is squared, the possible remainders are only zero and one.

Recall that any multiple of 3 leaves remainder zero when divided by 3, so we can write equation (5) as

$$r_a^3 + r_b^2 + r_c^2 \equiv 0 \qquad (6)$$

where the ≡ symbol means that the left side of equation (6) is a multiple of three. We have seen that r_a^3 can have the values 0, 1, or 2, while r_b^2 and r_c^2 can only be 0 or 1. What are the possibilities?

One possibility for solving equation (6) is to have all three remainders on the left side equal to zero. This means that a, b, and c are all multiples of 3 in equation (1). Let’s see if we can solve equation (1) with this assumption. Let a = 3m, b = 3n, and c = 3j, where m, n, and j are integers. Substitute these values into equation (1) with the result:

$$27m^2 + 9n^2 + 9j^2 = 2013$$

Now divide each term by 3 and get

$$9m^3 + 3n^2 + 3j^2 = 671 \qquad (7)$$

Note that the left side of equation (7) is a multiple of 3, but the right side, 671, is not. This means there is no solution of equation (1) with a, b, and c all multiples of 3.

Examine equation (6) again and observe that if $r_a^3 = 0$, there is no solution because the largest possible value of r_b^2 and r_c^2 is one, but we can't get a multiple of three on the left side with these values (1 + 1 = 2). It follows that r_a^3 cannot be zero, so r_a itself cannot be zero.

Looking at equation (6) again, we see that if $r_a^3 = 2$, then either $r_b^2 = 1$ and $r_c^2 = 0$, or the reverse. Since b and c are interchangeable in equation (1), we will not miss any solutions if we assume that $r_c^2 = 0 \rightarrow r_c = 0$. Recall that since $2^3 = 8 = 6 + 2 \equiv 2$, we saw that $r_a = 2$ is the only possibility that gives $r_a^3 \equiv 2$. Recall that $a \leq 12$, so in this case, integer a can only take the values: 2, 5, 8 and 11. These are the integers less than 12 that leave remainder 2 when divided by three. We have saved a bit of work, since we now know we need not check any multiples of 3 for a in equation (1).

Since $1^3 = 1$, we see that $r_a = 1$ is the only possibility that gives $r_a^3 = 1$. In equation (6), if $r_a^3 = 1$, then we need both r_b^2 and r_c^2. But now it follows that $r_b = r_c = 1$.

Let me summarize what we have found so far. If $r_a = 2$, then integer c must be a multiple of 3. If $r_a = 1$, then $r_c = 1$ also, and c = 1, 4, 7, 10, …, 43. We stop at 43 since $\sqrt{2013} \approx 44.9$.

Finally, here's how we will conduct the search for solutions: Re-write equation (1) as

$$b^2 = 2013 - a^3 - c^2 \qquad (8)$$

We substitute in turn a = 1, 2, 4, 5, 7, 8, 10, 11 and for a = 2, 5, 8, 11, try multiples of 3 for integer c and look for a perfect square on the left side of equation (8).

For a = 1, 4, 7, 10, try c = 1, 4, 7, 10, …, 43.

Notice that by using remainders and division by three, (called modular arithmetic), we have saved a bit of work. We do not have to check every integer. Let's begin the search:

For a = 1, equation (8) becomes: $b^2 = 2012 - c^2$, with c = 1, 4, 7, …. I checked all such values of c up to c = 43 and did not find a perfect square for b^2.

a = 2: Equation (8) becomes $b^2 = 2005 - c^2$, with c = 3, 6, 9, 12, …, 42. For this case, I found two solutions: (a, b, c) = (2, 41, 18), and (2, 39, 22). However, the problem states that we need either b or c a multiple of five (recall that b and c are interchangeable in equation (1)). On to the next case.

a = 4: Equation (8) becomes $b^2 = 1949 - c^2$, with c = 1, 4, 7, …, 43. In this case, I found the solution (a, b, c) = (4, 10, 43). Since b = 10 is a multiple of five, we have the desired solution and we see that a + b + c = 4 + 10 + 43 = 57, and the answer is (B).

I stopped searching at this point, but if you wanted to find all remaining solutions, you could finish a = 4 and continue with a = 5, 7, 8, 10, 11, and c alternating between multiples of three for a = 5, 8, and 11, and c = 1, 4, 7, …, 43 for a = 7 and 10. This would exhaust all possibilities.

8. The factors in the right side of the given equation are all prime numbers, having only themselves and one as factors, so by the fundamental theorem of arithmetic, this factorization is unique. In particular, since there are no factors of two, the product must be odd. This means that none of the letters of AMATYC can be represented by even numbers. Also, since no primes greater than eleven appear in the factorization, none of the letters can be represented by a prime greater than eleven.

Also note that eleven multiplied by any of the other factors is greater than 27, so one of the letters must be represented by eleven. Since eleven is squared in the factorization, and the only letter appearing twice in AMATYC is "A", then we have A = 11.

Now we have the product $MTYC = 3^2 \cdot 5^2 \cdot 7$. At this point, I just did trial and error, since there are relatively few combinations. I started with M = 3, T = 5, Y = 7, and this leaves a remaining factor of fifteen, so I assigned C = 15. To my surprise, this first choice gave a solution satisfying all the conditions above. Note that any permutation of these values, such as M = 7, T = 3, Y = 15, and C = 5, etc., would also work. The problem asks for the sum M + T + Y + C, which in this case works out to 3 + 5 + 7 + 15 = 30, and the answer is (A).

9. A general third-degree polynomial is given as:$ax^3 + bx^2 + cx + d$. We see immediately that P(0) = d, and P(3) = 27a + 9b + 3c + d. We express the product P(0) x P(3) as:

$$d(27a + 9b + 3c + d) = 139 \qquad (1)$$

Now observe that 139 is a prime number, having only 1 and 139 as factors. Since the coefficients a, b, c, and d are stated to be non-negative integers, the expression in parentheses in equation (1) must be greater than one. It follows that

$$P(0) = d = 1 \qquad (2)$$

and

$$P(3) = 27a + 9b + 3c + 1 = 139 \qquad (3)$$

Simplify equation (3) with the result

$$9a + 3b + c = 46 \qquad (4)$$

where a common factor of 3 was divided out.

We are also given $P(1) \cdot P(2) = 689 = 13 \cdot 53$, using the unique prime factorization of 689. Again, since a, b, c, and d are non-negative integers, it follows that $P(2) > P(1)$, and we see that $P(1) = 13$ and $P(2) = 53$. Using d = 1, this gives the following two equations:

$$a + b + c = 12 \qquad (5)$$

and

$$4a + 2b + c = 26 \qquad (6)$$

If you subtract equation (5) from equation (4), the result is

$$4a + b = 17 \qquad (7)$$

Now subtract equation (6) from equation (4) and get

$$5a + b = 20 \qquad (8)$$

Solve equations (7) and (8) for the two unknowns a and b and get a = 3, and b = 5. Then equation (5) gives c = 4.

Now we can directly evaluate P(-1) as

$$P(-1) = -a + b - c + d = -3 + 5 - 4 + 1 = -1$$

and the answer is (B).

10. If t is measured in radians, then $\frac{180}{\pi} t$ is the same angle measured in degrees. If we plot both $\cos t$ and $\cos \frac{180}{\pi} t$ on the same graph, we find numerous intersections occurring in pairs with t in the first quadrant. The graphs are shown in the figure below.

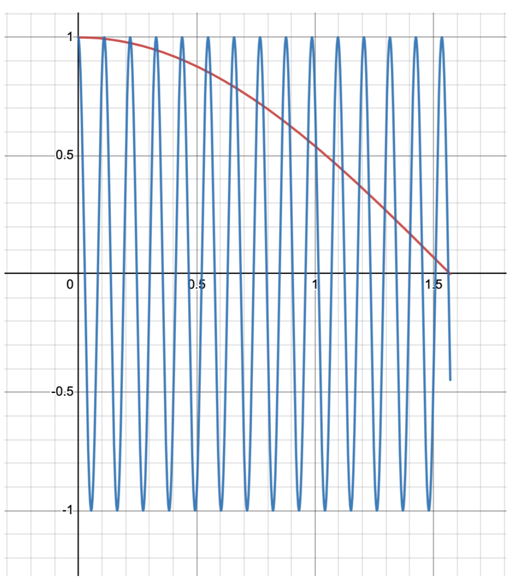


Figure 1

In Figure 1, the slowly varying curve in red is $\cos t$ and the rapidly oscillating curve in blue is $\cos\frac{180}{\pi}$. This makes sense since the period of $\cos t$ is $2\pi \approx 6.283$, while the period of $\cos\frac{180}{\pi}t$ is obtained by the equation $\frac{180}{\pi}t = 2\pi$, which gives a period of $T = \frac{2\pi^2}{180} \approx 0.1097$. Recall that the reciprocal of the period is the frequency of oscillation, so a smaller period gives a larger frequency.

Notice in Figure 1 that the intersections come in closely spaced pairs. To investigate this, see Figure 2 below, where I zoomed in to the first two intersections.

