Math 104 Midterm 1 Sep 22, 2021

20+10+10+10 = 50 points total.

1. (20 points, 2 points each) True or False? No explanation required.
   (1) The set of irrational numbers in $\mathbb{R}$ forms a field.
   (2) Let $S \subset \mathbb{R}$ be a bounded subset. If S consists of only irrational numbers, then $\sup(S)$ is a irrational number.
   (3) Let $A, B \subset \mathbb{R}$ be two bounded subsets, then $\sup(A \cup B) = \max(\sup(A), \sup(B))$.
   (4) If a convergent sequence (s_n) takes value 0 infinitely many times, then (s_n) converges to 0.
   (5) Let (s_n) be a sequence such that $\limsup(s_n) = 0$, then there exists an $N > 0$, such that for all $n > N$, $s_n \leq 0$.
   (6) Let (s_n) converge to 1, then it is still possible that s_n can take value $1/2$ infinitely many times.
   (7) Let (a_n) be a sequence in $\mathbb{R}$, if for any $\epsilon > 0$, there exists an $N > 0$, such that $|a_n - a_{n+1}| < \epsilon$ for all $n > N$, then (a_n) is a Cauchy sequence.
   (8) Let (s_n) be a sequence of real number. Suppose that for any $\epsilon > 0$ and for any integer $N > 0$, there exists $n, m > N$, such that $|s_n| < \epsilon$ and $|s_m - 1| < \epsilon$. Then the sequence (s_n) is not convergent.
   (9) Let (a_n) be a bounded sequence, and $L_+ = \limsup a_n, L_- = \liminf a_n$, then there are infinitely many $n \in \mathbb{N}$ such that $L_- \leq a_n \leq L_+$.
   (10) Let (a_n) be a bounded sequence, and $L_+ = \limsup a_n$, then for any $\epsilon > 0$, there are finitely many $n \in \mathbb{N}$, such that $a_n > L_+ + \epsilon$.
2. (10 points, 2.5 each) Construct the following sequences. No proof is required that they satisfy the desired property.
   (1) Give an example of a Cauchy sequence (a_n) in $\mathbb{Q}$, such that $a_n \neq 1/2$ for any n, but $\lim a_n = 1/2$.
   (2) Give an example of a bounded sequence (a_n) in $\mathbb{R}$, such that $|a_n - a_{n+1}|$ is a monotone strictly increasing.
   (3) Give an example of a sequence (a_n) in $\mathbb{R}$, such that (a_n) is unboudned above and unbounded below.
   (4) Give an example of a sequence (a_n) in $\mathbb{R}$, such that for any $k \in \mathbb{N}$, k is a limit point of (a_n). (no need to give a formula, just describe how you would construct (a_n).)
3. (10 points) Let s_n and t_n be Cauchy sequences in $\mathbb{R}$. Use only the definition of Cauchy sequence to prove that $2s_n + 3t_n$ is also a Cauchy sequence.
4. (10 points) Given any $x, y \in \mathbb{R}$ and $t \in (0, 1)$. Let (a_n) be given by $a_1 = x, a_2 = y$ and $a_n = ta_{n-1} + (1-t)a_{n-2}$ for $n \geq 3$. Prove that a_n is convergent. If you like, you can set $x = 0, y = 1, t = 1/2$. Hint: you may use the homework result that, if there is an $N > 0$ and $0 < r < 1$, such that $|a_{n+1} - a_n|/|a_n - a_{n-1}| < r$ for all $n > N$, then (a_n) is convergent.

0.1 Solution

1. FFTTF FFTFT

2.1 $1/2 + 1/n$;

2.2 $a_n = (-1)^n(1 - 1/n)$, then $|a_{n+1} - a_n| = (1 - 1/(n + 1)) + (1 - 1/n)$, hence is strictly monotone increasing.

2.3 $a_n = (-1)^n n$

2.4 Take a_n to be the following sequence (the gap is only for illustration of the grouping)

$$1, \quad 1, 2, \quad 1, 2, 3, \quad 1, 2, 3, 4, \quad 1, 2, 3, 4, 5; \cdots$$

3. see lecture note or textbook (Ross)

4. Consider the difference

$$a_{n+1} - a_n = ta_n + (1 - t)a_{n-1} - a_n = (1 - t)(-a_n + a_{n-1})$$

hence

$$\frac{|a_{n+1} - a_n|}{|-a_n + a_{n-1}|} = 1 - t < 1$$

By our homework, this is convergent.