Math 104 Final Dec 13, 2021

10 problems, 10 points each.

(1) Let $a_n = [(0.3)^n + (0.7)^n]/[1 + (0.8)^n]$. Compute $\lim_n a_n^{1/n}$.

(2) Suppose $a_n > 0$ and $\sum_n a_n$ converges. Prove that

$$\sum_n ((-1)^n a_n + a_{2n})/2$$

converges.

(3) Fix a positive number a, and choose $x_1 > \sqrt{a}$. Define $x_2, x_3, \cdots$ iteratively by

$$x_{n+1} = (x_n + a/x_n)/2.$$

Prove that x_n converges to a. Hint: for any $x, y > 0$, we have $(x + y)/2 \geq \sqrt{xy}$.

(4) Let $K = [0, 1]$, and $C(K)$ be the set of all real valued continuous functions on K. Recall that the distance function

$$d(f, g) = \sup_{x \in K} |f(x) - g(x)|, \quad \forall f, g \in C(K)$$

makes $C(K)$ a metric space. Show that $C(K)$ is a complete metric space, and show that $C(K)$ is not sequentially compact (i.e. give an example of a sequence in $C(K)$ where there is no convergent subsequence.)

(5) Is it true that, if $f : (0, 1) \to \mathbb{R}$ is uniformly continuous then f is bounded. If false, give a counter-example; if true, give a proof.

(6) Let f be a periodic continuous function on $\mathbb{R}$ with period $T = 1$, i.e. $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$. Show that for any $c \in (0, 1)$, there exists an $x \in \mathbb{R}$, such that $f(x) = f(x + c)$. (If you wish, you can take $c = 1/2$)

(7) Suppose f is differentiable on $\mathbb{R}$, and $1 \leq f'(x) \leq 2$ for all $x \in \mathbb{R}$ and $f(0) = 0$. Show that $x \leq f(x) \leq 2x$ for all $x > 0$.

(8) Let X be a metric space, and A, B be two compact subsets of X. Is it true that $A \cap B$ is compact? If so, give a proof; if not, give a counter-example.

(9) Let $h(x) = 1/(e^x - 1) - 1/x$. Compute $\lim_{x\to 0} h(x)$

(10) Let $f_n = \frac{n+\sin(x)}{2n+\sin^2 x}$ for all $x \in \mathbb{R}$. Show that f_n converges uniformly on $\mathbb{R}$. And compute $\lim_{n\to\infty} \int_1^2 f_n(x)dx$. Hint: you may use the following 'squeezing lemma': if there exists sequences of functions

$g_n(x), h_n(x)$, such that $g_n(x) \le f_n(x) \le h_n(x)$ and g_n and h_n converges to f uniformly, then $f_n \to f$ uniformly.

1. Answer Key

1. In the numerator, the term $(0.7)^n$ is more important than $(0.3)^n$, in the term 1 is more important than $(0.8)^n$, so intuitively, one can approximate the expression for a_n as $(0.7)^n$. More precisely, we can write

$$a_n = (0.7)^n \frac{1 + (0.3/0.7)^n}{1 + (0.8)^n} = 0.7^n b_n$$

We can show $b_n \to 1$ as $n \to \infty$, and for any $x > 0$, we have $x^{1/n} \to 1$ as $n \to \infty$, hence $\lim_n b_n^{1/n} = 1$. Hence

$$\lim a_n^{1/n} = \lim_n [(0.7)^n]^{1/n} = 0.7.$$

2. Because $\sum_n (-1)^n a_n$ absolute converges, and $\sum_n a_{2n} < \sum_n a_n < \infty$ hence also converges, hence the sum converges.

3. We first prove that $x_n \ge \sqrt{a}$. By induction, assume $x_k \ge \sqrt{a}$ for $k = 1, \cdots, n-1$, then

$$x_n = (x_{n-1} + a/x_{n-1})/2 \ge \sqrt{x_{n-1} a / x_{n-1}} = \sqrt{a}.$$

We then prove that $x_{n+1} \le x_n$, indeed

$$x_n - x_{n+1} = (x_n - a/x_n)/2 = (x_n^2 - a)/(2x_n) \ge 0.$$

Since (x_n) is a monotone decreasing sequence, and is bounded from below, hence x_n converges. Let x denote the limit, since $x_n \ge \sqrt{a}$, the limit $x \ge \sqrt{a}$. Consider the relation $x_{n+1} = (x_n + a/x_n)/2$, and let $n \to \infty$, then we have $x = (x + a/x)/2$, solve for it, one gets $x = \pm\sqrt{a}$, and use the constraint $x \ge \sqrt{a}$, we get $x = \sqrt{a}$.