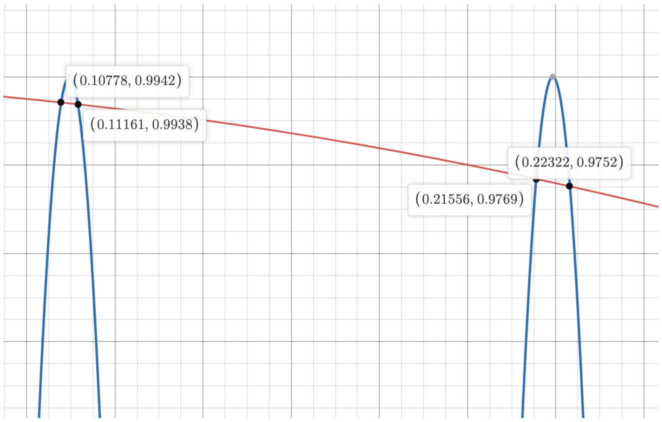


Figure 2

Observe in Figure two that the first solution of a pair occurs just before $\cos\frac{180}{\pi}t$ completes an oscillation, and the second solution of the pair occurs just after the oscillation is completed. Why does it work this way? To see this see the unit circle in Figure 3 below.

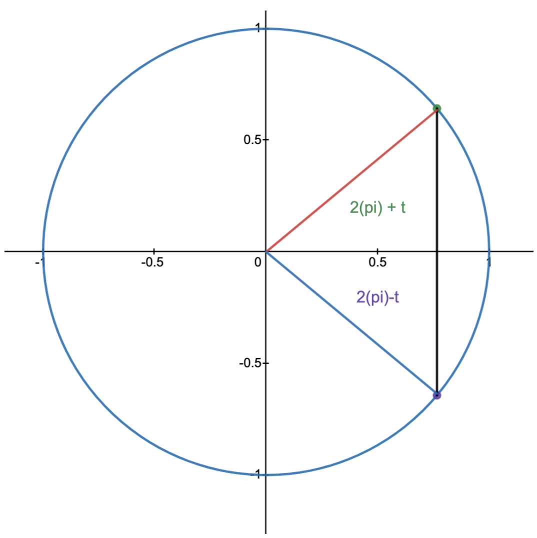


Figure 3

Figure 3 illustrates the trigonometric identity: $\cos(2\pi + t) = \cos(2\pi - t)$. (Recall that the cosine is the x-coordinate of the point on the unit circle.)

So to find the first two intersection points in Figure 2, we have two equations:

$$2\pi - t = \frac{180}{\pi} t \qquad (1)$$

and

$$2\pi + t = \frac{180}{\pi} t \qquad (2)$$

Equations (1) and (2) have solutions

$$t = \frac{2\pi}{\left(\frac{180}{\pi} \pm 1\right)} \approx 0.10778, 0.1161 \qquad (3)$$

At first, I thought this meant $t \approx 0.10778$ is the desired smallest positive solution for which the cosine of t is the same regardless of whether t is in radians or degrees. But you can verify using a calculator, first in radian mode, then in degree mode, that cos 0.10778 has different values in these two cases. What went wrong?

The problem is that in all the work above, t is in radians and we did not make any connection with t being in degrees. Let us look at equation (1) again. In this equation, t is in radians on both sides. But now let us suppose that t is in degrees on the left side of equation (1). Then we should replace 2π on the left with 360, and the equation becomes

$$360 - t = \frac{180}{\pi} t \qquad (4)$$

Now we are taking t in degrees on the left side and t in radians on the right side of equation (4). Solving equation (4) for t yields

$$t = \frac{360}{\left(\frac{180}{\pi} + 1\right)} \approx 6.175404172 \cdots \qquad (5)$$

Using the nine decimal places of the irrational solution to equation (5), I found that the cosine function gave the same answer either in degree or radian mode to the tenth decimal place. So I am confident that in the limit as we include more and more decimal places on the right side of equation (5), we will approach exact agreement of the cosine functions in degree and radian mode. To three decimal places, the solution is 6.175.

Notice that equation (2) with 2π replaced by 360 would give the next highest solution. Then the next pair of higher solutions is obtained by replacing 360 with 360 x 2 = 720, etc.

We began the solution by assuming that t was in radians. What if we instead started with the assumption that t is in degrees?

Then equation (1) above would become

$$360 - t = \frac{\pi}{180} t \qquad (6)$$

In equation (6), t is in degrees on both sides. Notice that the conversion factor on the right side is now the reciprocal because we are going from degrees to radians. But now assume that t is in radians on the left side and replace 360 with 2π. Equation (6) becomes

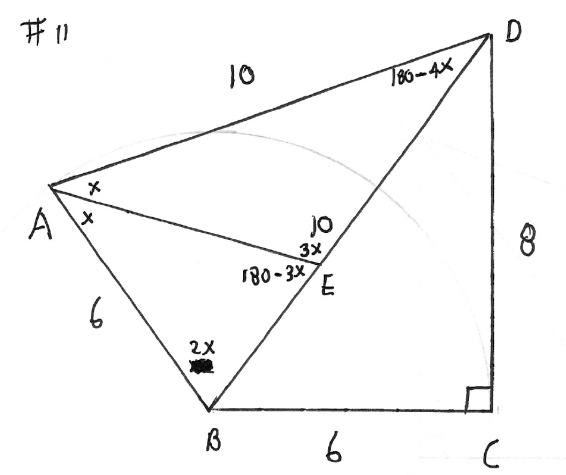
$$2\pi - t = \frac{\pi}{180}t \qquad (7)$$

Equation (7) has the solution

$$t = \frac{2\pi}{\left(1 + \frac{\pi}{180}\right)} \approx 6.17504172\cdots \qquad (8)$$

We see that the same solution is obtained as above.

11. The given quadrilateral is shown in the figure below:



The angle bisector AE divides angle DAB into two equal angles of measure x. By the Pythagorean theorem on triangle BCD, we find BD = 10. So triangle ABC is isosceles, so with angle DAB = 2x, we see that angle ABD = 2x.

Then by sum of interior angles applied to triangle ABE, we find that angle BEA = 180 – 2x, and so angle AED = 2x, being a linear pair with BEA.

Then by sum of interior angles on triangle AED, we find that angle ADE = 180 – 4x. These results in terms of x are shown in the figure. We are trying to find BE.

Apply the Law of Sines on triangle ABD:

$$\frac{\sin 2x}{10} = \frac{\sin(180 - 4x)}{6} = \frac{\sin 4x}{6} \qquad (1)$$

I used a trigonometric identity for the last equality: $\sin(180 - a) = \sin a$. Now apply the double angle formula $\sin 2a = 2 \sin a \cos a$ to the right side of equation (1) with the result

$$\frac{\sin 2x}{10} = \frac{2 \sin 2x \cos 2x}{6}$$

Cancel the common term $\sin 2x$, which cannot be zero and we have

$$\cos 2x = \frac{3}{10} \qquad (2)$$

Then since the sum of sine squared and cosine squared is one, we have

$$\sin 2x = \sqrt{1 - \left(\frac{3}{10}\right)^2} = \frac{\sqrt{91}}{10} \qquad (3)$$

Now use the double angle formula for cosine, written in two-ways as $\cos 2x = 1 - 2(\sin x)^2$ and $\cos 2x = 2(\cos x)^2 - 1$ and get the results

$$\sin x = \frac{\sqrt{35}}{10} \quad \text{and} \quad \cos x = \frac{\sqrt{65}}{10} \qquad (4)$$

We are going to need $\sin 3x = \sin(2x + x)$, so we apply another trigonometric identity for sum of angles: $\sin(a + b) = \sin a \cos b + \sin b \cos a$. This yields

$$\sin 3x = \left(\frac{\sqrt{91}}{10}\right)\left(\frac{\sqrt{65}}{10}\right) + \left(\frac{3}{10}\right)\left(\frac{\sqrt{35}}{10}\right) = \frac{4\sqrt{35}}{25} \qquad (5)$$

where I used equations (2) – (4) above to make the substitutions in equation (5).

Now we can apply the Law of Sines to triangle ABE to get side BE:

$$\frac{BE}{\sin x} = \frac{6}{\sin 3x} \qquad (6)$$

Make substitutions from equations (4) and (5) and get

$$BE = \frac{6 \cdot 25}{40} = \frac{15}{4}$$

and the answer is (A).

12. The x-intercept is the location on the x-axis where the line crosses it, and similarly for the y-intercept. Recall that the slope is rise (or fall) over run, or change in y over change in x. Note that the given intercepts are not identified as being x or y-intercepts, so we have four possibilities, two choices for each line.

