Math 104 Midterm 2 Nov 10, 2021

20+15 + 15 = 50 points total.

All subsets of $\mathbb{R}$, for example $\mathbb{Z}$, $\mathbb{Q}$, are understood to have the induced topology and the induced metric. $\mathbb{R}^2$ is equipped with Euclidean metric.

1. (20 points, 2 points each) True or False? No explanation required.

F (1) Let X be a topological space. If $E \subset X$ is open and $F \subset X$ is closed, then $E \cap F$ is neither open or closed. $(0,1) \cap [-1,2]$ $[0,1] \cap (-1,2)$

T (2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then for any $p \in \mathbb{R}^2$ with $f(p) < 0$, there exists a $\epsilon > 0$, such that $f(B_\epsilon(p)) \subset (-\infty, 0)$. $\because f^{-1}((-\infty,0))$ is open.

F (3) Let $E \subset \mathbb{R}^2$ be a closed set. Then, for any $p \in E$, there exists a $r > 0$, such that the closed ball $\overline{B_r(p)} \subset E$. $E = \{(0,0)\}$

T (4) Let $X = \{(x, 0) \mid x \in [0, 2]\} \subset \mathbb{R}^2$, then X is a compact set. Heine-Borel. in $\mathbb{R}^n$ compact = closed + bounded

F (5) Let E be a bounded and closed subset of a metric space X, then E is compact.

F (6) Let $X = [0, 1]$, and let $Y = \mathbb{R}$, then we can find a continuous surjective map $f : X \to Y$. X is compact, f is continuous, then $f(X)$ is compact.

T (7) Let $X = \mathbb{Z}$ and let $Y = \mathbb{Q}$, then we can find a continuous surjective map $f : X \to Y$.

T (8) Let X be a metric space, $E \subset X$ be a closed set, $x \in X$ be a point not in E. Then, $\inf\{d(y, x) \mid y \in E\} > 0$. Yes. $\because x \in E^c$, E^c open $\therefore$

F (9) Let $\sum_{n=1}^{\infty} a_n$ be a series of with positive terms. Let $r = \limsup a_n^{1/n}$. If the series is divergent, then $r > 1$.

F (10) If $f : [0, 1] \cap \mathbb{Q} \to \mathbb{R}$ is continuous, then f is uniformly continuous. $f(x) = \begin{cases} 0 & x < \frac{\sqrt{2}}{2} \\ 1 & x > \frac{\sqrt{2}}{2} \end{cases}$

2. (15 points, 3 each) Construct examples that satisfies the following properties. No proof is required that they satisfy the given property.

(1) Give an example of a series $\sum_n a_n$, that converges but does not absolutely converge. $\sum (-1)^n \frac{1}{n}$

(2) Give an example of bounded monotone increasing function $f : (0, \infty) \to (0, \infty)$ such that f is discontinuous at all positive integers. $f(x) = 2 - \frac{1}{\lceil x \rceil}$

(3) Give an example of a bounded continuous real valued function on $(0, 1)$ that is not uniformly continuous. $\sin(\frac{1}{x})$

(4) Give an example of continuous map $f : \mathbb{R} \to \mathbb{R}$ and a open subset $E \subset \mathbb{R}$, such that $f(E)$ is not open. $f(x) = 0$, $E = (0,1)$

(5) Give an example of continuous map $f : \mathbb{R} \to \mathbb{R}$ and a compact subset $E \subset \mathbb{R}$, such that $f^{-1}(E)$ is not compact. $f(x) = 0$, $E = \{0\}$

3. (15 points, 5 each)

(1) Let $X = (-1, 1)$, and let $E = (0, 1)$ be a subset of X. What is the closure of E in X? Please justify your answer. $[0,1)$

(2) Let $X = (0, 1)$, and let $E = \{1/n \mid n = 2, 3, \cdots\}$ be a subset of X. Is E closed in X? Please justify your answer. yes, closed.

(3) Prove that the only connected subsets of $\mathbb{Q}$ are singleton sets, i.e., subset with only one element. (You may find the characterization of connected subset in $\mathbb{R}$ useful: $E \subset \mathbb{R}$ is connected if and only if the following condition is true: for any $x, y \in E$ with $x < y$, we have $[x, y] \subset E$.)