4. (See Rudin Thm 7.8) To show that $C(K)$ is complete, one need to show that every Cauchy sequence in $C(K)$ is converngent. Suppose (f_n) is a Cauchy sequence in $C(K)$, then for each $x \in K$, $(f_n(x))$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, hence Cauchy sequence in $\mathbb{R}$ has limit, let for each $x \in [0,1]$, let $f(x) = \lim_n f_n(x)$, i.e. we construct f so that $f_n \to f$ pointwise. Now we need to show that $f_n \to f$ uniformly. By Cauchy property of f_n, we know for any $\epsilon > 0$,there exists $N > 0$, such that for all $n, m > N$ and any $x \in K$, we have

$$|f_n(x) - f_m(x)| \le \epsilon.$$

Fix n and let $m \to \infty$ in the above relation, we get

$$|f_n(x) - f(x)| \le \epsilon, \forall x \in K, \forall n > N$$

This shows $f_n \to f$ uniformly, thus f is the limit of f_n in the metric space $C(K)$.

To see that $C(K)$ is not sequentially compact, we can just take the sequence of functions $f_n(x) = n$, then this sequence has no convergent subsequence.

5. It is true. If f is uniformly continuous, then for any $\epsilon > 0$, there is a $\delta > 0$, such that for any $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Fix $\epsilon = 1$ and obtain the corresponding $\delta > 0$. Let integer $N > 0$ be large enough, such that $1/\delta < N$. Then for any $x, y \in (0, 1)$, say $x < y$, we can let $x_0 = x, x_N = y$, and $x_k = (y - x)(k/N) + x_0$ for $k = 1, \cdots, N - 1$. Then $|x_k - x_{k-1}| = (y - x)/N < 1/N < \delta$, hence $|f(x_k) - f(x_{k-1})| \leq \epsilon = 1$. Thus, we have

$$|f(y) - f(x)| = |\sum_{k=1}^{N} f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{N} 1 = N.$$

Thus, the image $f(0, 1)$ is a bounded subset in $\mathbb{R}$.

6. Define function $g(x) = f(x + c) - f(x)$, we just need to show that $g(x)$ equal to 0 for some x. Since $f(x)$ is periodic with period 1, then $g(x)$ is also periodic with period 1. Furthermore, for any $a \in \mathbb{R}$, we have

$$\int_a^{a+1} g(x)dx = \int_{a+c}^{a+c+1} f(x)dx - \int_a^{a+1} f(x)dx = 0.$$

Hence, either $g(x) = 0$ for all x, in which case we have nothing to prove; or $g(x) > 0$ for some x, then by the above integral condition, we have $g(x) < 0$ for some other x. Thus, by intermediate value theorem, $g(x) = 0$ for some x.

In the case $c = 1/2$, we can avoid using integral. We note that $g(x) = f(x + 1/2) - f(x) = f(x - 1/2) - f(x) = -g(x + 1/2)$. Assume $g(x) \neq 0$ for some x. Then, if $g(x) > 0$ for some x, then $g(x + 1/2) < 0$, thus there is a $x' \in (x, x + 1/2)$, such that $g(x') = 0$.

7. We can consider mean value theorem. For $x > 0$, consider the interval $(0, x)$, then we have

$$f(x) - f(0) = f'(c)(x - 0)$$

for some $c \in (0, x)$. Using $f(0) = 0$, we have

$$f(x)/x = f'(c) \in [1, 2].$$

Thus $x \leq f(x) \leq 2x$ for all $x > 0$.

8. True. Since B is compact, hence B is closed. We know closed set intersect compact set is compact, hence $A \cap B$ is compact.

9. We may clear denominator and apply L'hopital rule twice. Answer is $-1/2$.

10. We can show $f_n \to 1/2$ uniformly, and use integral commute with uniform convergence, to compute $\int_1^2 (1/2)dx = 1/2$.