If the x and y-intercepts of a line are c and d, then the line contains the points, (c,0) and (0,d), where (c,0) is on the x-axis, and (0,d) is on the y-axis. Let us suppose that both c and d are positive, as is the case for the numbers given. The "run" is obtained by taking (c - 0), which is the distance traveled to the right. The corresponding change in y is (0 - d), which is a "fall" in this case. So the slope is

$$m = -\frac{d}{c} \qquad (1)$$

So the line with the slope in equation (1) and y-intercept d can be written in slope-intercept form as

$$y = -\frac{d}{c}x + d \qquad (2)$$

Let us now list the four possible equations for the two lines:

First line: x-intercept = 4, y-intercept = 2, so c = 4, and d = 2.

$$y = -\frac{1}{2}x + 2 \qquad (3)$$

First line: x-intercept = 2, y-intercept = 4, so c = 2, and d = 4.

$$y = -2x + 4 \qquad (4)$$

Second line: x-intercept = 6, y-intercept = 4, so c = 6 and d = 4.

$$y = -\frac{2}{3}x + 4 \qquad (5)$$

Second line: x-intercept = 4, y-intercept = 6, so c = 4, and d = 6.

$$y = -\frac{3}{2}x + 6 \qquad (6)$$

For the first intersection point, set the right sides of equations (3) and (5) equal with the result x = 12, and then substitution into either equation gives y = -4. So we have (a, b) = (12, -4). Then we can compute

$$3a + b = 32 \qquad (7)$$

Second intersection point, use equations (3) and (6) with the result x = a = 4 and y = b = 0 and we have

$$3a + b = 12 \qquad (8)$$

Third intersection point, use equations (4) and (5) and get x = a = 0 and y = b = 4, which yields

$$3a + b = 4 \qquad (9)$$

Final intersection point, use equations (4) and (6) with the result x = a = -4 and y = b = 12 with the result

$$3a + b = 0 \qquad (10)$$

We see that the only answer missing is 3a + b = 8, so the answer is (C).

13. The year 2012 is a leap year, so the month of February has 29 days and the year is 366 days long. The first three months of the year have total length: 31 + 29 + 31 = 91 days, so we need the first trip to be less than 91 days in length. Let the length of the first trip be the integer a, and the number of trips given by the integer n. Then we can expression the sum of the n trips in days as

$$a + \big(a + 2(1)\big) + \big(a + 2(2)\big) + \cdots + \big(a + 2(n-1)\big) = 366 \qquad (1)$$

The left side of equation (1) has n terms, so we can re-write this as

$$na + \big(2 + 4 + 6 + \cdots + 2(n-1)\big) = 366$$

which we re-write as

$$na + 2\big(1 + 2 + 3 + \cdots + (n-1)\big) = 366 \qquad (2)$$

There are (n – 1) terms in parentheses on the left side of equation (2), and it is the sum of the first (n – 1) positive integers. Let us get an expression for this sum.

Let $S = 1 + 2 + 3 + \cdots + (n-1)$, and write S again with terms in reverse order: $S = (n-1) + (n-2) + \cdots + 3 + 2 + 1$.

If you add these two expressions for S together by adding corresponding terms, you get (n – 1) terms, each of which is equal to n:

$$2S = [(n-1) + 1] + [(n-2) + 2] + \cdots + [1 + (n-1)] \qquad (3)$$

There are (n – 1) terms in equation (3) and each term is equal to n. It follows that 2S = (n – 1) n and we have the result:

$$S = 1 + 2 + 3 + \cdots + (n-1) = \frac{(n-1)n}{2} \qquad (4)$$

We substitute equation (4) into equation (2) and get

$$na + n(n-1) = n(a + n - 1) = 366 \qquad (5)$$

The expression (a + n -1) in equation (5) is an integer, so we see that the number of trips n must be a factor of 366. The prime factorization of 366 is

$$366 = 2 \cdot 3 \cdot 61 \qquad (6)$$

We substitute n = 2 into equation (5) and get: 2(a + 1) = 366, with the solution a = 182. We reject this solution because of the requirement above that a < 91.

With n = 3 in equation (5), we get a = 120, still too big, so we try $n = 2 \cdot 3 = 6$, and this gives

$$a = 56$$

and since each subsequent trip is two days longer, the series of six trip lengths is: 56, 58, 60, 62, 64, 66. As a check, we add the lengths: 56 + 58 + 60 + 62 + 64 + 66 = 366, as needed. Since 58 is one of the listed answers, we conclude that the answer is 58, (B).

Any other solutions would need to use the factor 61from equation (6), so let us try n = 61 in equation (5). But this gives a = -54, so we will not find any other solutions.

14. Brief solution: Notice that, for example, the two strings 110000 and 000011, which are reversals, do not add up to 111111, so reversal and digit-flip are distinct operations. One way to do this problem is to note that if the strings are binary representations of regular base ten integers, then strings ending in zero are even, while those ending in a one are odd. To add up to 111111 requires that an even be paired with an odd. So we can eliminate complements (pairs adding up to 111111) by just using either the evens or the odds, which are 32 in number. But some of these 32 (either the even or the odd set) form reversal pairs. Suppose you are using the even set of strings and you eliminate one member of a reversal pair. Then since each member of the pair has a complement from the odd set, you can now add the odd string whose even complement has been eliminated, and the total stays at 32. So every time you eliminate one member of an even reversal pair, you can then put back in the odd complement of the even string, which was deleted. So the answer is 32. The correct answer was not listed among the answers given, as the national coordinator of the contest exam informed me. She informed me that the correct answer, 32, was not given.

My corrected more detailed solution to this problem:

Sometimes, to understand a complex problem, it helps to look first at a simpler problem of the same nature. So let us examine the case with binary strings of length three rather than six. Counting from zero to seven in binary we have the following list of eight:

000, 001, 010, 011, 100, 101, 110, 111.

Note that in binary numbers, each place goes up in value by a factor of two, rather than ten, as in our decimal system. So, for example, the string 010, converted to decimal is:

in our usual decimal notation.

If we start with 000 and include every other entry in the list, we have the list of four:

000, 010, 100, 110.

If you examine this list, you will see that no two of the numbers add up to 111, and no two of them are the reverse of each other. For example, the reverse of 011 would be 110. But if we try to include any of the missing numbers, such as 001, they will either add up to 111 (001 + 110 = 111) or they will be the reverse of a number on the list of four. So, for example, we cannot add 001 to the list of four, since 001 is the reverse of 100, which is on the list.

Note that we could have used an alternate list of four, starting with 001:

001, 011, 101, 111.

So the examination of this simpler problem makes it seem like perhaps the desired number of distinct binary strings with no additions to 111111 and no reversals is half of the total possible number of strings.

With strings of length six, the total number of strings possible is . So if we start with 000000, and use every other string as in the list of eight above, we will, according to this conjecture, obtain a list of 32 strings satisfying the conditions given. Or use the alternate list of 32 strings, starting with 000001, and including every other string.

So is the answer 32?

Unfortunately, one of the risks in using a simpler problem as a model for a more complex one is that you may not be illustrating all aspects of the more complicated problem. We need to understand a few things, but that is where the fun begins!

First of all, let us ask ourselves why it is that using only every other binary string, starting from either zero or one, will ensure that we eliminate all pairs that add up to 111111. Recall from decimal numbers that zero is even, and every other number starting with zero will be even. Similarly, one is odd, and every other number, starting with one is odd.

Now here is the important point. A binary string representing an even number must end with a zero digit. To see this, consider the number:

$$111 = 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \qquad (1)$$

where $2^0 = 1$.

By the way, you might be wondering why a number to the zero power should be one, rather than zero. After all, if you have zero factors of a number, isn't that zero? Well, everyone agrees that if you divide any non-zero number by itself, you should get one (the case 0/0 is a famous exception, and is one of the things calculus is all about). So, for example, $\frac{2^5}{2^5} = 2^{5-5} = 2^0 = 1$, since we agree that dividing a non-zero number by itself yields one. If you like logarithms, also note that $Ln2^0 = 0 \cdot Ln2 = 0$, and ln 1 = 0, so we need $2^0 = 1$ for consistency with properties of logarithms too. There are other reasons, but you see why we need to define numbers to the zero power this way. Definitions in mathematics are not coincidental or arbitrary, but are constrained by consistency requirements and logical necessity. That is one of the beautiful aspects of mathematics. It reflects nature, we are not just making this up as we go.

Getting back to the problem at hand, you see that each term on the right for the expansion in powers of two of the number 111 in equation (1) has at least one factor of two except the last one. Note that zero has a factor of two because zero over two is zero, an integer. So numbers like 101 are odd too. So any string ending with a "1" on the right must represent an even number plus one, which is odd. For example, the binary number 111111 represents the odd number $63 = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0$. We also see that if the last digit on the right is a zero, each term in the expansion of the binary string in powers of two has a common factor of two, and the number is even.

Now note that the sum of any two odd numbers is even, because the last digits on the right add together to give another "two" instead of a "one". Each term in the number now has a common factor of two. Also, note that the sum of two even numbers is even, because each number has a factor of two, so the sum has a common factor of two.

So if we use every other binary string, starting with zero, we are using only the even numbers, and no pair of such numbers can possibly add up to the odd number 111111. Similarly, if we start our list with one, we are using only the odd numbers, and no pair of these can add up to the odd number 111111. So by using every other number in the list of strings, we ensure that no two of them will add to 111111.

However, there is another condition, that no pair of strings be the reverse of each other. First of all, note that if the first digit is a one, and the last digit a zero, the number must be even, as discussed above. If we reverse such a number, the last digit will be a one, and the number becomes an odd number, which will already have been eliminated from our list if we are using only the even numbers. Similarly, if the first digit is zero, and the last a one, reversing the string changes the number from odd to even, and we will have already eliminated all evens if we are using the list of odds.

Thus, to look for reversals in our list, we only need to examine strings with equal first and last digits, so that reversing the string does not change the number from even to odd or odd to even (sometimes, we call this the parity of the integer).

In our simplified problem using three digits, the only numbers with equal first and last digits are 000, 010, 101, 111. Note that these are all symmetric with respect to reversing (you get the same number again when you reverse the string), so no further strings were eliminated by reversing in the case of three-digit strings. However, if we allow longer strings, there are more possibilities, so using only three digits did not illustrate this aspect of the overall problem.

For example, using four digits: If we use the list of even numbers, we have one pair of reversals, 0100, and 0010. So we have to eliminate one of these from our list. However, if we eliminate, say, 0100, then we can re-admit 1011 from the list of odd strings, since its complement has been eliminated. So the total stays at 32.

Similarly, using the odd list, we have the pair of reversals: 1011, and 1101, so again one of these must be removed. But if we remove 1101, then we can re-admit 0010 from the even list, and the total stays at 32.

Using five digits, I found two pairs of reversals in the even list: (01100 and 00110), and (01000 and 00010). There are two pairs in the odd list too, just replace the first and last zeros in the pairs above with one's. But as above, if we remove one member of a reversal pair, say 01100, then we can re-admit its complement, 10011, from the odd list. Similarly, if we remove 00010 from the even list, then we can re-admit 11101 from the odd list. In both cases, the total number stays at 32.

Now we arrive at our problem, six digits. For this case, I found four pairs of reversals in the even list: (010000 and 000010), (001000 and 000100), (011000 and 000110), and (010100 and

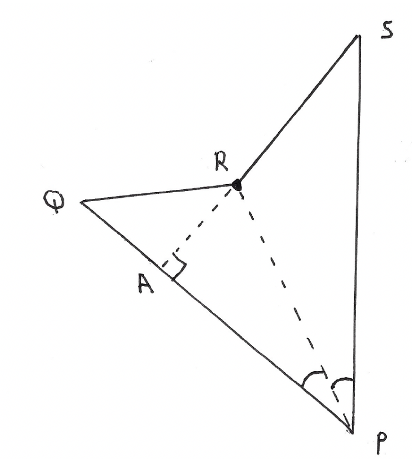
001010). However, Dr. Ho pointed out that I missed two more reversals: (011100 and 001110), and (011010 and 010110)

So we remove one member of each pair of reversals, and then re-admit the complement of the sequence eliminated, and the total remains at 32.

Dr. Ho also supplied a proof that the set of 32 sequences is maximal: Consider any 6 digit binary sequence s not in this set of 32. Since E contains all the sequences ending in 0, the sequence s must end in 1. Now, the complement, s', of s, obtained by shifting the digits of s from 1 to 0 and 0 to 1, is a sequence ending in 0, and therefore, is already in E. Consequently, s cannot be added to E any more, and E is therefore a maximal complement-free set.

Notice that this correct answer is not listed as one of the choices given. I was informed of this mistake by the national coordinator of the exam.

15. A quadrilateral satisfying the conditions for this problem is shown in the figure below



Angles Q, P and S are all 45 degrees, and the added segment RP is an angle bisector of angle P. Notice that since the sum of interior angles of a quadrilateral is 360 degrees, then with Q, P, and S all equal to 45, this forces angle QRS = 225 degrees. If segment SR is extended to point A, then since angles S and P are both 45, it follows that triangle PSA is a 45-45-90 one, and segments AS and AP are equal.

So the desired quadrilateral is the combination of two right triangles, PSA and QRA. We are given $PR = 8\sqrt{2}$. PR is the hypotenuse of right triangle RAP, and angle RPA = 22.5 degrees since RP is the bisector of angle P. Using trigonometry, we have

$$AP = RP\cos\frac{45}{2} \qquad (1)$$

We can find $\cos\frac{45}{2}$ using the double angle formula:$\cos 2\theta = 2(\cos\theta)^2 - 1$. Put $\theta = \frac{45}{2}$, and. Solve for $\cos\theta$ with the result

$$\cos\frac{45}{2} = \frac{\sqrt{2+\sqrt{2}}}{2} \qquad (2)$$

From equation (2), using $(\cos\theta)^2 + (\sin\theta)^2 = 1$, we then find

$$\sin\frac{45}{2} = \frac{\sqrt{2-\sqrt{2}}}{2} \qquad (3)$$

Then using substituting equation (2) and $RP = 8\sqrt{2}$ into equation (1), we get

$$AP = 4\sqrt{2\left(2+\sqrt{2}\right)} \qquad (4)$$

Now we get the area of triangle PSA as

$$\frac{1}{2}AP \cdot AS = 16\left(2+\sqrt{2}\ \right) \qquad (5)$$

where I used the values of AP and AS = AP from equation (4).

Since triangle QRA is also 45-45-90, we find

$$AR = AQ = RP\sin\frac{45}{2} = 4\sqrt{4-2\sqrt{2}} \qquad (6)$$

Since Q is a 45 degree angle, its tangent is one, so QA = AR. Using these values, we find the area of triangle QAR as

$$\frac{1}{2}QA \cdot AR = \ 16\left(2-\sqrt{2}\ \right) \qquad (7)$$

Add equations (5) and (7) to get the desired area, which is

$$Area = 64$$

We see that the exact area is the integer 64, and the answer is (E). In practice, while doing this exam under a time constraint, the student would have used a calculator to get an approximate value of the sine and cosine of 45/2, hence the need for rounding to the nearest integer.

16. Writing the given condition as an equation, we have

$$a^2 = 2b^2 + 2 \qquad (1)$$

Observe that the right side of equation (1) is even, so the square of a is even on the left side. I claim that a must also be even. To see this, suppose a is odd so we have a = 2k + 1 for some integer k. If you take the square of a, you get: $a^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. This contradicts equation (1), which shows a^2 is even.

Note that since a is even, then its square is a multiple of 4 so that $\frac{a^2}{2}$ is also even. Let us re-write equation (1) as

$$b^2 = \frac{a^2}{2} - 1 \qquad (2)$$

Since $\frac{a^2}{2}$ is even, it follows from equation (2) that b^2 is odd, therefore b itself is odd by an argument similar to the one above that showed a was even. (Assume b is even and arrive at a contradiction to equation (2), which shows b^2 is odd.).

Observe that (a, b) = (10, 7) is a solution to equations (1) and (2), but we need a > 10. Since b is the smaller number, and b is odd, I will do a search for the next solution using equation (1), starting with a substitution of b = 9 into the equation, and looking for a perfect square on the left side. Successive substitutions of b = 9, 11, 13, …, into equation (1), I finally found a solution at b = 41, yielding a = 58. We see that a – b = 17, and the answer is (C).

17. First Solution by MF: Think of constructing the five-digit numbers by placing a number into each of five slots arranged in a horizontal row. After thinking about this for a while, I decided that slots #2 and #4 were the important ones to consider:

Case I: Slot 2 contains 1 and slot 4 contains 5: The number has this form:

_ 1 _ 5 _ .

You see that 2 can only be placed on the far right, and 4 only on the far left, so we have 4 1 3 5 2. The case with slots 2 and 4 interchanged is just the reverse of this:
2 5 3 1 4. So we have two possibilities so far.

Case II: Slot 2 has a "1" and slot 4 a "4". Now the form is: _ 1 _ 4 _
We see that 3 can only go on the far left, but then there is no place to put the 5. Still only two possibilities.

Case III: If you use numbers 1 and 3 for slots 2 and 4, there is no place to put the 2.

Case IV: Using 2 and 4 for slots 2 and 4, there is no place for the 3.

Case V: Using 2 and 5 for slots 2 and 4, the 1 can only go at the far right, and then there is no place to put the 3.

Case VI: Using 3 and 5 for slots 2 and 4, the 2 can only go at the far right, and then there is no place for the 4.

Holding steady at two possibilities so far. The remaining cases have consecutive integers for slots 2 and 4. Maybe we can see a pattern here.

Case VII: Use numbers 1 and 2 for slots 2 and 4. The form is: _ 1 _ 2 _

We see that the 3 has to go on the far left. This leaves two possibilities for 4 and 5. For each of these, there is the reverse, which has 2 in slot 2 and 1 in slot 4. So we have added four possibilities and now have a total of six.

Case VIII: Using 2 for slot 2 and 3 for slot 4: The form is _ 2 _ 3 _

Note that the 1 has to go on the far right, and the 4 on the far left, so there is only the possibility 4 2 5 3 1, and its reverse. We are up to eight total.

Case IX: Using 3 for slot 2 and 4 for slot 4: The form is _ 3 _ 4 _

The 5 must go on the far left, and the 2 on the far right, so only one possibility again, 5 3 1 4 2, and its reverse, and we are up to ten total.

Case X: Using 4 for slot 2 and 5 for slot 4: The form is _ 4 _ 5 _

The 3 has to go on the far right, leaving two possibilities for 1 and 2. Each of these has a reverse, so we are up to a total of fourteen possibilities.

The way to calculate the probability that an object has certain desired characteristics is to count the number of objects that have the characteristics and divide by the total number of objects.

To get this probability as a fraction, we take fourteen and divide it by the total number of five-digit numbers that are possible. This total number is obtained by observing that there are five ways to fill the first slot, then four ways to fill the second, etc., down to one way to fill the last slot, since there are the same number of slots as numbers. So the total is 5! = 5 x 4 x 3 x 2 x 1 = 120. The probability is

$$P = \frac{14}{120} = \frac{7}{60}$$

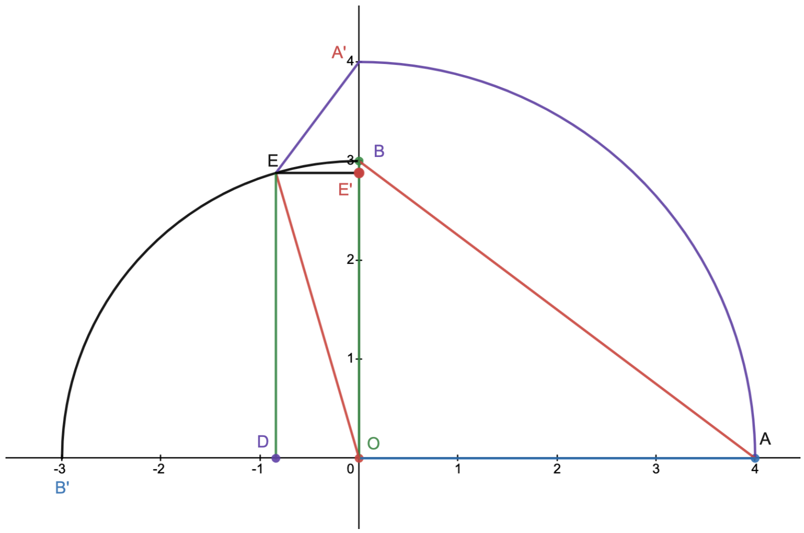
and the answer is (B).

Second solution: Dr. Vanniasegaram has a solution where you just do five cases, putting a number into the first slot, and counting up the possibilities:

There are 5! = 120 ways of selecting the five numbers and there are fourteen ways of selecting numbers where no two adjacent digits in the number are consecutive integers. The answer is 14/120 = 7/60 (B).

How I did get 14? There are two numbers where 1 is the first digit (13524 and 14253). By symmetry, there are also two numbers where 5 is the first digit (53142 and 52413). There are three numbers where 2 is the first digit (24153, 24135, and 25314). By symmetry, there are also three numbers where 4 is the first digit (42513, 42531, and 41352). Finally, there are four numbers where 3 is the first digit (35241, 31425, 31524, and 35142). So, the total number of numbers is 2 + 2 + 3 + 3 + 4 =14.

18. I assume that we are being asked to find the area swept out by the triangle when it is rotated. This area is shown in the figure below



As the given triangle rotates, the side OA sweeps out quarter circle $A'OA$ and side OB sweeps out quarter circle $B'OB$. The desired area has points $B', E, A', and\ A$ as vertices. From the figure, we see that this region consists of the two quarter circles $B'OB$ and $A'OA$ plus the almost triangular region $EA'B$, which has Arc EB as its base.

This region $EA'B$ can be obtained as the combination of the following areas: Quadrilateral $DEA'O$ minus Sector EOB minus triangle DEO.

We need the coordinates of point E. Note that segment EA' is perpendicular to segment BA because of the 90-degree rotation. The slope of segment BA has slope $-\frac{BO}{OA} = -\frac{3}{4}$, so segment EA' has the negative reciprocal, $\frac{4}{3}$, as its slope. Point A' is the y-intercept at (0, 4), so the equation of the line containing segment $EA'$is

$$Segment\ EA': y = \frac{4}{3}x + 4 \qquad (1)$$

The circular arc $B'B$ has the equation of a circle with radius 3 and center at the origin:

$$Arc\ B'B: x^2 + y^2 = 9 \qquad (2)$$

Eliminate y between equations (1) and (2) with the result

$$25x^2 + 96x + 63 = 0 \qquad (3)$$

The quadratic formula gives the solutions for x:

$$x = \frac{-96 \pm \sqrt{96^2 - 4(25)(63)}}{50} = -0.84, -3 \qquad (4)$$

Note that these are exact values because the discriminant under the radical sign in equation (4) is equal to 54^2. From the figure, we see that the segment EA', if extended, would intersect the full circle in two points, and point E has the larger value of x (less negative), so we have x = -0.84 for point E. Then the y coordinate is found from equation (1) as y = 2.88. We have the result

$$Point\ E\colon (x, y) = (-0.84, 2.88) \qquad (5)$$

Now we can get the area of quadrilateral $DEA'O$. Segment EE' is constructed to be horizontal, so point E' has coordinates (x, y) = (0, 2.88). From the figure, this quadrilateral is composed of the rectangle $DEE'O$ plus triangle $A'EE'$. These are easily found: From the figure, the rectangle has base 0.84 and height 2.88, and the triangle has base 0.84 and altitude (4 – 2.88).

$$Area\ DEE'O = 0.84 \cdot 2.88 = 2.4192 \qquad (6)$$

and

$$Area\ A'EE' = \frac{1}{2} \cdot 0.84 \cdot (4 - 2.88) = 0.4704 \qquad (7)$$

Note that the results in equations (6) and (7) are also exact. Quadrilateral $DEA'O$ is the sum of equations (6) and (7), so we have

$$Quadrilateral\ DEA'O = 2.4192 + 0.4704 = 2.8896 \qquad (8)$$

Now we need the area of the sector EOB subtended by angle EOB. From the figure, we see that

$$\sin EOB = \frac{EE'}{OE} = \frac{0.84}{3}$$

and using the inverse sine option on a T-84 calculator, I got $\angle EOB = \sin^{-1}\frac{0.84}{3}$. The area of the sector makes the same fraction with respect to the total area of the circle that the angle makes with 2π radians. We write this as

$$\frac{Sector\ Area}{9\pi} = \frac{\sin^{-1}\frac{0.84}{3}}{2\pi}$$

This gives the sector area as

$$Sector\ Area = \frac{9}{2} \cdot \sin^{-1}\frac{0.84}{3} \qquad (9)$$

Now we just need the area of triangle DEO, which has base 0.84 and altitude 2.88. The resulting area is

$$Area\ DEO = \frac{1}{2} \cdot 0.84 \cdot 2.88 = 1.2096 \qquad (10)$$

Recall that the region $EA'B$ in the figure can be obtained as the combination of the following areas: Quadrilateral $DEA'O$ minus Sector EOB minus triangle DEO. So we take equation (8) minus equation (9) minus equation (10) with the result

$$Area\ EA'B = 2.8896 - \frac{9}{2}\sin^{-1}\frac{0.84}{3} - 1.2096 = 1.68 - \frac{9}{2}\sin^{-1}\frac{0.84}{3} \qquad (11)$$

The two quarter circles $AOA'$and BOB'have areas $\pi\frac{16}{4}$ and $\pi\frac{9}{4}$ respectively, so their sum is $\pi\frac{25}{4}$. The desired area swept out by the rotation is the sum of the quarter circles plus the area in equation (11) with the result

$$Area = \pi\frac{25}{4} + 1.68 - \frac{9}{2}\sin^{-1}\frac{0.84}{3} \qquad (12)$$

Equation (12) is the exact result, we need an approximation to the nearest hundredth, so using a TI-84 calculator, I got

$$Area \approx 20.03788$$

To the nearest hundredth, this is 20.04, and the answer is (B).

19. Solution by MF: First, we give a definition of what it means for a polynomial to be factorable over the integers: It means that the polynomial can be written as the product of linear factors $(ax - b)$, where a and b are integers. The equation $ax + b = 0$ has only the rational solution $x = -\frac{b}{a}$.

But note that if any factor of a polynomial is zero, it makes the whole product of factors zero, so $-\frac{b}{a}$ is one of the roots of the polynomial. So an equivalent definition of being factorable over the integers completely to linear factors is that the polynomial has only rational roots. There is a close relationship between roots and factors of polynomials, expressed as the root-factor theorem:

Let P(x) be a polynomial and suppose that it has (x – r) as a factor. It follows that if you do a long division of P(x) by (x – r), you get remainder zero:

$$P(x) = q(x)(x - r) \qquad (1)$$

If you substitute x = r into equation (1), you get P(r) = 0. So if (x – r) is a factor, then r is a root of the polynomial. This is the root-factor theorem. Note that this theorem holds even if r is irrational and polynomials with integer coefficients can also have irrational roots. For example, the polynomial $P(x) = x^2 - 2$ has the two irrational roots $x = \pm\sqrt{2}$.

In this solution, I will use the rational roots of the given quadratic equations (they have such roots since they are assumed factorable over the integers). So the first quadratic equation is assumed to be factorable as

$$x^2 + mx + n = (x - r)(x - s) \qquad (2)$$

There is a similar equation for the other given quadratic. I claim that the roots r and s are integers. To see this, let us consider the general quadratic equation with integer coefficients, $ax^2 + bx + c$ and assume that the equation has a rational root $\frac{p}{q}$ with p and q integers and the fraction reduced to lowest terms so that p and q have no factors in common other than one. Substitute the rational root into the equation $ax^2 + bx + c = 0$, set it equal to zero, and then clear the denominators by multiplying both sides by q^2. The resulting equation is:

$$ap^2 + bpq + cq^2 = 0 \qquad (3)$$

Rewrite equation (3) as

$$q(bp + cq) = -ap^2 \qquad (4)$$

Since p and q have no common factors greater than one, then p^2 and q also have no common factors, since p^2 just uses the same factors as p twice. The quantity in parentheses on the left side of equation (4) is an integer. So equation (4) says that q multiplied by some integer gives $-ap^2$. Since q is not a factor of p^2, it must be a factor of a.

Similarly, if you rewrite equation (3) as

$$p(ap + bq) = -cq^2 \qquad (5)$$

Since p is not a factor of q^2, it must be a factor of c. This argument, given here for quadratics, can be generalized to polynomials of any higher degree with integer coefficients. Only the first and last coefficients of a polynomial determine the possible rational roots. Note that if a polynomial has rational coefficients, it can be converted into a polynomial with integer coefficients and the same roots just by multiplying the polynomial by the product of the denominators of the coefficients (clearing the denominators). So the rational root theorem applies to any polynomial with rational coefficients.

To summarize, if $\frac{p}{q}$ is a root of $ax^2 + bx + c$, then p is a factor of c and q is a factor of a. This is the rational root theorem applied to quadratics. Note that if the polynomial has irrational roots, then the theorem does not apply.

Now, observe that the leading (first) coefficient of the given quadratic (the x^2 term) is one, so q has to be a factor of one, which gives $q = \pm 1$. The fraction $\frac{p}{q}$ is equal to the integer $\pm p$. I'm almost ready to begin searching for solutions to this problem. Let us write

$$x^2 + mx + n = (x + r)(x + s) = x^2 + (r + s)x + rs$$

where r and s are integers. Subtract x^2 from both sides and compare corresponding coefficients with the result

$$m = (r + s) \qquad (6)$$

and

$$n = rs \qquad (7)$$

For the polynomial $x^2 + mx - n$, equation (6) is unchanged, while equation (7) just picks up a minus sign:

$$n = -rs \qquad (8)$$

We see that in equation (8), one of the integers r or s must be negative. Now I describe how to search for solutions.

For example, consider the case m = 11. Since r + s = m, from equation (6), the possible pairs (r, s) are (1, 10), (2, 9), (3, 8), (4, 7), and (5, 6) The reason they want to exclude zero as a possible root is discussed below. Note that the pair (6, 5) gives the same values of m and n in equation (6) – (8) as (5, 6). The order of the roots does not matter, so we can stop with the pair (5, 6).

So the possible products rs = n are: 10, 18, 24, 28, and 30.

For the other polynomial $x^2 + mx - n$ we take r < 0 and s > 0 such that r + s = m and rs = -n (equations (7) and (8)) for one of the possible values of n. The possible pairs (r, s) are (-1, 12), and (-2, 13). I can stop with these two because for (-3, 14), -3 x 14 = -42, which is larger than the largest possible value of -n, which is -30 in the list above. After this, they just keep getting larger in magnitude.

The possible products here are -12 and -26, neither of which is equal to 10, 18, 24, 28, or 30. So we have discovered that for m = 11, it is not possible for both radicals $\sqrt{m^2 \pm 4n}$ to be integers for any n. So m = 11 is eliminated.

Proceeding in this manner, starting with m = 1, I did not have a success until m = 5. In this case, the pair (m, n) = (5, 6) allows the following pairs of roots for the given polynomials. To see this, note that 2 + 3 = 5, and 2 x 3 = 6, while -1 + 6 = 5, and -1 x 6 = -6.

Now I claim any multiple of 5 for m must also provide at least one solution. To see this, note that the two pairs of solutions for the two given polynomials are, by quadratic formula or equivalently, by completing the square:

$$x = \frac{-m \pm \sqrt{m^2 - 4n}}{2}$$

and

$$x = \frac{-m \pm \sqrt{m^2 + 4n}}{2}$$

There is a theorem that says the square root of any positive integer is either another integer, or it is irrational. The proof is omitted here, but it proceeds in a straightforward manner by assuming that a square root is a non-integer rational number $\frac{a}{b}$ expressed in lowest terms with a and b integers. This assumption leads to a contradiction. So we have the two equations

$$m^2 - 4n = v^2 \qquad (9)$$

and

$$m^2 + 4n = u^2 \qquad (10)$$

where u and v are integers.

Consider the radical $\sqrt{m^2 - 4n}$. If we multiply m by 2 and n by 4, the radical becomes: $\sqrt{(2m)^2 - 4(4n)} = 2\sqrt{m^2 - 4n}$. So if the radical on the right is an integer, then so is the radical on the left. It follows that since the pair (5, 6) gives a solution, then so must the pair (10, 24). More generally, if (m, n) is a solution, then (km, k^2n) is also a solution for any scale factor k.

But we can also multiply m by 3 and n by 9, yielding $\sqrt{(3m)^2 - 4(9n)} = 3\sqrt{m^2 - 4n}$, so the pair (15, 54) also gives a solution (15 = 5 x 3, and 54 = 9 x 6). Finally, I can multiply m = 5 by 4 and n = 6 by 16 to give the solution (20, 96). I can stop here with this method of generating solutions, since the next one would give n > 100.

So we have (5, 6), (10, 24), (15, 54), and (20, 96). Four working pairs so far. Unfortunately, it seems I still have to check m = 10, 15, and 20 when I get there in case they yield additional solutions. I did not see a way to shorten the search, so I checked every value of m up to m = 51. I will explain below why we can stop there.

The next pair I found was (13, 30), since 3 x 10 = 30, and -2 x 15 = -30, and the factors both add up to 13 for (r, s) = (3, 10) and (-2, 15). Five solutions now.

Then I found (17, 60), since 5 x 12 = 60, and -3 x 20 = -60, and the factors add up to 17. Six solutions. This occurred for (r, s) = (5, 12) and (-3, 20).

The seventh and final solution came from m = 25, using (r, s) = (4, 21) and (-3, 28) Note that m = 25 also generates a second solution (m, n) = (25, 150) by virtue of m being a multiple of 5 so it can be scaled up from the first solution m = 5. But we threw this one out since n > 100. So the final pair is (m, n) = (25, 84).

The seven solutions are: (5, 6), (10, 24), (13, 30), (15, 54), (17, 60), (20, 96), and (25, 84).

Why are these the only ones with m, n < 100? I checked each value of m up to m = 51. We can stop there because any potential solution would give n > 100. To see this, consider the case m = 52. The possible product is 1 x 51, since the next one, 2 x 50, is already at n = 100. The first possible negative product is -1 x 52, which cannot be equal to 1 x 51. The next possible negative product, -2 x 53, is already larger than 100. You can see that this situation will persist from m = 51 up to m = 100.

So the answer is (D). However, you can see that this involved a great deal of tedious searching and is really not very satisfying. Also note that for each solution found, the integer n is a multiple of six. If we could have proven that before beginning the search, then we could check all the multiples of 6 less than 100 (sixteen of them) and reduce the search.

Gregory Melblom and Chungwu Ho came up with better solutions:

If you add equations (9) and (10) together, the result is

$$u^2 + v^2 = 2m^2 \tag{11}$$

Subtracting equation (9) from equation (10) yields

$$8n = u^2 - v^2 \tag{12}$$

The left side of equation (11) can be factored using complex numbers of the form $p \pm q\boldsymbol{i}$, where $\boldsymbol{i} = \sqrt{-1}$ and p and q are integers. Such numbers are called Gaussian integers. The left side of equation (11) can be factored as

$$u^2 + v^2 = (u + v\boldsymbol{i})(u - v\boldsymbol{i})$$

An integer such as m can also be factored using Gaussian integers as

$$m = (p + q\boldsymbol{i})(p - q\boldsymbol{i}) = p^2 + q^2 \tag{13}$$

Finally, the number 2 can be factored as

$$2 = (1 + \boldsymbol{i})(p - q\boldsymbol{i})$$

Putting these last three factorizations into equation (11) gives

$$(u + v\boldsymbol{i})(u - v\boldsymbol{i}) = (1 + \boldsymbol{i})(p - q\boldsymbol{i})^2(p - q\boldsymbol{i})(p + q\boldsymbol{i})^2 \tag{14}$$

We can equate corresponding factors on the left and right sides as follows:

$$u + v\boldsymbol{i} = (1 + \boldsymbol{i})(p - q\boldsymbol{i})^2 = (p^2 + 2pq - q^2) + (p^2 + 2pq - q^2)\boldsymbol{i} \tag{15}$$

and

$$u - v\boldsymbol{i} = (1 - \boldsymbol{i})(p + q\boldsymbol{i})^2 = (p^2 + 2pq - q^2) - (p^2 + 2pq - q^2)\boldsymbol{i} \tag{16}$$

By equating corresponding terms on both sides of equations (15) and (16), we find

$$u = p^2 + 2pq - q^2 \tag{17}$$

and

$$v = p^2 - 2pq - q^2 \tag{18}$$

We already know from equation (13) that m is the sum of squares of p and q, but if we substitute equations (17) and (18) into equation (12) and work out the algebra, the result is

$$n = pq(p + q)(p - q) \tag{19}$$

Now recall from the discussion below equation (10) that if (m, n) is a solution, then so is (km, k^2n) for any scale factor k. So we add the scale factor to equations (13) and (19):

$$m = k(p^2 + q^2) \tag{20}$$

and

$$n = k^2 pq(p + q)(p - q) \tag{21}$$

I claim that n has to be a multiple of 6. Note that since 6 = 2 x 3, any multiple of six must be even and also be a multiple of 3. Let the scale factor k = 1 in equation (21). If either of p or q are even, then n is even, so suppose both p and q are odd. But then the sum and difference $p \pm q$ are both even, so n is even regardless of the value of the scale factor.

To show that n is a multiple of three, recall the concept of division with remainder. We can write $p = 3Q + R$, where Q is the quotient when p is divided by 3, and R is the remainder, which can take on the values 0, 1, or 2 only. We can write a similar expression for q.

In equation (21), if either p or q are multiples of three (R = 0), then n is a multiple of three. If both p and q have remainder one or remainder two, then (p – q) is a multiple of three (the two remainders cancel). The only other possibilities are that p leaves remainder one and q remainder two or vice versa. In this case, (p + q) is a multiple of three since the two remainders add to three and we add one to the quotient, leaving remainder zero for (p + q). This exhausts the possibilities, so n is even and it is a multiple of three, so it is a multiple of six.

Gregory and Chungwu found the seven solutions using equations (20) and (21) and searching using p, q, and k directly. I decided to investigate another strategy for searching:

Let us now search for solutions using the restriction that n is a multiple of six. Before substituting values for n into equations, let us see what we can learn from equations (9) and (10). Rewrite equation (10) as

$$4n = u^2 - m^2 = (u - m)(u + m) \tag{22}$$

If we factor 4n on the left side of equation (22) as $1 \cdot 4n$, then we can take (u – m) = 1, and (u + m) = 4n. But then solving for m gives m = 4n – 1, which is not an integer, so we skip this factorization of 4n.

If we factor 4n as $2 \cdot 2n$, setting (u – m) = 2 and (u + m) = 2n in equation (22), we get m = n – 1, and u = n + 1. Substitute this value of n into equation (9) with the result

$$n^2 - 6n + 1 = v^2 \tag{23}$$

I tried all multiples of six for n less than 100 in equation (23) and arrived at the solution (m, n) = (5, 6), which gives v = 1. Equation (23) did not give any other solutions for n < 100. However, recall from equations (20) and (21) that m and n can be scaled up by k and k^2 respectively. This generates three more solutions: (m, n) = (10, 24), (15, 54), and (20, 96).

Another possible factorization of 4n without having to specify n is: $4 \cdot n$. In equation (22), we set (u – m) = 4, and (u + m) = n with the result: $m = \frac{n}{2} - 2$. Substitution of this value of m into equation (9) yields

$$\frac{n^2}{4} - 6n + 4 = v^2 \qquad (24)$$

Here, I found n = 24 gives v = 2 and m = 10, but we have already found that solution. However, n = 30 gives v = 7 and m = 13, so we have a new solution (m, n) = (13, 30) and now five altogether: We get no further solutions by scaling because k = 2 already gives n > 100. I found no further solutions by substituting the remaining multiples of six into equation (24). Any other factorizations of 4n require factoring n itself.

It seems like we can find all solutions if we consider all possible factorizations of 4n. But how many factors are possible for the first 16 multiples of six (n < 100)? If we write $4n = 4f \cdot \frac{n}{f}$, where f is an integer and allow n to be as large as 100, it seems we have to check all values of f from two to fifty. (Equation (24) satisfies the case f = 1.)

So let $4n = 4f \cdot \frac{n}{f}$ and then from equation (22), we have: $u - m = 4f$, and $u + m = \frac{n}{f}$, which gives $u = \frac{n}{2f} + 2f$ and $m = \frac{n}{2f} - 2f$. Note that if we reverse the assignment of factors for (u – m) and (u + m), it will leave u unchanged and reverse the sign of m. But in equation (9), m only appears as m^2, so any solution with m < 0 gives a corresponding one with m > 0.

Substitution of $m = \frac{n}{2f} - 2f$ into equation (9) gives

$$\frac{n^2}{4f^2} - 6n + 4f^2 = v^2 \qquad (25)$$

We will use this equation for our searching, but we only need to search values of n for which the left side of equation (25) is positive. To determine this region, we find the x-intercepts using the quadratic formula with the results:

$$n_0 = \left(12 \pm 8\sqrt{2}\right)f^2 \approx 0.69f^2, 23.3f^2 \qquad (26)$$

Since the quadratic opens up, we do not have to search in-between the boundaries of equation (26), only outside those boundaries. However, as f increases, the lower limit, $0.69\ f^2$, becomes greater than 100 and provides no further limitations. But also as f increases, the first term in equation (25) must be greater than or equal to one for the possibility of a solution.

The search strategy will be: Choose a value for f, then substitute the allowed values of n into equation (25) and look for a perfect square. For each f, the allowed values of n are the multiples of six that are also multiples of 2f, since $m = \frac{n}{2f} - 2f$. We have found five solutions so far:

(m, n) = (5, 6), (10, 24), (13, 30), (15, 54), and (20, 96).

The limits on n given in equation (26) are helpful when f is small. For example, when f = 2, the lower limit is 2.76 < 6, and the upper limit is 93.2, so you only have to check n = 96.

Let f = 2, so n must be a factor of 4 from equation (25). Only n = 96 is allowed, and this yielded the solution (20, 96), which we already have.

For f = 3, the limits of equation (26) give n < 25. The upper limit is already bigger than 100. So we just have to check n = 6, 12, and 24. We obtain the previous solution (m, n) = (5, 6), which gives the other three previous solutions when it is scaled using factor k and k^2 respectively for m and n.

Unfortunately, as f gets larger, the lower limit in equation (26) also gets larger. The upper limits are now always greater than 100. For f = 10, the lower limit is n < 69, and n > 20 in order to keep the first term in equation (25) greater than one. Here, I found the new solution (17, 60).

For f = 14, the lower limit of equation (26) is greater than 100 but the requirement that the first term in equation (25) be greater than or equal to one gives n greater than 27. Here, I found the solution (m, n) = (25, 84), and the seven solutions are

(m, n) = (5, 6), (10, 25), (13, 30), (15, 54), (17, 60), (20, 96), (25, 84).

For f > 14, it is no longer necessary to check prime numbers like f = 17, because the smallest possible n = 17 x 6 is already larger than 100.

For f = 16, we have to check n = 16 x 3 = 48 and 16 x 6 = 96. So it seems like to be sure there are no further solutions, you have to check values of f up to f = 48. For f > 48, the first term is no longer greater than one, even for n = 96.

So perhaps a better search strategy would be to use p and q and the scale factor k directly from equations (20) and (21). Gregory's and Chungwu's solutions are given below and they use this strategy. Since interchanging the values of p and q changes the sign of n from equation (21), you can take p > q. Also, equation (20) shows that with k = 1, the largest value of p that you need to consider is p = 9, because p = 10 gives m > 100.

Here is the solution Gregory sent to me:

Hi Marc,

This is a nasty problem. □ You'll see why.

For how many pairs of positive integers (m, n) with m and n both < 100 are both of the polynomials $x^2 + mx + n$ and $x^2 + mx - n$ factorable over the integers?

Here is what I came up with.

Solution:

I checked by programming a trial and error method to find the solutions; So there are exactly 7 solutions to the problem.

In order for both polynomials to be factorable over the integers, the discriminants of both polynomials must be perfect squares. Thus:

$$v^2 = m^2 - 4n \text{ and } u^2 = m^2 + 4n \text{ for integers u and v.}$$

Adding these two equations together produces: $u^2 + v^2 = 2m^2$. The left side can be factored as $(u + v\cdot i)(u - v\cdot i)$ and 2 can be factored as $(1 + i)(1 - i)$ thus:

$$(u + v\cdot i)(u - v\cdot i) = (1 + i)(1 - i)\cdot m^2 .$$

It remains therefore to make $(u + v\cdot i) = (1 + i)$ times a perfect (Gaussian) square and $(u - v\cdot i) = (1 - i)$ times a perfect (Gaussian) square, which is easy.

(*I take the liberty of pairing up factors this way so that* $u > v$.)

$$\text{Let } (u + v\cdot i) = (1 + i)(p - q\cdot i)^2 = (p^2 + 2\cdot p\cdot q - q^2) + (p^2 - 2\cdot p\cdot q - q^2)\cdot i$$

$$\text{and } (u - v\cdot i) = (1 - i)(p + q\cdot i)^2 = (p^2 + 2\cdot p\cdot q - q^2) - (p^2 - 2\cdot p\cdot q - q^2)\cdot i$$

which shows that we can take $u = (p^2 + 2\cdot p\cdot q - q^2)$ and $v = (p^2 - 2\cdot p\cdot q - q^2)$

Substituting these values into $u^2 + v^2 = 2m^2$ produces:

$$(p^2 + 2\cdot p\cdot q - q^2)^2 + (p^2 - 2\cdot p\cdot q - q^2)^2 = 2(p^2 + q^2)^2 = 2\cdot m^2.$$

Thus we may let $m = (p^2 + q^2)$.

Now, $u^2 - v^2 = 8n$. Substituting the values of u and v into this equation produces:

$$(p^2 + 2\cdot p\cdot q - q^2)^2 - (p^2 - 2\cdot p\cdot q - q^2)^2 = 8\cdot p\cdot q\cdot(p + q)\cdot(p - q) = 8n$$

which means that: $n = p\cdot q\cdot(p + q)\cdot(p - q)$.

With the constraints that $(p^2 + q^2) = m < 100$ and $p\cdot q\cdot(p + q)\cdot(p - q) = n < 100$,
the only numbers p & q that work are:

(p, q) = (2, 1) –> (m, n) = (5, 6)
(p, q) = (3, 1) –> (m, n) = (10, 24)
(p, q) = (4, 1) –> (m, n) = (17, 60)
(p, q) = (3, 2) –> (m, n) = (13, 30)
(p, q) = (4, 2) –> (m, n) = (20, 96)
(p, q) = (4, 3) –> (m, n) = (25, 84)

The one that is missing is (m, n) = (15, 54); and the reason is because m = 15 cannot be represented as the sum of two squares – which seems strange. But it can be found; It has to do with common factors.

Observe:

Suppose that p is replaced by k·p’ and q is replaced by k·q’ in

$m = (p^2 + q^2)$ and $n = p·q·(p + q)·(p – q)$

After factoring carefully, we get: $m = (p^2 + q^2) = k·$ **[**$k·(p'^2 + q'^2)$**]** and

$n = p·q·(p + q)·(p – q) = k^2·$ **[**$k^2·[p'·q'·(p' + q')·(p' – q')]$**]**

Define $m' = m/k$ and $n' = n/k^2$. Then these will produce a solution to the problem.

Because (p, q) = (2, 1) taken from above and works to give (m, n) = (5, 6),
let (p, q) = (6, 3). In this case k = 3 and we get $m = (6^2 + 3^2) = 45$ and
$n = 6·3·(6 + 3)·(6 – 3) = 486$.
Therefore $m' = m/k = 45/3 = 15$ and $n' = n/k^2 = 486/9 = 54$.
And (m', n') = (15, 54) is the missing solution but is still found using the formula to the Diophantine equation. It would have been easy to overlook.

After thought:
Notice that if $x^2 + mx + n$ factors as $(x + c)(x + d)$
Then $x^2 + k·mx + k^2·n$ factors as $(x + k·c)(x + k·d)$
So when you find a solution (m, n), then $(k·m, k^2·n)$ also works.
Thus (m, n) = (5, 6) produces (m', n') = (10, 24) using k = 2,
and produces (m", n") = (15, 54) using k = 3,
and produces (m''', n''') = (20, 96) using k = 4.
The "primitive" solutions are: (m, n) = (5, 6), (13, 30), (17, 60), (25, 84).

I think there should be another method for solving this, using a different perspective.
– Gregory

Here are Chungwu's comments:

Problem 19:
It appears that the values of "m" and "n" can be found by using positive integers u > v and k:
m = k*(u^2 + v^2) and n = k^2*u*v*(u + v)*(u - v)
These produce all the solutions Marc found and beyond.
I believe these formulas produce ALL the solutions for the problem without putting upper limits on m and n.
Since u > v ≥ 1 then u > 1 and v ≥ 1 then
u*v^2 > v^2 and u^2*v ≥ u^2
adding the inequalities together produces
u^2*v + u*v^2 > u^2 + v^2
The left side factors to show
u*v*(u + v) > u^2 + v^2

Since u > v and both are integers, u - v ≥ 1.
Therefore u*v*(u + v)*(u - v) > u^2 + v^2
Also since k ≥ 1, then k^2 ≥ k
so that k^2*u*v*(u + v)*(u - v) > k*(u^2 + v^2)
This shows that n > m as Marc conjectured in his solutions.

20. We need a theorem from geometry. Consider Figure 1 below:

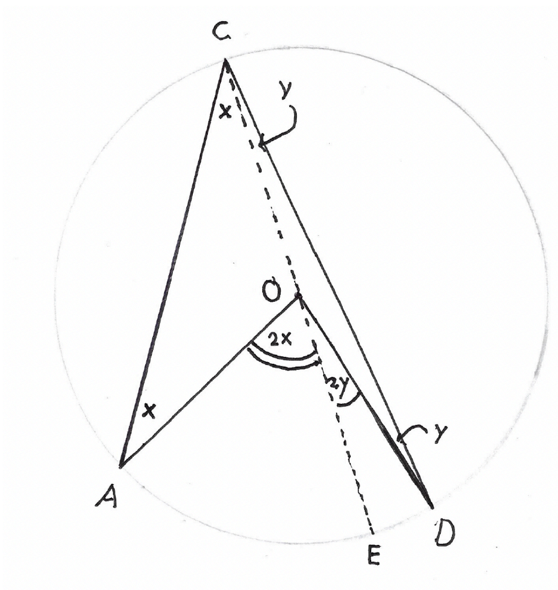


Figure 1

Angle ACD is inscribed in a circle, and angle AOD is the corresponding central angle less than 180 degrees with the same intersection points with the circle. The triangles AOC and DOC are isosceles since they each have two sides equal to the radius of the circle. So angle ACO = angle CAO = x and angle OCD = angle ODC = y. By the remote exterior angle theorem, we see that angle AOE is twice the size, 2x, of angle ACO. Similarly, angle EOD is twice the size of angle OCD, 2y.

From Figure 1, we can see that the inscribed angle, ACD is x + y, while the corresponding central angle, AOD, is 2x + 2y. This establishes the theorem that any inscribed angle is one-half the measure of the corresponding central angle less than 180 degrees.

It follows that as we let points A and D approach their maximum separation in Figure 1, when segment AOD is a diameter of the circle, the central angle approaches 180 degrees, so any inscribed angle with the same intersection point with the circle approaches 90 degrees. So the two triangles in the present problem are both right triangles.

The semicircle with the two triangles is shown in Figure 2 below:

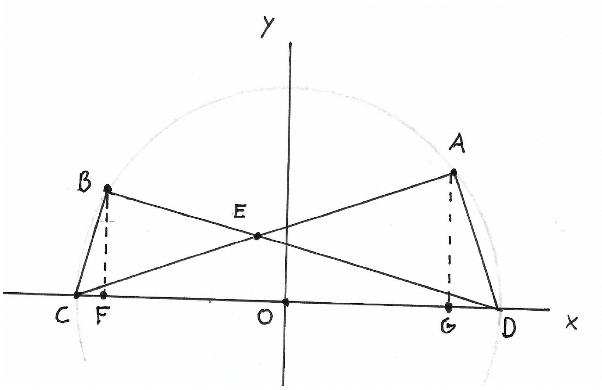


Figure 2

The hypotenuse of each triangle, ACD and BCD is 50. We are given AD = 14, and BD = 40. With these lengths as given, I did not see how it is possible for AB to be less than 25, so I'm puzzled as to why that was given as an additional condition. Recall that triangles ACD and BCD are right triangles with two sides known, so we can apply the Pythagorean Theorem to find

$$BC = 30 \qquad (1)$$

and

$$AC = 48 \qquad (2)$$

Two altitudes, the dashed lines in Figure 2, have been drawn, forming right triangles BFC and AGD. Observe that triangle BFC shares angle BCD with triangle DBC. It follows that triangle BFC is similar to triangle DBC, so we set up the proportion:

$$\frac{BD}{CD} = \frac{40}{50} = \frac{BF}{BC} = \frac{BF}{30} \qquad (3)$$

Equation (3) gives BF = 24.
Also, triangles AGD and CAD are similar. We set up the proportion:

$$\frac{AC}{CD} = \frac{48}{50} = \frac{AG}{AD} = \frac{AG}{14} \qquad (4)$$

Equation (4) gives AG = 13.44.

The Pythagorean Theorem applied to triangle BFC gives:

$$BF^2 + CF^2 = BC^2 \qquad (5)$$

With BF = 24 and BC = 30, we get CF = 18. So the coordinates of point B in Figure 2 are: B = (-7, 24). The line containing BD contains the two points, B, and D = (25, 0). These two points give us the equation of the line containing BD:

$$y = -0.75x + 18.75 \qquad (6)$$

Similarly, we have

$$AG^2 + GD^2 = AD^2$$

With AG = 13.44, and AD = 14, we get GD = 3.92. So the coordinates of point A are A = (21.08, 13.44). These two points give the equation of the line containing AC:

$$y = \frac{7}{24}x + \frac{175}{24} \qquad (7)$$

Point E is the intersection of these two lines. Setting the right sides of equations (6) and (7) equal yields x = 11. Substituting x = 11 into either equation (6) or (7) gives $y = \frac{21}{2}$. So the coordinates of point E are $\left(11, \frac{21}{2}\right)$. So the altitude of triangle CED in Figure 2 is $E_y = \frac{21}{2}$.

Now we can determine the areas of triangles BCD, DAC, and CED:

$$\Delta BCD = \frac{1}{2} CD \cdot BF = \frac{1}{2} \cdot 50 \cdot 24 = 600 \quad (8)$$

$$\Delta DAC = \frac{1}{2} CD \cdot AG = \frac{1}{2} \cdot 50 \cdot 13.44 = 336 \quad (9)$$

$$\Delta CED = \frac{1}{2} CD \cdot E_y = \frac{1}{2} \cdot 50 \cdot \frac{21}{2} = 262.5 \quad (10)$$

The desired area is sum of equations (8) and (9) minus the area in equation (10). This works because if we add the two triangles BCD and DAC, we include the area of intersection, triangle CED, twice so we must subtract the intersection once.

So the desired area is: 600 + 336 – 262.5 = 673.5, and the answer is (D).